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Generic properties for random repeated quantum iterations

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Abstract We denote by M^n the set of n by n complex matrices. Given a fixed density matrix $\beta : \mathbb{C}^n \to \mathbb{C}^n$ and a fixed unitary operator $U : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$, the transformation $\Phi : M^n \to M^n$

$$Q \to \Phi(Q) = \mathrm{Tr}_2(U(Q \otimes \beta)U^*)$$

describes the interaction of Q with the external source β. The result of this operation is $\Phi(Q)$. If Q is a density operator then $\Phi(Q)$ is also a density operator. The main interest is to know what happens when we repeat several times the action of Φ in an initial fixed density operator Q_0. This procedure is known as random repeated quantum iterations and is of course related to the existence of one or more fixed points for Φ. In Nechita and Pellegrini (Probab Theory Relat Fields 52:299–320, 2012), among other things, the authors show that for a fixed β, there exists a set of full probability for the Haar measure such that the unitary operator U satisfies the property that for the associated Φ there is a unique fixed point Q_Φ. Moreover, there exists convergence of the iterates $\Phi^n(Q_0) \to Q_\Phi$, when $n \to \infty$, for any given initial Q_0. We show here that there is an open and dense set of unitary operators $U : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ such that the associated Φ has a unique fixed point. We will also consider a detailed analysis of the case when $n = 2$. We will be able to show explicit results. We consider the C^0 topology on the coefficients of U. In this case, we will exhibit the explicit expression on the coefficients of U which assures the existence of a unique fixed point for Φ. Moreover, we present the explicit expression of the fixed point Q_Φ.

Keywords Random repeated quantum iterations · Density matrices · Unitary operators · External source · Fixed point · Generic property · Kraus decomposition · Stinespring dilation

1 Introduction

We denote by M^n the set of n by n complex matrices. Given a fixed density matrix $\beta : \mathbb{C}^n \to \mathbb{C}^n$ and a fixed unitary operator $U : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$, the transformation $\Phi : M^n \to M^n$

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$$Q \to \Phi(Q) = \mathrm{Tr}_2(U(Q \otimes \beta)U^*)$$

describes the interaction of Q with the external source β.

We assume that all eigenvalues of β are strictly positive.

In [4], the model is precisely explained: Q is in the small system and β describes the environment. Then $\Phi(Q)$ gives the output of the action of β in Q given the action of the unitary operator U.

Other related papers are [2,3]. Our proof is of quite different nature than these other papers.

The main question is about the convergence of the iterates $\Phi^n(Q_0)$, when $n \to \infty$, for any given Q_0. It is natural to expect that any limit (if exists) is a fixed point for Φ.

Our purpose is to show the following theorem:

Theorem 1 *Given a fixed density matrix $\beta : \mathbb{C}^n \to \mathbb{C}^n$, for an open and dense set of unitary operators $U : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ the transformation $\Phi : M^n \to M^n$*

$$Q \to \Phi(Q) = \mathrm{Tr}_2(U(Q \otimes \beta)U^*)$$

has a unique fixed point Q_Φ. In the case $n = 2$, we present explicitly the analytic characterization of such family of U and also the explicit formula for Q_Φ.

This result implies one of the main results in [4] that we mentioned before.

2 The general dimensional case

Suppose V is a complex Hilbert space of dimension n ≥ 2 and $\mathscr{L}(V)$ denotes the space of linear transformations of V in itself.

Then, $\mathrm{Tr}_2 : \mathscr{L}(V \otimes V) \to \mathscr{L}(V)$, given by $\mathrm{Tr}_2(A \otimes B) = \mathrm{Tr}(B)A$.

There is a canonical way to extend the inner product on V to $V \otimes V$.

We fix a density matrix $\beta \in \mathscr{L}(V)$. For each unitary operator $U \in \mathscr{L}(V \otimes V)$, we denote by $\Phi_U : \mathscr{L}(V) \to \mathscr{L}(V)$ the linear transformation

$$\Phi_U(A) = \mathrm{Tr}_2(U(A \otimes \beta)U^*).$$

We denote by $\Gamma \subset \mathscr{L}(V)$ the set of density operators. It will be shown that Φ_U preserves Γ. As Γ is a convex compact space, it has a fixed point.

The set of unitary operators is denoted by $\mathscr{U}$.

If A is such that $\Phi_U(A) = A$, then it follows that the range of $\Phi_U - I$ is smaller or equal to $n^2 - 1$.

We will show that there exists a proper real analytic subset $X \subset \mathscr{U}$ such that if U is not in X, then range of $\Phi_U - I = n^2 - 1$. In this case, the fixed point is unique. More precisely

$$X = \{U \in \mathscr{U} : \text{ range } (\Phi_U - I) < n^2 - 1\}.$$

This $X \subset \mathscr{U}$ is an analytic set because it is described by equations given by the determinant of minors equal to zero. It is known that the complement of an analytic set, also known as a Zariski open set, is empty or is open and dense on the analytic manifold (see [1]). Therefore, to prove our main result, we have to present an explicit U such that range of $(\Phi_U - I)$ is $n^2 - 1$.

This will be the purpose of our reasoning described below.

The bilinear transformation $(A, B) \to \mathrm{Tr}(B)A$ from $\mathscr{L}(V) \times \mathscr{L}(V)$ to $\mathscr{L}(V)$ induces the linear transformation

$$\mathrm{Tr}_2 : \mathscr{L}(V \otimes V) = [\mathscr{L}(V) \otimes \mathscr{L}(V)] \to \mathscr{L}(V).$$

Denote by $e_1, e_2, \dots, e_n$ an orthonormal basis for V. We also denote $L_{ij} \in \mathscr{L}(V)$ the transformation such that $L_{ij}(e_j) = e_i$ and $L_{ij}(e_k) = 0$ if $k \neq j$.

The L_{ij} provides a basis for $\mathscr{L}(V)$.

If $A \in \mathscr{L}(V)$, we can write $A = \sum_{i,j} a_{ij} L_{ij}$ and we call $[a_{ij}]_{1 \le i,j \le n}$ the matrix of A.

Note that $e_i \otimes e_j$, $1 \le i, j \le n$ is an orthonormal basis of $V \otimes V$. Moreover,

$$L_{ik} \otimes L_{jl}(e_k \otimes e_l) = e_i \otimes e_j,$$

and

$$L_{ik} \otimes L_{jl}(e_p \otimes e_q) = 0 \quad \text{if } (p,q) \neq (k,l).$$

It is also true that:

(a) $L_{ij} L_{pq} = 0$ if $j \neq p$,
(b) $L_{ij} L_{pj} = L_{iq}$,
(c) Tr $(L_{ij}) = 0$ if $i \neq j$ and Tr $(L_{ii}) = 1$.

One can see that $L_{ik} \otimes L_{jl}$, $1 \le i, k, j, l \le n$ is a basis for $\mathscr{L}(V \otimes V)$.

Given $T \in \mathscr{L}(V \otimes V)$ denote $T = \sum t_{i,j,k,l} L_{ik} \otimes L_{jl}$. Then,

$$\mathrm{Tr}_2(T) = \sum t_{i,j,k,j} L_{ik} = \sum_{ik} \left(\sum_j t_{i,j,k,j} \right) L_{ik}.$$

In the appendix, we give a direct proof that: if $A \in \Gamma$, then $\Phi_U(A) \in \Gamma$, for all $U \in \mathscr{U}$.

Now we will express Φ_U in coordinates. We choose an orthonormal base $e_1, e_2, \dots, e_n \in V$ which diagonalize β. That is

$$\beta = \sum_q \lambda_q L_{qq}, \quad \lambda_q > 0, \;\; 1 \le q \le n, \;\; \sum_q \lambda_q = 1.$$

Given r, s, $1 \le r, s \le n$, we will calculate $\Phi_U(L_{rs})$.

Suppose $U = \sum u_{i,j,k,l} L_{ik} \otimes L_{jl}$, then $U^* = \sum \overline{u_{i,j,k,l}} L_{ik} \otimes L_{jl}$ and

$$(L_{rs} \otimes \beta) U^* = \left(\sum_q \lambda_q L_{rs} \otimes L_{qq} \right) U^* = \sum_j \lambda_j \overline{u_{k,l,s,j}} L_{rk} \otimes L_{jl}.$$

Now, we write $U = \sum u_{\alpha,\beta,\gamma,\delta} L_{\alpha\gamma} \otimes L_{\beta\delta}$. Then, we get

$$U(L_{rs} \otimes \beta) U^* = \sum \lambda_j u_{\alpha,\beta,r,j} \overline{u_{k,l,s,j}} L_{\alpha k} \otimes L_{\beta l}.$$

Finally,

$$\Phi_U(L_{rs}) = \sum \lambda_j u_{\alpha,l,r,j} \overline{u_{k,l,s,j}} L_{\alpha k} = \sum_{\alpha,k} \left(\sum_{j,l} \lambda_j u_{\alpha,l,r,j} \overline{u_{k,l,s,j}} \lambda_j \right) L_{\alpha k}.$$

As Γ is convex and compact and ϕ_U is continuous as we said before there exists a fixed point $A \in \Gamma$. In particular, the range of ϕ_U is smaller or equal to $n^2 - 1$.

We will present an explicit U such that range of $(\Phi_U - I)$ is $n^2 - 1$.

This will be described by a certain kind of circulant unitary operator

Suppose $u_1, u_2, \ldots, u_{n^2}$ are complex numbers of modulus 1. We define U in the following way

$$U(e_1 \otimes e_1) = u_1(e_1 \otimes e_2), U(e_1 \otimes e_2) = u_2(e_1 \otimes e_3), \ldots, U(e_1 \otimes e_n) = u_n(e_2 \otimes e_1),$$
$$U(e_2 \otimes e_1) = u_{n+1}(e_2 \otimes e_2), U(e_2 \otimes e_2) = u_{n+2}(e_2 \otimes e_3), \ldots, U(e_2 \otimes e_n) = u_{2n}(e_3 \otimes e_1),$$
$$\ldots$$
$$U(e_n \otimes e_1) = u_{n^2-n+1}(e_n \otimes e_2), U(e_n \otimes e_2) = u_{n^2-n+2}(e_n \otimes e_3), \ldots, U(e_n \otimes e_n) = u_{n^2}(e_1 \otimes e_1),$$

We will show that for some convenient choice of $u_1, u_2, \ldots, u_{n^2}$ we will get that the range of $\Phi_U - I$ is $n^2 - 1$. Suppose

$$U = \sum u_{i,j,k,l} L_{ik} \otimes L_{jl},$$

in this case

$$U(e_k \otimes e_l) = \sum_{i,j} u_{i,j,k,l} e_i \otimes e_j.$$

By definition of U, we get

(a) if $l < n$, then $u_{i,j,k,l} \neq 0$, if and only if, $i = k,\ j = l + 1$;
(b) if $k < n$, then $u_{i,j,k,n} \neq 0$, if and only if, $i = k + 1,\ j = 1$;
(c) $u_{i,j,n,n} \neq 0$, if and only if, $i = j = 1$.

For fixed r, s such that $1 \leq, r, s \leq n$ we get from (a)–(c):
$1 \leq r < n,\ 1 \leq s < n$, implies

$$\Phi_U(L_{rs}) = \left(\sum_{j=1}^{n-1} u_{r,j+1,r,j} \overline{u_{s,j+1,s,j}} \lambda_j \right) L_{rs} + u_{r+1,1,r,n} \overline{u_{s+1,1,s,n}} \lambda_n L_{(r+1)(s+1)},$$

$1 \leq s < n$, implies

$$\Phi_U(L_{ns}) = \left(\sum_{j=1}^{n-1} u_{n,j+1,n,j} \overline{u_{s,j+1,s,j}} \lambda_j \right) L_{ns} + u_{1,1,n,n} \overline{u_{s+1,1,s,n}} \lambda_n L_{1(s+1)},$$

$1 \leq r < n$, implies

$$\Phi_U(L_{rn}) = \left(\sum_{j=1}^{n-1} u_{r,j+1,r,j} \overline{u_{n,j+1,n,j}} \lambda_j \right) L_{rn} + u_{r+1,1,r,n} \overline{u_{1,1,n,n}} \lambda_n L_{(r+1)1}.$$

In particular for $1 \leq r < n$, we have $\Phi_U(L_{rr}) = (1-\lambda_n) L_{rr} + \lambda_n L_{(r+1)(r+1)}$. To show that the range of $\Phi_U - I$ is $n^2 - 1$,we will show that the $\phi_U(L_{rs}) - L_{rs}$ are linearly independent for $(r, s) \neq (n, n)$

Suppose that

$$\sum_{(r,s)\neq(n,n)} c_{rs}(\phi_U(L_{rs}) - L_{rs}) = 0.$$

The coefficient of L_{11} is $-\lambda_n c_{11}$, then $c_{11} = 0$. The coefficient of L_{22} is $\lambda_n c_{11} - \lambda_n c_{22}$, then $c_{22} = 0$.

...

The coefficient of L_{nn} is $\lambda_n c_{(n-1)(n-1)}$, then $c_{(n-1)(n-1)} = 0$.

Then, we get that

$$\sum_{r\neq s} c_{rs}(\phi_U(L_{rs}) - L_{rs}) = 0. \tag{1}$$

We will divide the proof in several different cases.
(a) Case $n = 2$.

$$\sum_{r \neq s} c_{rs}(\phi_U(L_{rs}) - L_{rs}) = c_{12}(\phi_U(L_{12}) - L_{12}) + c_{21}(\phi_U(L_{21}) - L_{21}).$$

By definition of U, we have that $u_{1,2,1,1} = u_1$, $u_{2,1,1,2} = u_2$, $u_{2,2,2,1} = u_3$, $u_{1,1,2,2} = u_4$.
Therefore,

$$\phi_U(L_{12}) - L_{12} = (u_1\overline{u_3}\lambda_1 - 1)L_{12} + u_2\overline{u_4}\lambda_2 L_{21}$$

and

$$\phi_U(L_{21}) - L_{21} = (u_3\overline{u_1}\lambda_1 - 1)L_{21} + u_4\overline{u_2}\lambda_2 L_{12}.$$

From (1), it follows that

$$(u_1\overline{u_3}\lambda_1 - 1)c_{12} + u_4\overline{u_2}\lambda_2 c_{21} = 0$$
$$u_2\overline{u_4}\lambda_2 c_{12} + (u_3\overline{u_1}\lambda_1 - 1)c_{21} = 0.$$

Taking U such that $u_1 = i$, $u_2 = u_3 = u_4 = 1$, it is easy to see that the determinant of the above system is not equal to zero. Then we get that $c_{12} = c_{21} = 0$.

Then, we get a U with maximal range.
(b) Case $n > 2$.

We choose $u_1, u_2, \ldots, u_{n^2}$ according to Lemma 1 below.

The equations we consider before can be written as
$1 \le r < n$, $1 \le s < n$, $r \neq s$, then, $\Phi_U(L_{rs}) - L_{rs} = (a_{rs} - 1)L_{rs} + b_{rs}L_{(r+1)(s+1)}$,
$1 \le s < n$, then, $\Phi_U(L_{ns}) - L_{ns} = (a_{ns} - 1)L_{ns} + b_{ns}L_{1(s+1)}$,
$1 \le r < n$, then, $\Phi_U(L_{rn}) - L_{rn} = (a_{rn} - 1)L_{rn} + b_{rn}L_{(r+1)1}$.

For instance

$$a_{rs} = \sum_{j=1}^{n-1} u_{r,j+1,r,j}\overline{u_{s,j+1,s,j}}\lambda_j,$$

and

$$b_{rs} = u_{r+1,1,r,n}\overline{u_{s+1,1,s,n}}\lambda_n.$$

Note that $u_{r,j+1,r,j}\overline{u_{s,j+1,s,j}}$ has modulus one and also $u_{r+1,1,r,n}\overline{u_{s+1,1,s,n}}$.

Moreover, $|b_{rs}| = \lambda_n > 0$ and $|a_{rs}| < \lambda_1 + \cdots + \lambda_{n-1}$. Indeed, note first that the products $u_{r,j+1,r,j}\overline{u_{s,j+1,s,j}}$ are different by the choice of the $u_{i,j,k,l}$ (see Lemma 1). Furthermore, by Lemma 2, we get that $|a_{rs}|$ can not be equal to $\lambda_1 + \cdots + \lambda_{n-1}$.

Therefore, $|a_{rs} - 1| \ge 1 - |a_{rs}| > 1 - \sum_{q=1}^{n-1} \lambda_q = \lambda_n = |b_{ij}| > 0$, for all r, s, i, j and $r \neq s$, $i \neq j$.

Suppose $2 \le k \le n$.

Remember that the L_{ij} define a linear independent set.

The coefficient of L_{1k} in (1) is

$$c_{1k}(a_{1k} - 1) + c_{n(k-1)}b_{n(k-1)} = 0.$$

The coefficient of $L_{n(k-1)}$ in (1) is

$$c_{n(k-1)}(a_{n(k-1)} - 1) + c_{(n-1)(k-2)}b_{(n-1)(k-1)} = 0.$$

The coefficient of $L_{(n-k+2)1}$ in (1) is

$$c_{(n-k+2)1}(a_{(n-k+2)1} - 1) + c_{(n-k-1)n}b_{(n-k+1)n} = 0.$$

The coefficient of $L_{(n-k+1)n}$ in (1) is

$$c_{(n-k+1)n}(a_{(n-k+1)n} - 1) + c_{(n-k)(n-1)}b_{(n-k)(n-1)} = 0.$$

The coefficient of $L_{(n-k)(n-1)}$ in (1) is

$$c_{(n-k)(n-1)}(a_{(n-k)(n-1)} - 1) + c_{(n-k-1)(n-2)}b_{(n-k-1)(n-2)} = 0.$$

...

The coefficient of $L_{2(k+1)}$ in (1) is

$$c_{2(k+1)}(a_{2(k+1)} - 1) + c_{1k}b_{1k} = 0.$$

If $c_{1k} \neq 0$, then, from above, we get $|c_{1k}| < |c_{n(k-1)}| < \cdots < |c_{2(k+1)}| < |c_{1k}|$.
Then, we get a contradiction. It follows that $c_{1k} = 0$.
Therefore,

$$c_{n(k-1)} = c_{(n-1)(k-2)} = \cdots = c_{(n-k+2)1} = c_{(n-k+1)n} = c_{(n-k)(n-1)} = \cdots = c_{2(k+1)} = 0.$$

From this, it follows that $c_{rs} = 0$ for all r, s, when $r \neq s$. This shows that for such U, we have maximal range equal to $n^2 - 1$.

Now we will prove two Lemmas that we used before.

Lemma 1 *Given $m \geq 2$, there exist complex numbers $u_1, \ldots, u_m$ of modulus 1, such that, if $1 \leq i \neq j \leq m$, $1 \leq k \neq l \leq m$ and $u_i\overline{u_j} = u_k\overline{u_l}$, then $i = k$, $j = l$.*

Proof The proof is by induction on m

For $m = 2$, just take $u_1\overline{u_2}$ not in $\mathbb{R}$.

Suppose the claim is true for $m \geq 2$ and $u_1, \ldots, u_m$ the corresponding ones.

Consider

$$S = \{u_i\overline{u_j} | 1 \leq i, j \leq m\}$$

and

$$T = \{u_p u_q | 1 \leq p, q \leq m\}.$$

Then, take u_{m+1} such that $u_{m+1}\overline{u_p}$ is not in S for all $1 \leq p \leq m$, and u_{m+1}^2 is not in T.

Then, $u_1, \ldots, u_m, u_{m+1}$ satisfy the claim.

□

Lemma 2 *Consider $\lambda_1, \ldots, \lambda_m$, real positive numbers and $z_1, \ldots, z_m$, complex numbers of modulus 1.*
Suppose $|\sum_{j=1}^m \lambda_j z_j| = \sum_{j=1}^m \lambda_j$, then $z_1 = z_2 = \cdots = z_m$.

Proof The proof is by induction on m.

It is obviously true for $m = 1$.

Suppose the claim is true for $m - 1$ and we will show is true for m.

Note that

$$\sum_{j=1}^m \lambda_j = \left|\sum_{j=1}^m \lambda_j z_j\right| \leq \left|\sum_{j=1}^{m-1} \lambda_j z_j\right| + \lambda_m \leq \sum_{j=1}^m \lambda_j.$$

From this follows that

$$\left|\sum_{j=1}^{m-1} \lambda_j z_j\right| = \sum_{j=1}^{m-1} \lambda_j.$$

Then, $z_1 = z_2 = \cdots = z_{m-1} = z$.

Therefore,

$$\sum_{j=1}^{m} \lambda_j = \left| z \sum_{j=1}^{m-1} \lambda_j + z_m \lambda_m \right| \leq \left| z \sum_{j=1}^{m-1} \lambda_j \right| + |z_m \lambda_m| = \sum_{j=1}^{m} \lambda_j .$$

Given v_1, v_2 complex numbers such that $|v_1 + v_2| = |v_1| + |v_2|$, then they have the same argument.

Then, there exists an $s > 0$ such that $z \sum_{j=1}^{m-1} \lambda_j = s z_m \lambda_m$.

Now, taking modulus on both sides of the expression above, we get

$$\sum_{j=1}^{m-1} \lambda_j = \left| z \sum_{j=1}^{m-1} \lambda_j \right| = |s z_m \lambda_m| = s \lambda_m .$$

From this follows that $z_m = z$

□

3 The two-dimensional case: explicit results

Our main interest in this section is to present the explicit expression of the unique fixed point U. We restrict ourselves to the two-dimensional case.

We will consider a two-by-two density matrix β such that is diagonal in the basis $f_1 \in \mathbb{C}^2$, $f_2 \in \mathbb{C}^2$. Without lost of generality, we can consider that

$$\beta = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix},$$

$p_1, p_2 > 0$. We will describe initially in coordinates some of the definitions which were used before in the paper.

If

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$$

and

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

then

$$R \otimes S = \begin{pmatrix} R_{11}S_{11} & R_{11}S_{12} & R_{12}S_{11} & R_{12}S_{12} \\ R_{11}S_{21} & R_{11}S_{22} & R_{12}S_{21} & R_{12}S_{22} \\ R_{21}S_{11} & R_{21}S_{12} & R_{22}S_{11} & R_{22}S_{12} \\ R_{21}S_{21} & R_{21}S_{22} & R_{22}S_{21} & R_{22}S_{22} \end{pmatrix}$$

and

$$\mathrm{Tr}_2(R \otimes S) = \begin{pmatrix} R_{11}(S_{11} + S_{22}) & R_{12}(S_{11} + S_{22}) \\ R_{21}(S_{11} + S_{22}) & R_{22}(S_{11} + S_{22}) \end{pmatrix}.$$

Given

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix}$$

then, in a consistent way, we have

$$\mathrm{Tr}_2(T) = \begin{pmatrix} T_{11} + T_{22} & T_{13} + T_{24} \\ T_{31} + T_{42} & T_{33} + T_{44} \end{pmatrix}$$

The action of an operator U on $M_2 \otimes M_2$ in the basis $e_1 \otimes f_1, e_2 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_2$ is given by a 4 by 4 matrix U denoted by

$$U = \begin{pmatrix} U_{11}^{11} & U_{11}^{12} & U_{12}^{11} & U_{12}^{12} \\ U_{11}^{21} & U_{11}^{22} & U_{12}^{21} & U_{12}^{22} \\ U_{21}^{11} & U_{21}^{12} & U_{22}^{11} & U_{22}^{12} \\ U_{21}^{21} & U_{21}^{22} & U_{22}^{21} & U_{22}^{22} \end{pmatrix}$$

and

$$U^* = \begin{pmatrix} \overline{U_{11}^{11}} & \overline{U_{11}^{21}} & \overline{U_{21}^{11}} & \overline{U_{21}^{21}} \\ \overline{U_{11}^{12}} & \overline{U_{11}^{22}} & \overline{U_{21}^{12}} & \overline{U_{21}^{22}} \\ \overline{U_{12}^{11}} & \overline{U_{12}^{21}} & \overline{U_{22}^{11}} & \overline{U_{22}^{21}} \\ \overline{U_{12}^{12}} & \overline{U_{12}^{22}} & \overline{U_{22}^{12}} & \overline{U_{22}^{22}} \end{pmatrix}$$

If U is unitary then $UU^* = I$. This relation implies the following set of equations:

(1) $U_{11}^{11}\overline{U_{11}^{11}} + U_{11}^{12}\overline{U_{11}^{12}} + U_{12}^{11}\overline{U_{12}^{11}} + U_{12}^{12}\overline{U_{12}^{12}} = 1,$

(2) $U_{11}^{11}\overline{U_{11}^{21}} + U_{11}^{12}\overline{U_{11}^{22}} + U_{12}^{11}\overline{U_{12}^{21}} + U_{12}^{12}\overline{U_{12}^{22}} = 0,$

(3) $U_{11}^{11}\overline{U_{21}^{11}} + U_{11}^{12}\overline{U_{21}^{12}} + U_{12}^{11}\overline{U_{22}^{11}} + U_{12}^{12}\overline{U_{22}^{12}} = 0,$

(4) $U_{11}^{11}\overline{U_{21}^{21}} + U_{11}^{12}\overline{U_{21}^{22}} + U_{12}^{11}\overline{U_{22}^{21}} + U_{12}^{12}\overline{U_{22}^{22}} = 0,$

(5) $U_{11}^{21}\overline{U_{11}^{11}} + U_{11}^{22}\overline{U_{11}^{12}} + U_{12}^{21}\overline{U_{12}^{11}} + U_{12}^{22}\overline{U_{12}^{12}} = 0,$

(6) $U_{11}^{21}\overline{U_{11}^{21}} + U_{11}^{22}\overline{U_{11}^{22}} + U_{12}^{21}\overline{U_{12}^{21}} + U_{12}^{22}\overline{U_{12}^{22}} = 1,$

(7) $U_{11}^{21}\overline{U_{21}^{11}} + U_{11}^{22}\overline{U_{21}^{12}} + U_{12}^{21}\overline{U_{22}^{11}} + U_{12}^{22}\overline{U_{22}^{12}} = 0,$

(8) $U_{11}^{21}\overline{U_{21}^{21}} + U_{11}^{22}\overline{U_{21}^{22}} + U_{12}^{21}\overline{U_{22}^{21}} + U_{12}^{22}\overline{U_{22}^{22}} = 0,$

(9) $U_{21}^{11}\overline{U_{11}^{11}} + U_{21}^{12}\overline{U_{11}^{12}} + U_{22}^{11}\overline{U_{12}^{11}} + U_{22}^{12}\overline{U_{12}^{12}} = 0,$

(10) $U_{21}^{11}\overline{U_{11}^{21}} + U_{21}^{12}\overline{U_{11}^{22}} + U_{22}^{11}\overline{U_{12}^{21}} + U_{22}^{12}\overline{U_{12}^{22}} = 0,$

(11) $U_{21}^{11}\overline{U_{21}^{11}} + U_{21}^{12}\overline{U_{21}^{12}} + U_{22}^{11}U_{22}^{11} + U_{22}^{12}\overline{U_{22}^{12}} = 1,$

(12) $U_{21}^{11}\overline{U_{21}^{21}} + U_{21}^{12}\overline{U_{21}^{22}} + U_{22}^{11}\overline{U_{22}^{21}} + U_{22}^{12}\overline{U_{22}^{22}} = 0,$

(13) $U_{21}^{21}\overline{U_{21}^{11}} + U_{21}^{22}\overline{U_{21}^{12}} + U_{22}^{21}\overline{U_{22}^{11}} + U_{22}^{22}\overline{U_{22}^{12}} = 0.$

(14) $U_{21}^{21}\overline{U_{11}^{11}} + U_{21}^{22}\overline{U_{11}^{12}} + U_{22}^{21}\overline{U_{12}^{11}} + U_{22}^{22}\overline{U_{12}^{12}} = 0.$

(15) $U_{21}^{21}\overline{U_{11}^{21}} + U_{21}^{22}\overline{U_{11}^{22}} + U_{22}^{21}\overline{U_{12}^{21}} + U_{22}^{22}\overline{U_{12}^{22}} = 0.$

(16) $U_{21}^{21}\overline{U_{21}^{21}} + U_{21}^{22}\overline{U_{21}^{22}} + U_{22}^{21}\overline{U_{22}^{21}} + U_{22}^{22}\overline{U_{22}^{22}} = 1.$

Equation (2) is equivalent to (5), equation (12) is equivalent to (13), equation (8) is equivalent to (15), equation (3) is equivalent to (9), equation (7) is equivalent to (10) and equation (4) is equivalent to (14). Then, we have six free parameters for the coefficients of U.

Using the entries U_{rs}^{ij} we considered above, we define

$$\tilde{L}(Q) = p_1 \sum_{i=1}^{2} \begin{pmatrix} \overline{U}_{11}^{i1} & \overline{U}_{21}^{i1} \\ \overline{U}_{12}^{i1} & \overline{U}_{22}^{i1} \end{pmatrix} Q \begin{pmatrix} U_{11}^{i1} & U_{12}^{i1} \\ U_{21}^{i1} & U_{22}^{i1} \end{pmatrix}$$
$$+ p_2 \sum_{i=1}^{2} \begin{pmatrix} \overline{U_{11}^{i2}} & \overline{U_{21}^{i2}} \\ \overline{U_{12}^{i2}} & \overline{U_{22}^{i2}} \end{pmatrix} Q \begin{pmatrix} U_{11}^{i2} & U_{12}^{i2} \\ U_{21}^{i2} & U_{22}^{i2} \end{pmatrix}$$

We can consider an auxiliary L_{ij} and express

$$\tilde{L}(Q) = \sum_{i=1}^{2} (\sqrt{p_1}(U^{i1})^*) Q (\sqrt{p_1} U^{i1}) + \sum_{i=1}^{2} (\sqrt{p_2}(U^{i2})^*) Q (\sqrt{p_2} U^{i2})$$
$$= \sum_{i=1}^{2} L_{i1}^* Q L_{i1} + \sum_{i=1}^{2} L_{i2}^* Q L_{i2} = \sum_{i.j=1}^{2} L_{ij}^* Q L_{ij}.$$

From the fact that $UU^* = I$, it follows (after a long computation) that

$$\tilde{L}(I) = I.$$

Note that $\tilde{L}$ preserve the cone of positive matrices.

Using the entries U_{rs}^{ij} described above, we denote

$$\hat{L}(Q) = p_1 \sum_{i=1}^{2} \begin{pmatrix} U_{11}^{i1} & U_{12}^{i1} \\ U_{21}^{i1} & U_{22}^{i1} \end{pmatrix} Q \begin{pmatrix} \overline{U_{11}^{i1}} & \overline{U_{21}^{i1}} \\ \overline{U_{12}^{i1}} & \overline{U_{22}^{i1}} \end{pmatrix}$$
$$+ p_2 \sum_{i=1}^{2} \begin{pmatrix} U_{11}^{i2} & U_{12}^{i2} \\ U_{21}^{i2} & U_{22}^{i2} \end{pmatrix} Q \begin{pmatrix} \overline{U_{12}^{i2}} & \overline{U_{21}^{i2}} \\ \overline{U_{12}^{i2}} & \overline{U_{22}^{i2}} \end{pmatrix} = \sum_{i.j=1}^{2} L_{ij} Q L_{ij}^*.$$

One can also show that $\hat{L}(Q) = \mathrm{Tr}_2[U(Q \otimes \beta)U^*]$ (see [4]).

The first expression is the Kraus decomposition and the second the Stinespring dilation.

Moreover $\hat{L}$ preserves density matrices. This is proved in the appendix but we can present here another way to get that. If Q is a density matrix, then

$$\mathrm{Tr}(\hat{L}(Q)) = \mathrm{Tr}\left(\sum_{i.j=1}^{2} L_{ij} Q L_{ij}^*\right) = \sum_{i.j=1}^{2} \mathrm{Tr}(L_{ij} Q L_{ij}^*) = \sum_{i.j=1}^{2} \mathrm{Tr}(Q L_{ij}^* L_{ij})$$
$$= \mathrm{Tr}\left(\sum_{i.j=1}^{2} Q L_{ij}^* L_{ij}\right) = \mathrm{Tr}\left(Q \sum_{i.j=1}^{2} L_{ij}^* L_{ij}\right) = \mathrm{Tr}(Q) = 1$$

We denote

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

Then,

$$U^{ij}\,Q\,(U^{ij})^* = \begin{pmatrix} U^{ij}_{11} & U^{ij}_{12} \\ U^{ij}_{21} & U^{ij}_{22} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \overline{U^{ij}_{11}} & \overline{U^{ij}_{21}} \\ \overline{U^{ij}_{12}} & \overline{U^{ij}_{22}} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{U^{ij}_{11}}(U^{ij}_{11}Q_{11} + U^{ij}_{12}Q_{21}) + \overline{U^{ij}_{12}}(U^{ij}_{11}Q_{12} + U^{ij}_{12}Q_{22}) & \overline{U^{ij}_{21}}(U^{ij}_{11}Q_{11} + U^{ij}_{12}Q_{21}) + \overline{U^{ij}_{22}}(U^{ij}_{11}Q_{12} + U^{ij}_{12}Q_{22}) \\ \overline{U^{ij}_{11}}(U^{ij}_{21}Q_{11} + U^{ij}_{22}Q_{21}) + \overline{U^{ij}_{12}}(U^{ij}_{21}Q_{12} + U^{ij}_{22}Q_{22}) & \overline{U^{ij}_{21}}(U^{ij}_{21}Q_{11} + U^{ij}_{22}Q_{21}) + \overline{U^{ij}_{22}}(U^{ij}_{21}Q_{12} + U^{ij}_{22}Q_{22}) \end{pmatrix},$$

We have to compute

$$\hat{L}(Q) = p_1[U^{11}Q(U^{11})^* + U^{21}Q(U^{21})^*] + p_2[U^{12}Q(U^{12})^* + U^{22}Q(U^{22})^*].$$

The coordinate a_{11} of $\hat{L}(Q)$ is

$$\begin{aligned} &p_1\big[\overline{U^{11}_{11}}(U^{11}_{11}Q_{11} + U^{11}_{12}Q_{21}) + \overline{U^{11}_{12}}(U^{11}_{11}Q_{12} + U^{11}_{12}Q_{22})\big] \\ &\quad + p_1\big[\overline{U^{21}_{11}}(U^{21}_{11}Q_{11} + U^{21}_{12}Q_{21}) + \overline{U^{21}_{12}}(U^{21}_{11}Q_{12} + U^{21}_{12}Q_{22})\big] \\ &\quad + p_2\big[\overline{U^{12}_{11}}(U^{12}_{11}Q_{11} + U^{12}_{12}Q_{21}) + \overline{U^{12}_{12}}(U^{12}_{11}Q_{12} + U^{12}_{12}Q_{22})\big] \\ &\quad + p_2\big[\overline{U^{22}_{11}}(U^{22}_{11}Q_{11} + U^{22}_{12}Q_{21}) + \overline{U^{22}_{12}}(U^{22}_{11}Q_{12} + U^{22}_{12}Q_{22})\big]. \end{aligned} \tag{2}$$

The coordinate a_{12} is

$$\begin{aligned} &p_1\big[\overline{U^{11}_{21}}(U^{11}_{11}Q_{11} + U^{11}_{12}Q_{21}) + \overline{U^{11}_{22}}(U^{11}_{11}Q_{12} + U^{11}_{12}Q_{22})\big] \\ &\quad + p_1\big[\overline{U^{21}_{21}}(U^{21}_{11}Q_{11} + U^{21}_{12}Q_{21}) + \overline{U^{21}_{22}}(U^{21}_{11}Q_{12} + U^{21}_{12}Q_{22})\big] \\ &\quad + p_2\big[\overline{U^{12}_{21}}(U^{12}_{11}Q_{11} + U^{12}_{12}Q_{21}) + \overline{U^{12}_{22}}(U^{12}_{11}Q_{12} + U^{12}_{12}Q_{22})\big] \\ &\quad + p_2\big[\overline{U^{22}_{21}}(U^{22}_{11}Q_{11} + U^{22}_{12}Q_{21}) + \overline{U^{22}_{22}}(U^{22}_{11}Q_{12} + U^{22}_{12}Q_{22})\big]. \end{aligned} \tag{3}$$

We will consider a parametrization of the density matrices taking $Q_{11} = 1 - Q_{22}$ and $Q_{12} = \overline{Q_{21}}$.

The variable Q_{11} is positive in the real line and smaller than one. Indeed, by positivity of Q, we have $0 \leq Q_{11}Q_{22} = Q_{11}(1 - Q_{11}) = Q_{11} - Q_{11}^2$.

Q_{12} is in $\mathbb{C} = \mathbb{R}^2$ but satisfying $Q_{11}(1 - Q_{11}) - Q_{12}\overline{Q}_{12} \geq 0$ because we are interested in density matrices which are positive operators.

The numbers p_1 and p_2 are fixed. Consider the function G such that

$$\begin{aligned} G(Q_{11}, Q_{12}) = (&p_1[\overline{U^{11}_{11}}(U^{11}_{11}Q_{11} + U^{11}_{12}\overline{Q_{12}}) + \overline{U^{11}_{12}}(U^{11}_{11}Q_{12} + U^{11}_{12}(1 - Q_{11}))] \\ &+ p_1[\overline{U^{21}_{11}}(U^{21}_{11}Q_{11} + U^{21}_{12}\overline{Q_{12}}) + \overline{U^{21}_{12}}(U^{21}_{11}Q_{12} + U^{21}_{12}(1 - Q_{11}))] \\ &+ p_2[\overline{U^{12}_{11}}(U^{12}_{11}Q_{11} + U^{12}_{12}\overline{Q_{12}}) + \overline{U^{12}_{12}}(U^{12}_{11}Q_{12} + U^{12}_{12}(1 - Q_{11}))] \\ &+ p_2[\overline{U^{22}_{11}}(U^{22}_{11}Q_{11} + U^{22}_{12}\overline{Q_{12}}) + \overline{U^{22}_{12}}(U^{22}_{11}Q_{12} + U^{22}_{12}(1 - Q_{11}))], \\ &p_1[\overline{U^{11}_{21}}(U^{11}_{11}Q_{11} + U^{11}_{12}\overline{Q_{12}}) + \overline{U^{11}_{22}}(U^{11}_{11}Q_{12} + U^{11}_{12}(1 - Q_{11}))] \\ &+ p_1[\overline{U^{21}_{21}}(U^{21}_{11}Q_{11} + U^{21}_{12}\overline{Q_{12}}) + \overline{U^{21}_{22}}(U^{21}_{11}Q_{12} + U^{21}_{12}(1 - Q_{11}))] \\ &+ p_2[\overline{U^{12}_{21}}(U^{12}_{11}Q_{11} + U^{12}_{12}\overline{Q_{12}}) + \overline{U^{12}_{22}}(U^{12}_{11}Q_{12} + U^{12}_{12}(1 - Q_{11}))] \\ &+ p_2[\overline{U^{22}_{21}}(U^{22}_{11}Q_{11} + U^{22}_{12}\overline{Q_{12}}) + \overline{U^{22}_{22}}(U^{22}_{11}Q_{12} + U^{22}_{12}(1 - Q_{11}))]) \end{aligned}$$

When there is a unique fixed point for G?

Example Suppose $U = e^{i\beta\sigma^x \otimes \sigma^x} = \cos(\beta)(I \otimes I) + i\sin(\beta)(\sigma_x \otimes \sigma_x)$. In this case

$$U = \begin{pmatrix} \cos\beta & 0 & 0 & i\sin\beta \\ 0 & \cos\beta & i\sin\beta & 0 \\ 0 & i\sin\beta & \cos\beta & 0 \\ i\sin\beta & 0 & 0 & \cos\beta \end{pmatrix}$$

Therefore,

$$G(Q_{11}, Q_{12}) = ((p_1 - p_1 Q_{11} + p_2 - p_2 Q_{11}),$$

$$\begin{aligned} & p_1(\cos\beta)^2 Q_{12} + p_1(\sin\beta)^2 \overline{Q_{12}} + p_2(\sin\beta)^2 \overline{Q_{12}} + p_2(\cos\beta)^2 Q_{12}) \\ & \quad = (1 - Q_{11}, p_1(\cos\beta)^2 Q_{12} + p_1(\sin\beta)^2 \overline{Q_{12}} + p_2(\sin\beta)^2 \overline{Q_{12}} + p_2(\cos\beta)^2 Q_{12}) \end{aligned}$$

One can easily see that given any $a \in \mathbb{R}$ we have that $Q_{11} = 1/2$, and $Q_{12} = a$ determine a fixed point for G. In order the fixed point matrix to be positive we need that $-1/2 < a < 1/2$.

In this case, the fixed point is not unique.

It is more convenient to express G in terms of the variables $Q_{11} \in [0, 1]$, and $(a, b) \in \mathbb{R}^2$, where $Q_{12} = a + bi$. As these parameters describe density matrices, there are some restrictions: $1/4 \geq Q_{11}(1 - Q_{11}) \geq (a^2 + b^2)$ and $1 \geq Q_{11} \geq 0$

We denote by $Re(z)$, the real part of the complex number z and by $Im(z)$ its imaginary part.

In this case, we get

$$\begin{aligned} & G(Q_{11}, a, b) = (Q_{11}\alpha_1 + \beta_1 + (a_{11} + a_{12})a + i(a_{11} - a_{12})b, \\ & \mathrm{Re}(Q_{11}\alpha_2 + \beta_2 + (a_{21} + a_{22})a + i(a_{21} - a_{22})b), \\ & \mathrm{Im}(Q_{11}\alpha_2 + \beta_2 + (a_{21} + a_{22})a + i(a_{21} - a_{22})b)). \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= p_1[\overline{U_{11}^{11}} U_{11}^{11} - \overline{U_{12}^{11}} U_{12}^{11} + \overline{U_{11}^{21}} U_{11}^{21} - \overline{U_{12}^{21}} U_{12}^{21}] \\ & \quad + p_2[\overline{U_{11}^{12}} U_{11}^{12} - \overline{U_{12}^{12}} U_{12}^{12} + \overline{U_{11}^{22}} U_{11}^{22} - \overline{U_{12}^{22}} U_{12}^{22}], \\ \beta_1 &= p_1[\overline{U_{12}^{11}} U_{12}^{11} + \overline{U_{12}^{21}} U_{12}^{21}] + p_2[\overline{U_{12}^{12}} U_{12}^{12} + \overline{U_{12}^{22}} U_{12}^{22}], \\ \alpha_2 &= p_1[\overline{U_{21}^{11}} U_{11}^{11} - \overline{U_{22}^{11}} U_{12}^{11} + \overline{U_{21}^{21}} U_{11}^{21} - \overline{U_{22}^{21}} U_{12}^{21}] \\ & \quad + p_2[\overline{U_{21}^{12}} U_{11}^{12} - \overline{U_{22}^{12}} U_{12}^{12} + \overline{U_{21}^{22}} U_{11}^{22} - \overline{U_{22}^{22}} U_{12}^{22}], \\ \beta_2 &= p_1[\overline{U_{22}^{11}} U_{12}^{11} + \overline{U_{21}^{21}} U_{12}^{21}] + p_2[\overline{U_{22}^{12}} U_{12}^{12} + \overline{U_{22}^{22}} U_{12}^{22}], \\ a_{11} &= p_1[\overline{U_{12}^{11}} U_{11}^{11} + \overline{U_{12}^{21}} U_{11}^{21}] + p_2[\overline{U_{12}^{12}} U_{11}^{12} + \overline{U_{12}^{22}} U_{11}^{22}], \\ a_{12} &= p_1[\overline{U_{11}^{11}} U_{12}^{11} + \overline{U_{11}^{21}} U_{12}^{21}] + p_2[\overline{U_{11}^{12}} U_{12}^{12} + \overline{U_{11}^{22}} U_{12}^{22}], \\ a_{21} &= p_1[\overline{U_{22}^{11}} U_{11}^{11} + \overline{U_{22}^{21}} U_{11}^{21}] + p_2[\overline{U_{22}^{12}} U_{11}^{12} + \overline{U_{22}^{22}} U_{11}^{22}], \\ a_{22} &= p_1[\overline{U_{21}^{11}} U_{12}^{11} + \overline{U_{21}^{21}} U_{12}^{21}] + p_2[\overline{U_{21}^{12}} U_{12}^{12} + \overline{U_{21}^{22}} U_{12}^{22}], \end{aligned}$$

α_1 is a real number. As Φ takes density matrices to density matrices, we have that β_1 is also real.

Note that $|\alpha_1| < 1$ and $1 > \beta_1 > 0$.

It is easy to see from the above equations that $(a_{11} + a_{12})$ and $i(a_{11} - a_{12})$ are both real numbers.

We are not able to say the same for $(a_{21} + a_{22})a$ or $i(a_{21} - a_{22})b$.

To find the fixed point, we have to solve

$$\begin{aligned} & Q_{11}\alpha_1 + \beta_1 + (a_{11} + a_{12})a + i(a_{11} - a_{12})b = Q_{11} \\ & Q_{11}\alpha_2 + \beta_2 + (a_{21} + a_{22})a + i(a_{21} - a_{22})b = a + bi, \end{aligned}$$

which means in matrix form

$$\begin{pmatrix} (\alpha_1 - 1) & a_{11} + a_{12} & i(a_{11} - a_{12}) \\ \alpha_2 & a_{21} + a_{22} - 1 & i(a_{21} - a_{22} - 1) \end{pmatrix} \begin{pmatrix} Q_{11} \\ a \\ b \end{pmatrix} = \begin{pmatrix} -\beta_1 \\ -\beta_2 \end{pmatrix}.$$

We are interested in real solutions Q_{11}, a, b.

In the case of the example mentioned above, one can show that $\alpha_1 = 1$ and $\alpha_0 = 0$ which means that in the expressions above, we get a set of two equation in two variables a, b,

Remember that we are interested in matrices such that $1/4 \geq Q_{11}(1-Q_{11}) \geq (a^2+b^2)$. Notice that $0 \leq Q_{11} \leq 1$. As Φ takes density matrices to density matrices, there is a fixed point for G by the Brower fixed point theorem. The main question is the conditions on U and β such that the fixed point is unique.

If there is a solution $(\hat{Q}_{11}, \hat{a}, \hat{b}) \neq (0, 0, 0)$ in $\mathbb{R}^3$ to the equations

$$\begin{aligned} &\hat{Q}_{11}(\alpha_1 - 1) + (a_{11} + a_{12})\hat{a} + i(a_{11} - a_{12})\hat{b} = 0 \\ &\hat{Q}_{11}\alpha_2 + (a_{21} + a_{22} - 1)\hat{a} + i(a_{21} - a_{22} - 1)\hat{b} = 0, \end{aligned} \tag{4}$$

then, the fixed point is not unique. The condition is necessary and sufficient.

A necessary condition for the fixed point to be unique is to be nonzero the determinant of the operator

$$K = \begin{pmatrix} a_{11} + a_{12} & i(a_{11} - a_{12}) \\ a_{21} + a_{22} - 1 & i(a_{21} - a_{22} - 1) \end{pmatrix}.$$

Notice that if (z_1, z_2) satisfies $K(z_1, z_2) = (0, 0)$, then $\frac{z_1}{z_2}$ is real (because $a_{11} + a_{12}$ and $i(a_{11} - a_{12})$ are real). From this follows that there exists a solution $(a, b) \in \mathbb{R}^2$ in the kernel of K. In this case, $(0, a, b)$ is a nontrivial solution of (4).

The condition $\det K \neq 0$ is an open and dense property on the unitary matrices U. Indeed, there are six free parameters on the coefficients U_{rs}^{ij}. Consider an initial unitary operator U. One can fix 5 of them and move a little bit the last one. This will change U and will move the determinant of K_U in such way that can avoid the value 0 for some small perturbation of the initial U.

Suppose U satisfies such property $\operatorname{Det} U \neq 0$. For each real value Q_{11}, we get a different $(a_{Q_{11}}, b_{Q_{11}})$ which is a solution of $K(a, b) = (-Q_{11}(\alpha_1 - 1), -Q_{11}\alpha_2)$.

In this way, we get an infinite number of solutions $(Q_{11}, a_{Q_{11}}, b_{Q_{11}}) \in \mathbb{R} \times \mathbb{C}^2$ to (4).

α_2 is not real.

But, we need solutions on $\mathbb{R}^3$. Denote by $S = S_U$ the linear subspace of vectors in $\mathbb{C}^2$ of the form $\rho(\alpha_1 - 1, \alpha_2)$, where ρ is complex.

Lemma 3 *For an open and dense set of unitary U, we get that $K^{-1}(S) \cap \mathbb{R}^2 = \{(0, 0)\}$. For such U, suppose (Q_{11}, a, b) satisfies Eq. (4), then the non-trivial solutions $(\hat{a}, \hat{b})$ of*

$$K(\hat{a}, \hat{b}) = (-Q_{11}(\alpha_1 - 1), -Q_{11}\alpha_2)$$

are not in $\mathbb{R}^2$.

Proof Suppose $\frac{1-\alpha_1}{\alpha_2} = \alpha + \beta i = z^0 = z_U^0$. Note that for a generic U, we have that $\alpha_2 \neq 0$.

We denote $C_{11} = a_{11} + a_{12}$, $C_{12} = i(a_{11} - a_{12})$, $C_{21} = a_{21} + a_{22} - 1$ and finally $C_{22} = i(a_{21} - a_{22} - 1)$.

Suppose $(Q_{11}, a, b) \in \mathbb{R}^3$ satisfies Eq. (4). We know that generically on U the value Q_{11} is not zero.

For each C_{ij}, we denote $C_{ij} = C_{ij}^1 + C_{ij}^2 i$, where $i, j = 1, 2$.

If $K(\hat{a}, \hat{b}) = (-Q_{11}(\alpha_1 - 1), -Q_{11}\alpha_2)$, then

$$C_{11}\hat{a} + C_{21}\hat{b} = z^0(C_{21}\hat{a} + C_{22}\hat{b}) = (\alpha + \beta i)(C_{21}\hat{a} + C_{22}\hat{b}).$$

In this case

$$\begin{aligned} C_{11}\hat{a} + C_{21}\hat{b} = &(\alpha C_{21}^1\hat{a} - \beta C_{21}^2\hat{a} - \beta C_{22}^1\hat{b} - \alpha C_{22}^2\hat{b}) \\ &+ i(\beta C_{21}^1\hat{a} + \alpha C_{21}^2\hat{a} + \alpha C_{22}^1\hat{b} - \beta C_{22}^2\hat{b}). \end{aligned}$$

If $\hat{a}$ and $\hat{b}$ are real, then, as C_{11} and C_{22} are real , then

$$(\beta C_{21}^1 + \alpha C_{21}^2)\hat{a} + (\alpha C_{22}^1 - \beta C_{22}^2)\hat{b} = 0. \tag{5}$$

Moreover,

$$(\alpha C^1_{21} - \beta C^2_{21} - C_{11})\hat{a} - (\beta C^1_{22} - \alpha C^2_{22} - C_{21})\hat{b} = 0 \quad (6)$$

If

$$\mathrm{Det}\begin{pmatrix} \beta C^1_{21} + \alpha C^2_{21} & \alpha C^1_{22} - \beta C^2_{22} \\ \alpha C^1_{21} - \beta C^2_{21} - C_{11} & \beta C^1_{22} - \alpha C^2_{22} - C_{21} \end{pmatrix} \neq 0,$$

then just the trivial solution (0, 0) satisfies (5) and (6).
The above determinant is nonzero in an open and dense set of U.
Then, the solution $(Q_{11}, a, b) \in \mathbb{R}^3$ of (4) has to be trivial. □

Under these two assumptions on U (which are open and dense), the fixed point for G is unique. Then, it follows that the density matrix $Q = Q_\Phi$ which is invariant for Φ is unique. Given an initial Q_0, any convergent subsequence $\Phi^{n_k}(Q_0)$, $\kappa \to \infty$ will converge to the fixed point (because is unique).

As

$$\begin{aligned} G(Q_{11}, a, b) = (&Q_{11}\alpha_1 + \beta_1 + (a_{11} + a_{12})a + i(a_{11} - a_{12})b, \\ &\mathrm{Re}(Q_{11}\alpha_2 + \beta_2 + (a_{21} + a_{22})a + i(a_{21} - a_{22})b), \\ &\mathrm{Im}(Q_{11}\alpha_2 + \beta_2 + (a_{21} + a_{22})a + i(a_{21} - a_{22})b)), \end{aligned}$$

one can find the explicit solution

$$Q_\Phi = \begin{pmatrix} Q_{11} & a + bi \\ a - bi & 1 - Q_{11} \end{pmatrix}$$

by solving the linear problem $G(Q_{11}, a, b) = (Q_{11}, a, b)$.

Appendix

Lemma 4 *Given $A, B \in \mathscr{L}(V)$, then $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(\mathrm{Tr}_2(A \otimes B))$. Moreover, $\mathrm{Tr}(\mathrm{Tr}_2(T)) = \mathrm{Tr}(T)$, for all $T \in \mathscr{L}(V \otimes V)$.*

Proof Indeed,

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B) = \mathrm{Tr}(Tr(A)B) = \mathrm{Tr}(\mathrm{Tr}_2(A \otimes B)).$$

□

Lemma 5 *Given $T \in \mathscr{L}(V \otimes V)$,*

(a) *if T is selfadjoint, then, Tr_2 is also selfadjoint,*
(b) *moreover, if T is also positive semidefinite then $\mathrm{Tr}_2(T)$ is semidefinite.*

Proof (a) If T is selfadjoint, then, $t_{ijkl} = \overline{t_{klij}}$. This implies that $\sum_j t_{ijkj} = \sum_j \overline{t_{kjij}}$. Therefore, Tr_2 is selfadjoint.
(b) If T is postive semidefinite, then $\langle T(x \otimes x'), x \otimes x' \rangle \geq 0$, for all $x, x' \in V$. In particular, $\langle T(x \otimes e_q), x \otimes e_q \rangle \geq 0$, for all $x = c_1 e_1 + \cdots + c_n e_n \in V$ and $1 \leq q \leq n$.
As $T(x \otimes e_q) = \sum t_{ijkl} L_{ik}(x) \otimes L_{jl}(e_q) = \sum t_{ijkq} c_k (e_i \otimes e_j)$, then

$$\langle T(x \otimes e_q), x \otimes e_q \rangle = \sum_{i,k} t_{iqkq} c_k \overline{c_i}, \quad q = 1, 2, \ldots, n.$$

From this follows that $\sum_{i,k,q} t_{iqkq} c_k \overline{c_i} = \sum_{i,k} (\sum_q t_{iqkq}) c_k \overline{c_i} \geq 0$.
Then, $\langle \mathrm{Tr}_2(T)(x), x \rangle \geq 0$. □

Note that the analogous property for positive definite T is also true.

Lemma 6 *If $A \in \Gamma$, then $\Phi_U(A) \in \Gamma$, for all $U \in \mathscr{U}$.*

Proof As A and β are selfadjoint and positive semidefinite the same is true for $A \otimes \beta$. Then, the same is true for $U(A \otimes \beta)U^*$. From Lemma 5 we get that $\Phi_U(A) = \mathrm{Tr}_2(U(A \otimes \beta)U^*)$ is selfadjoint.

By Lemma 4 $\mathrm{Tr}(\Phi_U(A)) = \mathrm{Tr}(U(A \otimes \beta)U^*) = \mathrm{Tr}(A \otimes b) = \mathrm{Tr}(A)\mathrm{Tr}(B) = 1.$ □

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