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RESEARCH ARTICLE

Strong dictatorship via ratio-scale measurable utilities: a simpler proof

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Abstract
Tsui and Weymark (Econ Theory 10:241–256, 1997, https://doi.org/10.1007/s001990050156) have shown that the only continuous social welfare orderings on the whole Euclidean space which satisfy the weak Pareto principle and are invariant to individual-specific similarity transformations of utilities are strongly dictatorial. Their proof relies on functional equation arguments which are quite complex. This note provides a simpler proof of their theorem.

Keywords Arrow's impossibility theorem · Strong dictatorship, social welfare orderings · Ratio-scale measurability · Informational invariance conditions

JEL Classification D71

1 Introduction

A social welfare ordering is a transitive and complete binary relation on a set of utility vectors, assumed here to be the n-fold Euclidean space $\mathbb{R}^n$. Such an ordering is *strongly dictatorial* just in case there is some individual i such that, for any $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, u is at least as good as v if and only if $u_i \geq v_i$. The class of strongly dictatorial social welfare orderings can be axiomatically characterized using a continuous variation on Arrow (1951)'s impossibility theorem (see Bossert and Weymark 2004, Theorem 10.1). This familiar characterization uses an informational invariance condition associated with the view that individual utilities are measurable on interpersonally noncomparable ordinal or cardinal scales.

Tsui and Weymark (1997, Theorem 6), however, have shown that an even weaker invariance condition, associated with ratio-scale measurability (again, without interpersonal comparability) is sufficient. This condition requires the social welfare

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ordering to be invariant to similarity transformations of utilities which can differ between individuals.[1] Since this invariance condition is strictly weaker than those associated with ordinal and cardinal measurability, Tsui and Weymark's result is the most general axiomatic characterization of strongly dictatorial social welfare orderings on $\mathbb{R}^n$; the other axiomatizations of strong dictatorship are obvious corollaries of theirs. Moreover, the informational environment of ratio-scale measurability is by no means recherché: Skyrms and Narens (2019), for example, defend precisely the view that welfare is measurable on interpersonally noncomparable ratio scales. Tsui and Weymark's result has also been relied upon by subsequent impossibility theorems in the literature (such as Khmelnitskaya 2002, Theorem 3.5) and it would seem to provide the best case for the view that an ethically defensible continuous social welfare ordering on $\mathbb{R}^n$ requires interpersonal comparisons of welfare (though there are considerable subtleties to such an argument; see Baccelli forthcoming).

Unfortunately, Tsui and Weymark's proof is, as Bossert and Weymark (2004, p. 1148) put it, "long and difficult," relying on functional equation arguments which are quite complex. It is, to my knowledge, the only proof of their theorem in the existing literature. This note provides a simpler proof of their theorem using more familiar methods, which are of a piece with those used in extant proofs of Arrow's theorem (see Blackorby et al. 1984, 1990).

2 Axioms

Let $N = \{1, \dots, n\}$ be a set of individuals. Let $\succcurlyeq$ ("at least as good as") be a social welfare ordering on $\mathbb{R}^n$. $\succ$ ("better than") denotes the asymmetric part of $\succcurlyeq$, $\sim$ ("equally good") its symmetric part. The strict vector inequality $u \gg v$ means that $u_i > v_i$ for every $i \in N$.

We impose three axioms. The first is the weak version of the Pareto principle:

Weak Pareto For all $u, v \in \mathbb{R}^n$, if $u \gg v$, then $u \succ v$.

Second, the social welfare ordering is continuous:

Continuity For all $u \in \mathbb{R}^n$, the sets $\{v \in \mathbb{R}^n : v \succcurlyeq u\}$ and $\{v \in \mathbb{R}^n : u \succcurlyeq v\}$ are closed.

Third, the ordering is invariant to individual-specific similarity transformations of utilities:

Ratio-Scale Invariance For all $u, v, u', v' \in \mathbb{R}^n$, if for each $i \in N$ there is a positive real number k_i such that $u'_i = k_i u_i$ and $v'_i = k_i v_i$, then $u \succcurlyeq v$ if and only if $u' \succcurlyeq v'$.

Ratio-Scale Invariance is so called because it is associated with the view that individual welfares are measurable on intrapersonal ratio scales, which are unique up to similarity transformation.[2]

[1] See Krantz et al. (1971, ch. 3.) for standard representation and uniqueness theorems for ratio scales.

[2] The general inference from measurability/comparability assumptions to the associated invariance requirement is questioned by Morreau and Weymark (2016) and Nebel (2021, 2022, forthcoming).

3 Theorem

We begin by comparing each utility vector to the origin $\mathbf{0} = (0, \ldots, 0)$. An open orthant U is a subset of $\mathbb{R}^n$ such that, for some $u \in \mathbb{R}^n$ for which $u_i \neq 0$ for every $i \in N$, $U = \{v \in \mathbb{R}^n \mid \text{sgn}(v_i) = \text{sgn}(u_i) \text{ for every } i \in N\}$. Our first lemma says that all vectors in any given open orthant must compare to the origin in the same, strict way.

Lemma 1 *If an ordering $\succcurlyeq$ on $\mathbb{R}^n$ satisfies Weak Pareto and Ratio-Scale Invariance, then for any open orthant U of $\mathbb{R}^n$, either (a) $u \succ \mathbf{0}$ for every $u \in U$ or (b) $u \prec \mathbf{0}$ for every $u \in U$.*[3]

Proof Take any open orthant U of $\mathbb{R}^n$ and any $u, v \in U$. For every $i \in N$ there is some positive real number k_i such that $v_i = k_i u_i$, so by Ratio-Scale Invariance, $u \succcurlyeq \mathbf{0}$ iff $v \succcurlyeq \mathbf{0}$. Given the completeness of $\succcurlyeq$ it follows that either (a) $u \succ \mathbf{0}$ for every $u \in U$, (b) $u \prec \mathbf{0}$ for every $u \in U$, or (c) $u \sim \mathbf{0}$ for every $u \in U$. (c) would imply $u \sim v$ for every $u, v \in U$ by the transitivity of $\sim$. But this would violate Weak Pareto because there are $u, v \in U$ such that $u \gg v$. So the only possibilities are (a) and (b). □

Next, we show that some individual's utilities fully determine how each vector compares to the origin:

Lemma 2 *If an ordering $\succcurlyeq$ on $\mathbb{R}^n$ satisfies Weak Pareto, Continuity, and Ratio-Scale Invariance, then there is some $i^* \in N$ such that, for any $u \in \mathbb{R}^n$, $u \succcurlyeq \mathbf{0}$ iff $u_{i^*} \geq 0$.*

Proof We introduce the following notation. For any $i \in N$, let U^i_{++} denote the open orthant of $\mathbb{R}^n$ in which i has positive utility and all others have negative utilities—i.e., $U^i_{++} := \{u \in \mathbb{R}^n \mid u_i > 0 \text{ and } u_j < 0 \text{ for every } j \in N \backslash \{i\}\}$—and let U^i_0 denote the boundary of U^i_{++} in which $u_i = 0$—i.e., $U^i_0 := \{u \in \mathbb{R}^n \mid u_i = 0 \text{ and } u_j < 0 \text{ for every } j \in N \backslash \{i\}\}$.

We first show that, for some $i^* \in N$, $u \succ \mathbf{0}$ whenever $u \in U^{i^*}_{++}$. Suppose for contradiction that that there is no such i^*. Then by Lemma 1, we must have $u \prec \mathbf{0}$ for every $i \in N$ and $u \in U^i_{++}$.

For every $i \in N$ and $v \in U^i_0$, there is some $u \in U^i_{++}$ such that $v \ll u$, which implies $v \prec u$ by Weak Pareto, and therefore $v \prec \mathbf{0}$ by the transitivity of $\prec$. Continuity then implies that, for any such i and v, there is a neighborhood about the origin $Z_{\mathbf{0}} \subseteq \mathbb{R}^n$ such that $v \prec v'$ for any $v' \in Z_{\mathbf{0}}$. Given any such neighborhood, and any $j \in N$, there will be some $v' \in U^j_0 \cap N_{\mathbf{0}}$. Thus, for every $i, j \in N$ and $v \in U^i_0$, there is some $v' \in U^j_0$ such that $v \prec v'$.

Let $v^1 = (0, -1, \ldots, -1)$. By the last sentence of the previous paragraph, there are $v^2 \in U^2_0, \ldots, v^n \in U^n_0$, and $v^{n+1} \in U^1_0$ such that $v^1 \prec v^2 \prec \cdots \prec v^n \prec v^{n+1}$. We derive a contradiction from this using Ratio-Scale Invariance. We transform $v^1, \ldots, v^n$ into $w^1, \ldots, w^n$ by multiplying each v^i_j by $|v^{n+1}_j|$ whenever $j \geq i$, and otherwise

[3] This generalizes Tsui and Weymark (1997, Lemma, p. 253) which includes (and their proof invokes) Continuity.

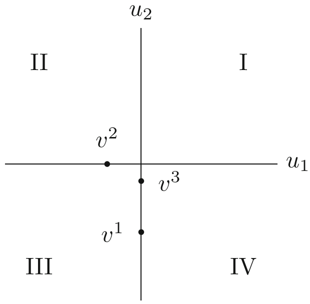


Fig. 1 Illustration of the central argument in the proof of Lemma 2 for $\mathbb{R}^2$

leaving v^i_j as is. This yields the following sequence of transformed vectors:

$$
\begin{array}{rl}
w^1 = & \left(0,\ -1|v_2^{n+1}|,\ -1|v_3^{n+1}|,\ -1|v_4^{n+1}|,\ \ldots,\ -1|v_{n-1}^{n+1}|,\ -1|v_n^{n+1}|\right) \\
w^2 = & \left(v_1^2,\ 0,\ v_3^2|v_3^{n+1}|,\ v_4^2|v_4^{n+1}|,\ \ldots,\ v_{n-1}^2|v_{n-1}^{n+1}|,\ v_n^2|v_n^{n+1}|\right) \\
w^3 = & \left(v_1^3,\ v_2^3,\ 0,\ v_4^3|v_4^{n+1}|,\ \ldots,\ v_{n-1}^3|v_{n-1}^{n+1}|,\ v_n^3|v_n^{n+1}|\right) \\
\ldots & \qquad\qquad \ldots \\
w^{n-1} = & \left(v_1^{n-1},\ v_2^{n-1},\ v_3^{n-1},\ v_4^{n-1},\ \ldots,\ 0,\ v_n^{n-1}|v_n^{n+1}|\right) \\
w^n = & \left(v_1^n,\ v_2^n,\ v_3^n,\ v_4^n,\ \ldots,\ v_{n-1}^n,\ 0\right)
\end{array}
$$

Consider each pair of adjacent vectors w^i and w^{i+1} for $i \in \{1, \ldots, n-1\}$. For every $j \in N$, there is a positive real number k^i_j such that $w^i_j = k^i_j v^i_j$ and $w^{i+1}_j = k^i_j v^{i+1}_j$. So, by Ratio-Scale Invariance, $w^1 \prec w^2 \prec \cdots \prec w^n$, and thus $w^1 \prec w^n$ by the transitivity of $\prec$. Notice, however, that $w^1 = v^{n+1}$ and $w^n = v^n$, so this result contradicts $v^n \prec v^{n+1}$.

There must therefore be some $i^* \in N$ such that $u \succ \mathbf{0}$ for every $u \in U^{i^*}_{++}$. Now take any $v \in \mathbb{R}^n$ for which $v_{i^*} > \mathbf{0}$. There exist $u \in U^{i^*}_{++}$ for which $v \gg u$, so we have $v \succ \mathbf{0}$ by Weak Pareto and the transitivity of $\succ$.

A parallel argument shows that there must be some $j^* \in N$ such that $u \prec \mathbf{0}$ whenever $u_{j^*} < 0$. It must be the case that $i^* = j^*$. Otherwise, we could find some $v \in \mathbb{R}^n$ such that $v_{i^*} > 0$ and $v_{j^*} < 0$; this would imply both $v \succ \mathbf{0}$ and $v \prec \mathbf{0}$, which is impossible.

Suppose without loss of generality that $i^* = 1$. Take any $u \in \mathbb{R}^n$ such that $u_1 = 0$. Since $(\epsilon, u_2, \ldots, u_n) \succ \mathbf{0}$ and $(-\epsilon, u_2, \ldots, u_n) \prec \mathbf{0}$ for any $\epsilon > 0$, Continuity implies $u \sim 0$. Thus for any $u \in \mathbb{R}^n$, $u \succcurlyeq \mathbf{0}$ iff $u_1 \geq 0$. □

The central argument in the above proof is illustrated, for the case of $\mathbb{R}^2$, in Fig. 1. If all points in quadrants II and IV are worse than the origin, then by Weak Pareto, so are all points along the negative half-axes. By Continuity, for some $\delta, \epsilon > 0$, $v^1 = (0, -1)$ is worse than $v^2 = (-\delta, 0)$, which is worse than $v^3 = (0, -\epsilon)$. But, by Ratio-Scale Invariance, $v^1 \prec v^2$ implies $v^3 \prec v^2$.

We prove the theorem by generalizing individual i^*'s dictatorial status away from the origin.

Theorem (Tsui and Weymark, Theorem 6) *A social welfare ordering $\succcurlyeq$ on $\mathbb{R}^n$ satisfies Weak Pareto, Continuity, and Ratio-Scale Invariance if and only if there is some $i^* \in N$ such that, for any $u, v \in \mathbb{R}^n$, $u \succcurlyeq v$ iff $u_{i^*} \geq v_{i^*}$.*

Proof Assume that $\succcurlyeq$ satisfies Weak Pareto, Continuity, and Ratio-Scale Invariance. By Lemma 2, there must be some $i^* \in N$ such that, for any $u \in \mathbb{R}^n$, $u \succcurlyeq \mathbf{0}$ iff $u_{i^*} \geq 0$. Again, without loss of generality, let $i^* = 1$.

We first consider vectors in which all components after the first are zero. Take any real numbers a and b with $a > b$. Let $\hat{u} = (a, 0, \ldots, 0)$ and $\check{u} = (b, 0, \ldots, 0)$. We prove $\hat{u} \succ \check{u}$ by cases.

Case 1 $\text{sgn}(a) \neq \text{sgn}(b)$. Then either $a > 0 \geq b$ or $a \geq 0 > b$, which respectively imply $\hat{u} \succ \mathbf{0} \succcurlyeq \check{u}$ or $\hat{u} \succcurlyeq \mathbf{0} \succ \check{u}$; either way, $\hat{u} \succ \check{u}$ as desired.

Case 2 $\text{sgn}(a) = \text{sgn}(b)$. Since $a > b$, neither is zero, and b/a is positive. Suppose for contradiction that $\hat{u} \preccurlyeq \check{u}$. Weak Pareto and Continuity jointly imply $\hat{u} \succcurlyeq \check{u}$, so $\hat{u} \sim \check{u}$. We then multiply the first component of both vectors by b/a. By Ratio-Scale Invariance, $(a, 0, \ldots, 0) \sim (b, 0, \ldots, 0)$ implies $(b, 0, \ldots, 0) \sim (b^2/a, 0, \ldots, 0)$, and more generally $(b^k/a^{k-1}, 0, \ldots, 0) \sim (b^{k+1}/a^k, 0, \ldots, 0)$ for every natural number k.

Therefore, $\hat{u} \sim (b^{k+1}/a^k, 0, \ldots, 0)$ for every natural number k, by the transitivity of $\sim$. Since $b/a < 1$, $\lim_{k\to\infty}(b^{k+1}/a^k) = 0$. So, by Continuity, $\hat{u} \sim \mathbf{0}$. This is impossible because $a \neq 0$, so either $\hat{u} \succ \mathbf{0}$ or $\hat{u} \prec \mathbf{0}$. Thus $\hat{u} \succ \check{u}$ after all.

Now take any vector $u = (a, u_2, \ldots, u_n)$. Suppose for contradiction that $u \preccurlyeq \check{u}$. This implies, by Ratio-Scale Invariance, that for any $u' = (a, u'_2, \ldots, u'_n)$ such that $\text{sgn}(u'_i) = \text{sgn}(u_i)$ for every $i \in \{2, \ldots, n\}$, $u' \preccurlyeq \check{u}$. By making $u'_2, \ldots, u'_n$ arbitrarily close to zero, Continuity implies $\hat{u} \preccurlyeq \check{u}$, contrary to what we just showed. Thus $u \succ \check{u}$ after all. By an exactly similar argument, for any real number c less than b, $(c, u_2, \ldots, u_n) \prec \check{u}$.

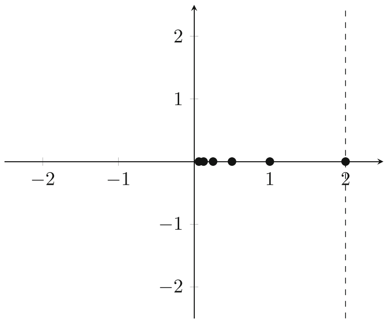
We have shown that, whenever $a > b > c$, $(a, u_2, \ldots, u_n) \succ (b, 0, \ldots, 0) \succ (c, u_2, \ldots, u_n)$. Continuity then requires that $(b, u_2, \ldots, u_n) \sim (b, 0, \ldots, 0)$. Therefore, for any $u \in \mathbb{R}^n$ and $b \in \mathbb{R}$, $u \succcurlyeq (b, 0, \ldots, 0)$ iff $u_1 \geq b$.

Now take any $u, v \in \mathbb{R}^n$. We have shown that $u \sim (u_1, 0, \ldots, 0)$ and $v \sim (v_1, 0, \ldots, 0)$, and that $(u_1, 0, \ldots, 0) \succcurlyeq (v_1, 0, \ldots, 0)$ iff $u_1 \geq v_1$. It follows that $u \succcurlyeq v$ iff $u_1 \geq v_1$.

Clearly, if $\succcurlyeq$ is strongly dictatorial, then it satisfies Weak Pareto, Continuity, and Ratio-Scale Invariance. □

A two-person instance of the central argument is again illustrated in Fig. 2. Suppose that $(a, 0) \succ \mathbf{0}$ for every $a > 0$. If $(2, 0) \sim (1, 0)$, then by Ratio-Scale Invariance, $(1, 0) \sim (1/2, 0) \sim \cdots \sim ((1/2)^k, 0)$ for every natural number k. Thus $(2, 0) \sim \mathbf{0}$ by the transitivity of $\sim$ and Continuity, contrary to hypothesis, so $(2, 0) \succ (1, 0)$ after all. Continuity and Ratio-Scale Invariance then imply that any point along the dashed line must be better than $(1, 0)$ as well.

Fig. 2 Illustration of the central argument in the proof of the Theorem for $\mathbb{R}^2$



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