Correction to: Irrationality issues for projective surfaces

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The purpose of this note is to fix an imprecision in Theorem 3.3 of the original paper, which affects the correctness of the statement and leads to a mistake in Corollary 3.5 of the original paper. In particular, given a smooth surface $S \subset \mathbb{P}^3$ of degree $d \geq 5$, it is not true that all the families of curves computing the covering and the connecting gonality of S are listed in Example 3.4 of the original paper, as it is shown in the example below. We refer to the original paper for the setting and notation.

According to [1], where the correct assertion of Theorem 3.3 of the original paper is included in [1, Corollary 1.7], we introduce a notion of equivalence for families of curves covering S and admitting a $\mathfrak{g}_k^1$, as follows. Let $\mathcal{C} \xrightarrow{\pi} T$ be a *covering family of k -gonal curves*, i.e. a family of irreducible curves $C_t = \pi^{-1}(t)$ endowed with a dominant morphism $f: \mathcal{C} \rightarrow S$ such that for general $t \in T$, $\text{gon}(C_t) = k$ and $f|_{C_t}$ is birational. Following [1, Proof of Corollary 1.7], we obtain a diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & B \times \mathbb{P}^1 \\ & \searrow & \downarrow \text{pr}_1 \\ & & B \end{array} \quad (1)$$

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where $B \rightarrow H$ is a finite cover of a subscheme $H \subset \text{Hilb}(S)$ of the Hilbert scheme of curves on S , the family of curves $\mathcal{E} \rightarrow B$ is the corresponding pullback of the universal family over $\text{Hilb}(S)$, and the restriction $F_b: E_b \rightarrow \{b\} \times \mathbb{P}^1$ is a base-point-free $\mathfrak{g}_k^1$. Therefore, denoting by $S^{(k)}$ the k -fold symmetric product of S , we may define a morphism $\gamma: B \times \mathbb{P}^1 \rightarrow S^{(k)}$ sending a point (b, y) to the 0-cycle $x_1 + \dots + x_k$ such that $F^{-1}(b, y) = \{x_1, \dots, x_k\}$. In particular, the image $\gamma(B \times \mathbb{P}^1)$ parameterizes the fibres of the k -gonal maps $F_b: E_b \rightarrow \{b\} \times \mathbb{P}^1$ as b varies in B . Then we give the following (cf. [1, Definition 3.3]).

Definition Let $\mathcal{C} \xrightarrow{\pi} T$ and $\mathcal{C}' \xrightarrow{\pi'} T'$ be two covering families of k -gonal curves. We say that the families are *equivalent* if—up to consider open subsets of $B \times \mathbb{P}^1$ and $B' \times \mathbb{P}^1$ —the images $(\gamma(B \times \mathbb{P}^1))_{\text{red}}$ and $(\gamma'(B' \times \mathbb{P}^1))_{\text{red}}$ coincide.

Therefore two covering families of k -gonal curves are equivalent if the two families of $\mathfrak{g}_k^1$ give the same 0-cycles of length k on S . Thanks to this notion, we can now state the correct assertion of Theorem 3.3 of the original paper (see [1, Corollary 1.7]).

Theorem 3.3 *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$. Then the covering gonality of S is $\text{cov. gon}(S) = d - 2$, and any family of irreducible curves computing the covering gonality is equivalent to (a subfamily of) one of the families described in Example 3.4 of the original paper.*

Example Assume that there exist two rational curves $R_1, R_2 \subset S$. Given a birational map $\varphi: R_1 \dashrightarrow R_2$, let Σ_φ be the ruled surface swept out by the lines ℓ_q joining $q \in R_1$ and $\varphi(q) \in R_2$, and let $C_\varphi := (\overline{S \cap \Sigma_\varphi}) \setminus (R_1 \cup R_2)$ be the curve cut out on S outside R_1 and R_2 . Then C_φ admits a map $C_\varphi \dashrightarrow R_1$ of degree $d - 2$, which sends to $q \in R_1$ the points on the line ℓ_q . Moreover, by considering a one dimensional family of maps $\varphi: R_1 \dashrightarrow R_2$, we obtain a covering family of $(d - 2)$ -gonal curves which does not appear in Example 3.4 of the original paper. However, this family is equivalent to the family of curves D_p in Example 3.4(ii) of the original paper with $R = R_1$ and $p \in R_2$, since both the families describe the same 0-cycles of length $d - 2$ on S .

Analogously, the assertion of Corollary 3.5 of the original paper can be fixed as follows. For the sake of completeness, we include a short proof.

Corollary 3.5 *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$. Then the connecting gonality of S is $\text{conn. gon}(S) = d - 2$, and any family of irreducible curves computing the connecting gonality is equivalent either to the family of tangent hyperplane sections in Example 3.4(i) of the original paper, or to the family described in Example 3.4(ii) of the original paper.*

Proof As observed in the discussion preceding Corollary 3.5 of the original paper, the family of tangent hyperplane sections computes the connecting gonality of S , so that $\text{conn. gon}(S) = d - 2$.

Now, let $\mathcal{C} \xrightarrow{\pi} T$ be a family of $(d - 2)$ -gonal curves computing the connecting gonality of S . Hence it induces a diagram as (1), with $\dim B \geq 2$ because for general $x, y \in S$ there is a curve E_b passing through them. As in [1, Proof of Corollary 1.7], the fibre $F^{-1}(b, y) = \{x_1, \dots, x_{d-2}\}$ over a general $(b, y) \in B \times \mathbb{P}^1$ consists of $d - 2$ collinear points. Let $\ell_{(b,y)} \subset \mathbb{P}^3$ be the line containing $F^{-1}(b, y)$, and let $x_{d-1}, x_d \in S$ be the points residual to $F^{-1}(b, y)$ in the intersection $S \cap \ell_{(b,y)}$.

If x_{d-1} and x_d coincide, then $\ell_{(b,y)}$ is tangent to S at x_d , so that the family $\mathcal{C} \xrightarrow{\pi} T$ is equivalent to the family of tangent hyperplane sections in Example 3.4(i) of the original paper. When instead x_{d-1} and x_d are distinct, we consider the ruled surface Σ_b swept out by the lines $\ell_{(b,y)}$ with $(b, y) \in \{b\} \times \mathbb{P}^1$, and we define the curve $Z_b := (\overline{S \cap \Sigma_b}) \setminus E_b$.

We claim that Z_b consists of rational components. If one of the points $x_{d-1}, x_d \in S$ is fixed as we vary $(b, y) \in \{b\} \times \mathbb{P}^1$, then Z_b is described by the other point, so that Z_b is an irreducible rational curve. If instead both x_{d-1} and x_d vary, we note that Z_b is dominated by the curve $\{(x, y) \in Z_b \times \mathbb{P}^1 \mid x \in \ell_{(b,y)}\}$, whose second projection gives a degree two map to $\mathbb{P}^1$. Thus Z_b admits a map $Z_b \dashrightarrow \mathbb{P}^1$ of degree two, and since Theorem 3.3 assures that $\text{cov. gon}(S) \geq 3$, we conclude that Z_b is fixed as b varies on some open subset of B . Therefore, either $Z = Z_b$ consists of two rational components, or it is irreducible. In the latter case, as we vary $(b, y) \in \{b\} \times \mathbb{P}^1$, the 0-cycle $x_{d-1} + x_d$ describes a rational curve in the second symmetric product $Z^{(2)}$ of Z . By varying also $b \in B$, we obtain a two-dimensional family of rational curve in $Z^{(2)}$, which forces the curve Z to be rational.

Since S is not covered by rational curves, there exists a rational curve R , which is a component of Z_b for any sufficiently general $b \in B$. In particular, for general $(b, y) \in B \times \mathbb{P}^1$, the line $\ell_{(b,y)}$ containing the fibre $F^{-1}(b, y)$ meets R , so that the family $\mathcal{C} \xrightarrow{\pi} T$ is equivalent to the family described in Example 3.4(ii) of the original paper. $\square$

Reference

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