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Remarks on Azarov's work on soluble groups of finite rank

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Abstract We present proofs of D. N. Azarov's recent three theorems determining precisely when a soluble group of finite rank is residually a finite π-group for a specified finite set π of primes. Our proofs seem to be substantially shorter; they also apply to groups with a somewhat weaker notion of finite rank.

Keywords Soluble group · Finite rank · Residually finite

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According to its English summary Azarov's paper [1] is devoted to proving the following. If π is a finite set of primes, then a soluble group of finite rank is a finite extension of a residually a finite π-group if and only if it is a finite extension of a residually finite nilpotent π-group, which happens if and only if it is a reduced FATR group with no π-divisible elements of infinite order (see below or see [3] for definitions). Further he proves that a soluble group of finite rank is residually a finite π-group for some finite set π of primes if and only if it is a reduced FATR group. (This is effectively an old result of D. J. S. Robinson, see [3] 5.3.8 or [4] Page 138.) We give below what presumably are much shorter proofs of these interesting results. In fact we prove the following. (Below τ(G) denotes the maximal periodic normal subgroup of a group G, FittG its Fitting subgroup and $\{\zeta_\alpha(G)\}$ its upper central series.)

Theorem *Let G be a finite extension of a soluble FAR group and let π be some finite set of primes. The following are equivalent.*

(a) *G is a finite extension of a residually finite π-group.*
(b) *$\tau(G)$ is finite and $\zeta_1(FittH)$ is π-reduced for some normal subgroup H of G of finite index.*
(c) *G is a finite extension of a residually finite nilpotent π-group.*

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(d) *G is a finite extension of a reduced soluble FATR group with no π-divisible elements of infinite order.*

Corollary *Suppose G is a finite extension of a soluble FAR group. The following are equivalent.*

(a) *There exists a finite set π of primes such that G is residually a finite π-group.*
(b) *G is reduced and a finite extension of an FATR group.*
(c) *$\tau(G)$ is finite and $\zeta_1(FittG)$ is reduced.*

Soluble FAR and FATR groups are defined in [3]. An equivalent definition, often more convenient, is the following. A soluble group G is FAR if it has finite Hirsch number and satisfies min-p for every prime p. (A group G has Hirsch number h if G has a series of finite length with exactly h of the factors infinite cyclic, the remaining factors of the series being locally finite; G satisfies min-p if it satisfies the minimal condition on p-subgroups.)

It is elementary that to within some normal subgroup of finite index, locally finite factors can essentially be moved down a series past torsion-free abelian factors of finite rank and finite factors can be moved up past torsion-free abelian factors of finite rank and past periodic abelian factors satisfying min-p for all primes p. Further divisible abelian factors in a periodic FAR group sink to the bottom (e.g. [2] 3.18). Thus it is elementary to see that a group G is a finite extension of a soluble FAR group if and only if it has a characteristic series

$$\langle 1\rangle = G_0 \leq G_1 \leq \cdots \leq G_r \leq \cdots \leq G_s \leq G,$$

where G_1 is periodic, divisible, abelian and satisfies min-p for all primes p, G_{i+1}/G_i for $1 \leq i < r$ is infinite periodic abelian with all its primary components (i.e. its Sylow subgroups) finite, G_{i+1}/G_i for $r \leq i <$ s is torsion-free abelian of finite rank and G/G_s is finite. The soluble FATR groups are exactly those G above with G_1 involving only finitely many primes, with r $= 1$ and, of course, with G/G_s soluble.

Suppose G is a finite extension of a soluble FAR group with its maximal periodic normal subgroup τ(G) finite. Then from the above, the following hold.

(a) G is (torsion-free)-by-finite and
(b) G is a finite extension of a soluble FATR group. Further
(c) FittG is nilpotent and G/FittG is abelian-by-finite (see [3] 5.2.2) and
(d) G/FittG is a finite extension of a free abelian group of finite rank (see [3] 5.2.3).

Note that in general FittG_s = $G_s \cap$FittG, FittG/FittG_s is finite and FittG is nilpotent if and only FittG_s is nilpotent. Also the soluble groups G of finite rank discussed by Azarov in [1] are exactly FAR groups above with G/G_s soluble and G_r of finite rank.

The proofs

We use the following elementary results.

Lemma 1 *Let A be a torsion-free abelian group.*

(a) *If π is any set of primes then A is residually a finite π-group if and only if A is π-reduced.*
(b) *If A is reduced and of finite rank, then for some finite set π of primes, A is residually a finite π-group.*

For example see [3] 5.3.4 *and* 5.3.5. *We also use the following, see* [4] 9.38.

Lemma 2 *Let G be a nilpotent FAR group and π any set of primes. Then G is residually a finite π-group if and only if $\zeta_1(G)$ is π-reduced.*

Thus in Lemma 2 if ζ_1(G) is π-reduced, then G is residually a finite π-group, so each G/ζ_i(G) is also residually a finite π-group by Learner's Lemma. Hence each ζ_{i+1}(G)/ζ_i(G) is π-reduced, something that is easy to see directly.

The proof of the theorem (a) implies (b). Let H be a normal subgroup of G that is residually a finite π-group. Then ζ_1(FittH) is π-reduced by Lemma 1. Also τ(H) is a π-group. If P is a p-subgroup of τ(H) then P is Chernikov, residually finite and hence finite, and π is finite. Consequently τ(H) is finite. Clearly τ(G)/τ(H) is finite, so τ(G) is finite.

(b) implies (c). (This is actually the core of the proof of the Theorem.) There exist normal subgroups N $\leq$ L of G with L $\leq$ H, G/L finite, L/N free abelian of finite rank and N = FittL torsion-free nilpotent of finite rank. Clearly N $\leq$ FittH. By (b) and Lemma 2 FittH is residually a finite π-group, so N is residually a finite π-group.

Set q = $\Pi_{p\in\pi}$ p and $M = \cap_i C_L(\zeta_{i+1}(N)^q\zeta_i(N)/\zeta_i(N))$. Clearly N $\leq$ M $\leq$ L and G/M is finite. We claim that M is residually a finite nilpotent π-group. If so then c) holds. Now M/N is free abelian, so M/N at least is residually a finite nilpotent π-group.

Let x $\in$ N$\backslash\langle 1\rangle$. Since N is residually a finite π-group, there exists a power m = q^μ of q with x $\notin$ N^m. Now N/N^m is finite and M/N is polycyclic, so there exists a torsion-free normal subgroup T/N^m in M/N^m with M/T finite. Also N/N^m lies in the hypercentre of M/N^m and M/N is abelian. Hence M/N^m is nilpotent, as therefore is its finite image M/T. Let S/T denote the Hall π'-subgroup of M/T. Then M/S is a finite nilpotent π-group. Further x $\notin$ S, since x is a non-trivial π-element modulo N^m and S/N^m is an extension of a torsion-free group by a π'-group. It follows that M is residually a finite nilpotent π-group.

(c) implies (a). This is trivial. Thus (a), (b) and (c) are equivalent.

(a) and (b) imply (d). G is residually finite, so G is reduced. Also τ(G) is finite, so G is a finite extension of a residually finite-π, reduced FATR group H. Since H is residually finite-π, so H contains no π-divisible elements of infinite order. Consequently neither does G.

(d) implies (b). By (d) G has a reduced soluble normal FATR subgroup H of finite index. Then H is (torsion-free)-by-finite, τ(H) is finite and consequently τ(G) is finite. Further we may choose H torsion-free. Then H has no non-trivial π-divisible elements by (d) and hence ζ_1(FittH) is π-reduced. Thus (b) holds.

The proof of the corollary If (a) holds, then so does (b) by the Theorem. Clearly (b) implies (c). Suppose (c) holds. By Lemma 1 there exists a finite set κ of primes such that ζ_1(FittG) is κ-reduced. Hence by the Theorem, (b) implies (a), there exists a normal subgroup H of G of finite index that is residually a finite κ-group. But then G is residually a finite π-group for $\pi = \kappa \cup$ {all prime divisors of (G:H)}. Thus (a) holds.

Remark In a special but still quite general case there is a slicker but less elementary proof of (b) implies (c), the main implication of the theorem.

With N as in the original proof let N_p denote the finite-p residual of N. Then $\cap_{p\in\pi} N_p = \langle 1\rangle$. The upper central factors of N/N_p are p-reduced (Lemma 2). If they are actually (torsion-free)-by-(a p'-group)-by-finite, then G/N_p embeds into GL(n, J) for some integer n and J the integers localized at p. Thus G/N_p is a finite extension of a residually finite p-group and consequently G is a finite extension of a residually finite nilpotent π-group.

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