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RESEARCH ARTICLE

# Substitution and size effect for factor demand revisited

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## Abstract

We reconsider the decomposition of the comparative statics effect of a factor price increase on (unconditional) factor demand into a substitution and a size (or level) effect. While for the own price effect the substitution effect and the size effect go into the same negative direction, the cross price effect cannot be signed unambiguously, in general. But for two cases of regularity, homotheticity and complementarity, the size effect is negative. We furthermore show by example that in both special cases the cross substitution effects are still ambiguous, but that under complementarity the unconditional effect is always negative. Hence, if the cross substitution effect happens to be positive, it is dominated by the negative size effect in this case. We further show by example that without such regularity assumptions the cross price effect is ambiguous. Finally, we study the impact of a factor price increase on the cost, which can also increase or decrease.

**Keywords** Slutsky equation · (Conditional)Factor demand · Comparative statics · Substitution effect · Size effect · Level effect

**JEL Classification** B16 · B21 · D11 · D21 · D41

## 1 Introduction

In a seminal but initially unnoticed paper, Slutsky (1915, 1952) suggested the decomposition of individual consumer demand change responding to a price increase into the substitution and the income effect.[1] A major conclusion from the Slutsky equa-

[1] Allen (1936) popularized and further developed Slutsy's work. See Dooley (1983), Weber (1999), and above all Chipman and Lenfant (2002) for a detailed history on Slutsky's and related work. For further applications of consumer demand for goods and financial assets see Bierwag and Grove (1968), Mundlak (1968), Fischer (1972), Kalman et al. (1974), Ellis (1976), Laitinen and Theil (1979), Sedaghat (1996), and Menezes and Wang (2005). For empirical evidence see Barten (1967).

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tion is the individual consumer's law of demand, stating that for normal goods the own price effect is always negative since substitution and income effect work into the same direction. By contrast, nothing interesting can be said about cross price effects. Typically, these are ambiguous or zero for special cases.

Similar decompositions have been suggested for factor demand, the earliest by Puu (1966) and replicated by Bertoletti (2005). These authors show that when a firm's technology is given by a strictly convex cost function, the unconditional factor demand can be decomposed into a factor substitution effect and a "size" (or "level") effect that determines how a factor price increase impacts on the production level.

In this article, we dive a little deeper into that decomposition by studying under what circumstances the substitution, the size (or level), and the total effect can be signed.

In case of the own price increase, similarly to the Slutsky equation for consumer demand of a normal good, the substitution effect and the level effect *always* work into the same direction (confirming the well known result that there is nothing like a Giffen factor). Regarding the cross price effect we show that, in general, both effects, and thus the total effect, cannot be signed unambiguously. But for two prominent special cases, homotheticity and complementarity, we show that the size (or level) effect is negative. As this is what is typically assumed in applications, we call this the *regular* case. Furthermore, we show by examples that in both special cases the cross substitution effects are still ambiguous, but that under complementarity for differentiable and strictly concave production functions the total effect is always strictly negative for an arbitrary number of input factors. While non-positivity has been shown using the tools of super-modularity (Topkis 1995, 1998), negativity follows from the Frobenius matrix algebra (see Takayama 1985, and more recently Amir 1996). Hence, if the cross substitution effect happens to be positive, it is dominated by the negative size effect in this case.

While it is easy to see that for homothetic production functions total output goes down when some factor price increases, it can also increase for sufficiently asymmetric isoquants. By contrast, maybe surprisingly, production costs may increase or decrease through a factor price increase.

In our analysis we start with a dual approach in Sect. 2 where we present our decomposition in terms of observable effects. In Sect. 3, we provide an overview under what conditions own and cross price effects can be signed and provide a graphical illustration. In Sect. 4, we study how factor price increases impact on costs. In Sect. 5, we wrap up and draw some policy conclusions.

## 2 Model and decomposition of factor demand

Consider a firm producing one output by means of $n$ inputs subject to a production function $y = f(x)$, where $x = (x_1, \dots, x_n)$. About $f$ we make the following assumption.

**Assumption 1** $f$ is strictly monotonic and unbounded in $x$, twice continuously differentiable, it has positive marginal products, i.e. $f_i \equiv \partial f/\partial x_i > 0$, and is strictly concave.

For output price $p$ and input prices $w_i$, $i = 1, \dots, n$, let $G(p, w) := \sup_x\{pf(x) - w \cdot x\}$ be the profit function, and let $C(w, y) := \inf_x\{w \cdot x | f(x) \geq y\}$ be the cost function. We denote by $x(p, w)$ the *factor demand* and by $\tilde{x}(w, y)$ the *conditional factor demand*, respectively. Moreover, let $y(p, w) := f(x(p, w))$ denote the firm's supply. Since $f$ is differentiable and strictly concave, $G(p, w)$ is strictly convex and differentiable. By Hotelling's Lemma we get $G_p = y$, $G_w = -x$ (denoting partials by subscripts), and the Hessian of $G$ is symmetrical positive-definite (S-PD), while $C$ is differentiable and strictly concave in $w$ with $C_w = \tilde{x}$, and its Hessian $C_{w,w}$ is S-ND almost everywhere.[2]

We thus get $y_p > 0$, $\frac{\partial x_i}{\partial w_i} < 0$ and

$$py_p = \sum_i w_i \frac{\partial x_i}{\partial p}. \tag{1}$$

This follows from differentiating

$$py - \sum_i w_i x_i = G$$

w.r.t. $p$ and using $G_p = y$ (suppressing the function arguments, for short). Partials of $x$ w.r.t. $p$ may be negative, but they cannot all be negative. In particular, if $f$ is homothetic, they are all positive, because they respond to $p$ in fixed proportions.

Similarly, we get $\frac{\partial \tilde{x}_i}{\partial w_i} < 0$ and

$$w_i \frac{\partial \tilde{x}_i}{\partial w_i} = -\sum_{j \neq i} w_j \frac{\partial \tilde{x}_j}{\partial w_i}. \tag{2}$$

Partial cross derivatives of the conditional factor demand $\tilde{x}$ may be negative, but they cannot all be negative. In particular, for only two inputs, this implies $\frac{\partial \tilde{x}_j}{\partial w_i} > 0, i \neq j$.

Define the inverse function of $y$ w.r.t. $p$ by $\pi(w, \bar{y}) := \{p | y(p, w) = \bar{y}\}$.[3] Since $y$ is monotonically increasing in $p$, $\pi(w, \bar{y})$ is a singleton.

The price function $\pi(w, \bar{y})$ can be considered as a compensating price for changes in factor prices $w$ to sustain a fixed output level $\bar{y}$. Then by taking the total differential of $\pi$, keeping output constant, we obtain

$$\pi_w = -\frac{y_w}{y_p}. \tag{3}$$

[2] I.e. negative-definiteness may fail only on a measure-zero subset of the domain.

[3] To see that $\{p | y(p, w) = \bar{y}\}$ is not empty, observe that $x(p, w) = \text{argmax}\{pf(x) - wx\}$, and $y(p, w) = f(x(p, w))$. If $p$ goes to infinity, $y(p, w)$ goes to infinity, and if $p$ becomes sufficiently small, $y(p, w)$ goes to zero. Since $y(p, w)$ is a function with $\mathbb{R}_0 = f[\mathbb{R}_0]$, $\{p | y(p, w) = \bar{y}\}$ is non-empty.

By definition of $\pi(\cdot)$, we can write $\tilde{x}(w, y) = x(w, \pi(w, y))$. Differentiating both sides with respect to the factor prices we get $\tilde{x}_w = x_w + x_p\pi_w$, or by rearranging, we obtain the *decomposition for factor demand* into a substitution effect and size (or level) effect:

$$x_w = \tilde{x}_w - x_p\pi_w. \tag{4}$$

## 3 Own and cross price effects

For the *own price effect* we have

$$0 > \frac{\partial x_i}{\partial w_i} = \frac{\partial \tilde{x}_i}{\partial w_i} + \left(-\frac{\partial x_i}{\partial p}\frac{\partial \pi}{\partial w_i}\right) \tag{5}$$

Both RHS terms are non-positive. See above for the first term. The proof for the second term goes as follows. From the symmetry of $G$'s Hessian it follows that

$$G_{p,w} = y_w = G_{w,p} = -x_p. \tag{6}$$

Thus, using (3), the second term in Eq. (5) becomes

$$-\frac{\partial x_i}{\partial p}\frac{\partial \pi}{\partial w_i} = -\frac{\partial x_i}{\partial p}\frac{\partial x_i}{\partial p}/y_p < 0. \tag{7}$$

Even though well known, we restate this result as follows:

**Proposition 1** *Under Assumption 1 the own price effect of a partial factor price increase is negative, and both the substitution and the size (or level) effect move into the same direction.*

One might suspect that $\frac{\partial \pi}{\partial w_i}$ is always non-negative. But we will show in Sect. 3.4 that this need not be the case.

Recall also that the last result stands in contrast to consumer demand, where income and substitution effect in the Slutsky equation can go into opposite directions.

Next we take a look at the *cross price effect*. Here we obtain

$$\frac{\partial x_i}{\partial w_j} = \frac{\partial \tilde{x}_i}{\partial w_j} - \frac{\partial x_i}{\partial p}\frac{\partial \pi}{\partial w_j} \tag{8}$$

$$= \frac{\partial \tilde{x}_i}{\partial w_j} + \frac{\partial x_i}{\partial p}\frac{\frac{\partial y}{\partial w_j}}{\frac{\partial y}{\partial p}} \tag{9}$$

where in the last equation we make use of (3).

In general, neither term can be signed unambiguously for $i \neq j$, except that the first term (i.e. the substitution term) must be positive for at least one $i$. To be more conclusive, we thus study two prominent cases in the following, homotheticity and complementarity. Both imply a negative size (or level) effect. As this is the typical assumption in applied work, we coin this the *regular* case. Despite regularity, the total effect remains ambiguous under homotheticity but can be shown to be negative under complementarity.

### 3.1 Homothetic production functions

Intuitively, output seems to be decreasing in input prices, inputs seem to be increasing in the output price, and the price compensation function seems to be increasing in input prices. We call these responses *regular*. The following result shows regularity to hold for homothetic production functions. But as shown later, it does not hold in general.

**Proposition 2** *If $f$ is homothetic, we get $\partial y/\partial w_i < 0$, $\partial x_i/\partial p > 0$, and $\partial \pi/\partial w_i > 0$, for all $i = 1, \ldots, n$, while both, the cross price substitution effect $\frac{\partial \tilde{x}_i}{\partial w_j}$ and the unconditional cross price effect $\frac{\partial x_i}{\partial w_j}$ are ambiguous.*

**Proof** Note that a production function $f$ is homothetic iff the cost function can be written as $C(w, y) = h(y)c(w)$ with convex $h$ and concave $c$. It is homogeneous, if, in addition, $h$ is a power function. Profit maximization implies

$$h'(y)c(w) = p \tag{10}$$

The solution in $y$ defines an implicit supply function $y(p, w)$. Differentiating this equation totally, we obtain

$$h''(y)c(w)\frac{\partial y}{\partial w_i} + h'(y)\frac{\partial c}{\partial w_i} = 0 \tag{11}$$

and thus

$$\frac{\partial y}{\partial w_i} = -\frac{h'(y)}{h''(y)}\frac{\frac{\partial c}{\partial w_i}}{c(w)} < 0 \tag{12}$$

as $\partial c/\partial w_i > 0$.

To see $\partial x_i/\partial p > 0$, we write (using Shepard's Lemma) $x_i(w, p) = \tilde{x}_i(w, y(p, w)) = \frac{\partial c}{\partial w_i}h(y(p, w))$. Thus:

$$\frac{\partial x_i}{\partial p} = \frac{\partial c}{\partial w_i}h'(y)y_p > 0 \tag{13}$$

Finally, $\partial \pi/\partial w_i > 0$ follows from (7).

The ambiguity of $\frac{\partial \tilde{x}_i}{\partial w_j}$ and $\frac{\partial x_i}{\partial w_j}$ is shown in the appendix by using the following Example 1. q.e.d.

***Example 1*** Let the production function be nested CD-CES, $f(x_1, x_2, x_3) = (x_1^\rho + x_2^\rho)^{\alpha\beta/\rho} x_3^{(1-\alpha)\beta}$, with CES composite $z^\rho = x_1^\rho + x_2^\rho$ that $x_1$ and $x_2$ symmetrically enter into and with parameters $0 < \alpha < 1$, $0 < \beta < 1$, and $\rho < 1$.

Note that Proposition 2 immediately implies that also the size (or level) effect $-\frac{\partial x_i}{\partial p}\frac{\partial \pi}{\partial w_j}$ is negative.

### 3.2 Complements

In this section we provide a condition under which cross price effects are negative. This holds even if the substitution effect is positive, meaning that in this case a negative size effect dominates the positive substitution effect. The respective condition is that all inputs are complements, i.e. an increase of the input amount of one factor increases the productivity of all other factors.

**Definition 1** The inputs of a differentiable production function are called pairwise strong complements if $f_{ij} > 0$ for all $i, j = 1, \ldots, n, i \neq j$.

From the Frobenius matrix algebra (see Takayama 1985, also Amir 1996) we can derive the following result.

**Proposition 3** (Takayama) *If all inputs are strong complements, then*

$$\frac{\partial x_i}{\partial w_j} < 0.$$

The proof follows from Proposition 4.D.3 in Takayama (1985) and the equivalence of part (IV"), which is satisfied under strict concavity of the production function, and (VIII"), which is our conclusion. It can also be derived from the strict version of Topkis's monotonocity proved by Amir (1996) (see also Theorem 2.8.5, Topkis 1998).

The result implies that if the substitution effect is positive, the size effect must be negative and must dominate the substitution effect. In fact, we can show that the size effect is always negative.

**Proposition 4** *If all inputs of a differentiable, strictly concave production function are pairwise strong complements, then $\partial x_i/\partial p > 0$ and $\partial \pi/\partial w_i > 0$ for all $i = 1, \ldots, n$.*

The proof is given in the appendix. With this result we can immediately obtain:

**Corollary 1** *If all inputs are pairwise strong complements, then the size (or level) effect is always negative.*

***Proof*** By the decomposition (8) the level effect is $-(\partial x_i/\partial p) \cdot (\partial \pi/\partial w_j) < 0$. □

Thus, if the substitution effect is negative, the substitution and the size effect enforce each other, as is the case for the own price effect.

In Fig. 1, we have, for the case of a Cobb-Douglas production function, visualized the decomposition of the comparative statics effect of a partial factor price increase of factor 2. Note that, contrasting from the case of a price increase in the consumer's choice problem, the iso-cost curve does not tilt around a fixed point on one axis, but also shifts.

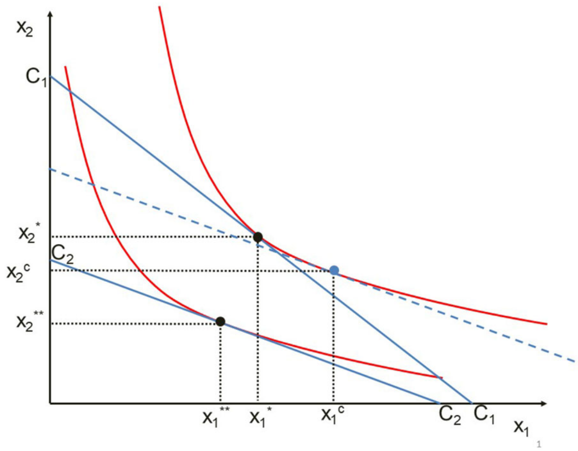


**Fig. 1** Decomposition of factor demand change into substitution and size effect: the regular case with homogenous production function. We denote: $(x_1^*, x_2^*)$ = original factor demand, $(x_1^c, x_2^c)$ = compensated factor demand, $(x_1^{**}, x_2^{**})$ = new factor demand, convex curves (red) = isoquants, $\overline{C_1C_1}$ = original iso-cost line, $\overline{C_2C_2}$ = new iso-cost line, dashed line = price compensated iso-cost line

### 3.3 Decomposition formulas for 2 and 3 inputs

We can further substantiate the decomposition for the cases of two and three inputs, assuming $f_{ij} > 0$ for all $i \neq j$ throughout. From micro textbooks (e.g. Varian 1992) it is well known that the comparative statics expressions for the unconditional factor demand, in the case of two factors, are given by:

$$\frac{\partial x_i}{\partial w_i} = \frac{f_{jj}}{p|H|} < 0 \tag{14}$$

$$\frac{\partial x_i}{\partial w_j} = -\frac{f_{ij}}{p|H|} < 0 \tag{15}$$

$$\frac{\partial x_i}{\partial p} = \frac{f_j f_{ij} - f_i f_{jj}}{p|H|} > 0 \tag{16}$$

where $|H| = f_{11} f_{22} - f_{12}^2 > 0$ is the determinant of the Hessian of $f$. By contrast, the comparative statics expressions for the *conditional factor demand* are given by

$$\frac{\partial \tilde{x}_i}{\partial w_i} = \frac{f_j^2}{\lambda|\tilde{H}|} < 0 \tag{17}$$

$$\frac{\partial \tilde{x}_i}{\partial w_j} = -\frac{f_1 f_2}{\lambda|\tilde{H}|} > 0 \tag{18}$$

where $|\tilde{H}| = f_1^2 f_{22} + f_2^2 f_{11} - 2 f_1 f_2 f_{12} < 0$ is the determinant of the bordered Hessian, which by the second-order condition needs to be negative, and $\lambda$ is the Langrange multiplier for the output constraint of the cost minimization problem.

By definition of $\pi(w, y)$ we have

$$f(x(w, \pi(w, y))) = y \tag{19}$$

Differentiating this with respect to $w_i$ we obtain:

$$\frac{\partial \pi}{\partial w_i} = \frac{-\sum_{j=1}^{n} f_j \frac{\partial x_j}{\partial w_i}}{\sum_{j=1}^{n} f_j \frac{\partial x_j}{\partial p}} \tag{20}$$

Using the last equation we obtain

$$\frac{\partial \pi}{\partial w_i} = \frac{f_i f_{jj} - f_j f_{ij}}{|\tilde{H}|} > 0 \tag{21}$$

as the denominator is exactly the determinant of the bordered Hessian. Thus, the total size effect for the own price change can be expressed as

$$-\frac{\partial x_i}{\partial p} \cdot \frac{\partial \pi}{\partial w_i} = -\frac{f_j f_{ij} - f_i f_{jj}}{p|H|} \cdot \frac{f_i f_{jj} - f_j f_{ij}}{|\tilde{H}|} = \frac{(f_i f_{jj} - f_j f_{ij})^2}{p|H||\tilde{H}|} < 0 \tag{22}$$

while for the cross-price effect it is given by

$$-\frac{\partial x_i}{\partial p} \cdot \frac{\partial \pi}{\partial w_j} = -\frac{f_j f_{ij} - f_i f_{jj}}{p|H|} \cdot \frac{f_j f_{ii} - f_i f_{ij}}{|\tilde{H}|} < 0 \tag{23}$$

Now, exploiting that due to (19) we have $\lambda = p$, and inserting (16), (17), and (22) into (5) and rearranging we can easily verify that

$$\frac{\partial x_i}{\partial w_i} = \frac{f_{jj}}{p|H|} = \frac{f_j^2}{|\tilde{H}|} - \frac{f_j f_{ij} - f_i f_{jj}}{p|H|} \cdot \frac{f_i f_{jj} - f_j f_{ij}}{|\tilde{H}|} = \frac{\partial \tilde{x}_i}{\partial w_i} - \frac{\partial x_i}{\partial p}\frac{\partial \pi}{\partial w_i} \tag{24}$$

holds, similarly for the cross price effect.

For $n \geq 3$ one can derive similar expressions. For instance for $n = 3$ the total cross price effect is given by

$$\frac{\partial x_i}{\partial w_j} = \frac{-f_{ij} f_{kk} + f_{ik} f_{jk}}{p|H|} < 0 \tag{25}$$

where again $|H| < 0$ is the determinant of the Hessian, while the substitution effect is now given by

$$\frac{\partial \tilde{x}_i}{\partial w_j} = \frac{f_k[-f_i f_{jk} - f_j f_{ik} + f_k f_{ij}] + f_i f_j f_{kk}}{p^2|\tilde{H}|} \tag{26}$$

where the determinant of the bordered Hessian, $|\tilde{H}|$, is negative. Note that in the numerator all terms except $f_k^2 f_{ij}$ are negative. Therefore, the substitution effect cannot be signed unambiguously (see Example 1). We know, however, that not all substitution effects can be negative.

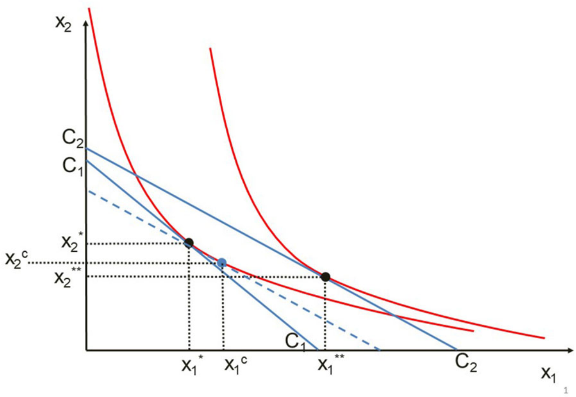


**Fig. 2** Decomposition of factor demand change into substitution and size effect: an irregular case. We denote: $(x_1^*, x_2^*)$ = original factor demand, $(x_1^c, x_2^c)$ = compensated factor demand, $(x_1^{**}, x_2^{**})$ = new factor demand, convex curves (red) = isoquants, $\overline{C_1C_1}$ = original iso-cost line, $\overline{C_2C_2}$ = new iso-cost line, dashed line = price compensated iso-cost line

### 3.4 Irregular behavior

We have seen that for both, homothetic productions functions and complements, increasing factor prices lower output, and the size effect is negative. The question arises whether cases exist with $\partial y/\partial w_i > 0$, and $\partial \pi/\partial w_i < 0$. The answer is "yes" if isoquants are sufficiently asymmetric. We show this by presenting an appropriate example.

***Example 2*** ($\frac{\partial y}{\partial w_2} > 0$, and thus also $\frac{\partial \pi}{\partial w_2} < 0$). Choose

$$y = f(x_1, x_2) = \sqrt{2(x_1 + \sqrt{2(x_1 + x_2)})} \tag{27}$$

The production function is concave because a non-decreasing concave function $f : \mathbb{R}^1 \to \mathbb{R}^1$ of a concave function is concave. For a profit maximum the non-negativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ need to be taken into account. Thus we choose prices such that the profit maximum is interior. For example, this is the case for $p = \sqrt{24}$, $w_1 = 3$ and $w_2 = 1$ yielding $x_1 = x_2 = 1$ and $y = \sqrt{6}$. Then, in the neighborhood of these values the firm's supply function is given by

$$y(p, w) = \frac{p}{w_1 - w_2}, \tag{28}$$

which is increasing in $w_2$. This case is illustrated in Fig. 2 where the new output level is on a higher isoquant, and thus on a higher iso-cost line. We also observe that the use of factor 1 goes up.

#### Summary of the effects

We can summarize our effects in Table 1.

To see that the total effect is ambiguous for homothetic production functions with $n = 2$, take $f(x_1, x_2) = (x_1^\rho + x_1^\rho)^{\beta/\rho}$ with $0 < \beta < 1$ and $\rho < 1$.

**Table 1** Summary of the effects

| Effects: $\rightarrow$ | Total | Substitution | Level |
|---|---|---|---|
| **Own price effects** | $-$ | $-$ | $-$ |
| **Cross price effects** | | | |
| Properties of | | | |
| Production function $\downarrow$ | | | |
| Homothetic, $n = 2$ | $+/-$ | $+$ | $-$ |
| Homothetic, $n > 2$ | $+/-$ | $+/-$ | $-$ |
| Complements, $n = 2$ | $-$ | $+$ | $-$ |
| Complements, $n > 2$ | $-$ | $+/-$ | $-$ |
| In general | $+/-$ | $+/-$ | $+/-$ |

## 4 Impact on cost

In Fig. 1 we have visualized the decomposition of the comparative statics effect of a partial factor price increase of factor 2 for the regular case. We observe that unlike the consumer budget line, the iso-cost line does not tilt around a fixed point on the horizontal axis, but also shifts downwards from $\overline{C_1C_1}$ to $\overline{C_2C_2}$. The level effect makes the firm to reduce output and thus to end up on a lower isoquant associated with a lower iso-cost line. The question is, whether the observed cost decrease holds in general. The answer is "no", but under certain conditions it does.

To study the impact on cost we define the firm's reduced cost function as $K(p, w) = w \cdot x(p, w)$. Differentiating w.r.t. $w_i$ we obtain

$$\frac{\partial K(p, w)}{\partial w_i} = x_i + \sum_{j=1}^{n} w_j \frac{\partial x_j}{\partial w_i} \tag{29}$$

We see that a factor price increase splits up into a positive direct effect and an (in general) ambiguous indirect effect which captures the size (or level) adjustment. While this overall effect is also ambiguous in general, it can be shown to be negative for homogeneous production functions.

Consider now the reduced cost function $K(p, w) = h(y(p, w))c(w)$ where $y(p, w))$ is the implicit supply function. Differentiating this with respect to $w_i$ and using (12) we get

$$\begin{aligned}\frac{\partial K}{\partial w_i} &= h'(y)c(w)\frac{\partial y}{\partial w_i} + h(y)\frac{\partial c}{\partial w_i} \\ &= h'(y)\left[-\frac{h'(y)}{h''(y)} + \frac{h(y)}{h'(y)}\right]\frac{\partial c}{\partial w_i}\end{aligned} \tag{30}$$

From this we see immediately that if $h(\cdot)$ is the exponential function,[4] $\frac{\partial K}{\partial w_i} = 0$, while if $h(\cdot)$ grows faster than exponentially, e.g. $h(y) = \exp(y^2)$, we get $\frac{\partial K}{\partial w_i} > 0$. If, by contrast, $h(\cdot)$ grows less than exponentially, which in particular is the case if $h(\cdot)$ is a power function, $h(y) = y^\beta$ with $\beta > 0$, implying homogeneity of $f$, we obtain $\frac{\partial K}{\partial w_i} < 0$.

**Proposition 5** *If $f$ is homogeneous, we get $\frac{\partial K}{\partial w_i} < 0$ for all $i$. If $f$ is homothetic with exponential growth of the scaling function $h$, we get $\frac{\partial K}{\partial w_i} = 0$. If $f$ is homothetic with stronger (less strong) than exponential growth of the scaling function $h$, we get $\frac{\partial K}{\partial w_i} > 0$ ($\frac{\partial K}{\partial w_i} < 0$).*

Example 2 and Fig. 2 show however that $\frac{\partial K}{\partial w_i} > 0$ is also possible for non-homothetic production functions. In that example the positive level effect also induces higher costs.

## 5 Conclusions and policy implications

In this note we have reconsidered the decomposition of factor demand into the substitution and the size (or level) effect. Regarding the own price effect, the substitution and the size effect go into the same negative direction. Regarding the cross price effect we have shown that, although in general both effects cannot be signed unambiguously, for two prominent special cases, homotheticity and complementarity, the size effect is negative.

Our findings, even though being implicit in other studies, e.g. in the literature on ecological tax reforms (see e.g. Parry 1995), have important policy implications. For example, they imply that if a government taxes a dirty production factor, this does not necessarily boost the employment of clean production factors. If, by contrast, politics wants to set positive incentives to use cleaner factors by subsidizing these, this can induce the (in this case positive) size effect to outweigh the (in this case negative) substitution effect and thus also increase the use of dirty factors.

A similar phenomenon can be observed through rebound effects identified in consumer demand (see Wirl 2000), where technological improvements to save energy per unit, result in higher levels of consumption which may outweigh the effect of energy savings per unit consumed.

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## Declarations

**Conflict of interest** The authors have no relevant financial or non-financial interests to disclose.

[4] The exponential function may look inconvenient because $h(0)$ equals one rather than zero. But note that we only need it for local behavior.

# A Appendix

## A.1 Proofs

### The ambiguity of cross price effects in Example 1

Consider, for inputs $x = (x_1, x_2, x_3)$, the strictly concave nested CD-CES production function

$$f(x) = (x_1^\rho + x_2^\rho)^{\alpha\beta/\rho} x_3^{(1-\alpha)\beta},$$

where $x_1$ and $x_2$ symmetrically enter into a CES function nested into an upper level CD function. For convenience we write this in nested form as

$$y = F^\beta, \quad F = z^\alpha x_3^{1-\alpha}, \quad z = (x_1^\rho + x_2^\rho)^{1/\rho},$$

with parameters $0 < \alpha < 1$, $0 < \beta < 1$, and $\rho < 1$. We find

$$f_i = \alpha\beta z^{\alpha\beta-\rho} x_i^{\rho-1} x_3^{(1-\alpha)\beta}, \quad i \in \{1, 2\},$$

implying that $f_{1,3} > 0$, $f_{2,3} > 0$ and $f_{1,2} > 0$ iff $\alpha\beta > \rho$. Hence, complementarity holds iff $\alpha\beta > \rho$, meaning that substitutability between factors 1 and 2 must not be too high.

Applying standard duality operations leads to the cost per unit of $F$,

$$c(w) = \min_x \{w \cdot x \mid F(x) \geq 1\} = a v^\alpha w_3^{1-\alpha}$$

with a constant $a = \alpha^{-\alpha}(1-\alpha)^{\alpha-1} > 0$, and the cost per unit of $z$,

$$v(w_1, w_2) = \min_{x_1, x_2} \{w_1 x_1 + w_2 x_2 \mid z(x_1, x_2) \geq 1\} = (w_1^{1-\sigma} + w_2^{1-\sigma})^{1/(1-\sigma)},$$

with elasticity of substitution $\sigma = 1/(1-\rho) > 0$. This yields the cost function $C(w, y) = y^{1/\beta} c(w)$ and its partial derivative

$$\tilde{x}_{w_1}(w, y) = y^{1/\beta} \alpha (c/v)(v/w_1)^\sigma = y^{1/\beta} \alpha a v^{\alpha-1} (v/w_1)^\sigma w_3^{1-\alpha} \tag{31}$$

implying $\tilde{x}_{w_1,w_2} > 0$ iff $\alpha + \sigma > 1$ and $\tilde{x}_{w_1,w_3} > 0$. Note that complementarity holds iff $\sigma < 1/(1-\alpha\beta)$. Thus, in the example, the sign of $\tilde{x}_{w_1,w_2}$ is not only ambiguous in general, but also under the complementarity restriction.

Now, to derive unconditional factor demand, we obtain $F = (c/\beta)^{1/(\beta-1)}$ from maximising profit $F^\beta - cF$, and thus, similar as in (31)

$$
\begin{aligned}
x_{w_1}(w, 1) &= (c/\beta)^{1/(\beta-1)}\alpha(c/v)(v/w_1)^\sigma \\
&= \beta^{1/(1-\beta)}\alpha c^{\beta/(\beta-1)} v^{\sigma-1} w_1^{-\sigma} \\
&= \beta^{1/(1-\beta)}\alpha a v^{\alpha\beta/(\beta-1)} v^{\sigma-1} w_1^{-\sigma} w_3^{\beta(1-\alpha)/(\beta-1)}.
\end{aligned}
$$

In general, $\alpha\beta/(\beta - 1) + \sigma - 1$ can be negative or positive; thus the cross price effect is ambiguous in general for a homogeneous production function. In case of complementarity, however, we have

$$
\sigma - 1 = \frac{\rho}{1-\rho} < \frac{\alpha\beta}{1-\alpha\beta} < \frac{\alpha\beta}{1-\beta}.
$$

Hence,

$$
\sigma - 1 - \frac{\alpha\beta}{1-\beta} = \sigma - 1 + \frac{\alpha\beta}{\beta - 1} < 0,
$$

as stated in Proposition 3 for the general case.

**Proof of Proposition 4**

Recall that $H$ denotes the Hessian of $f$, and let $x_w$ denote the matrix of all total factor price effects with typical element $x_w[i, j] = \partial x_i/\partial w_j$. Then $x_w = H^{-1}$. Let $x_p$ denote the vector of partial derivatives of unconditional factor demand with respect to the output price, and let $\nabla f$ be the vector of marginal products of $f$. Then $x_p$ solves $H \cdot x_p^T = -(\nabla f)^T$. Thus $x_p^T = H^{-1}(-\nabla f)^T$. Since by Proposition 3 all elements of $H^{-1}$ are negative, we obtain $x_p > 0$.

To show $\pi_w > 0$, observe that by positive marginal productivities and Proposition 3 we get

$$
\frac{\partial y}{\partial w_j} = \sum_i \frac{\partial y}{\partial x_i}\frac{\partial x_i}{\partial w_j} < 0.
$$

Using this and $y_p > 0$ in (3) yields $\pi_w > 0$.

### A.2 Weaker results when Assumption 1 does not hold.

In this section we briefly discuss how results change if we drop the Assumption 1 and instead assume

**Assumption 2** f is weakly monotonic and continuous.

In this case we can still define let $G(p, w) := \sup_x\{pf(x) - w \cdot x\}$ as the profit function, and $C(w, y) := \inf_x\{w \cdot x | f(x) \geq y\}$ as the cost function. Again assuming the extrema to be uniquely attained in some open set, we denote by $x(p, w)$ the *factor demand*, and by $\tilde{x}(w, y)$ the *conditional factor demand*, respectively, and by $y(p, w) := f(x(p, w))$ the firm's supply function. Under this weaker assumption it follows from Topkis (1998) that $\frac{\partial y}{\partial p} \geq 0$, $\frac{\partial x_i}{\partial w_i} \leq 0$, $\frac{\partial \tilde{x}_i}{\partial w_i} \leq 0$ and for the level effect we obtain $\frac{\partial x_i}{\partial y}\frac{\partial \pi}{\partial w_i} = \frac{\partial x_i}{\partial p}\frac{\partial \pi}{\partial w_i} \geq 0$.

Moreover, for the cross price effect, instead of Proposition 3 we then obtain from Topkis' (1998, Corollary 2.8.2):

**Proposition 6** (Topkis) *If $f$ is super-modular*[5] *then*

$$\frac{\partial x_i}{\partial w_j} \leq 0.$$

The result shows that factor demand for input $i$ is non-increasing in the other factors' prices under relatively weak conditions (super-modularity), which can be satisfied also for piece-wise linear production functions.

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[5] A function $f : \mathbb{R}^n \to \mathbb{R}$ is called super-modular if $f(x \vee z) + f(x \wedge z) \geq f(x) + f(z)$, where $x \vee z = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$, and $x \wedge z = (\min\{x_1, z_1\}, \ldots, \min\{x_n, z_n\})$. If $f$ is twice differentiable, super-modularity is equivalent to $f_{ij} \equiv \partial^2 f/(\partial x_i)(\partial x_j) > 0$ for all $i, j = 1, \ldots, n, i \neq j$.

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