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RESEARCH ARTICLE

# The dual of Bertrand with homogenous products is Cournot with perfect complements

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**Abstract**
The quantity-setting (Cournot) oligopoly with perfect complements is dual to the price-setting (Betrand) oligopoly with homogeneous goods. Under mild technical conditions the former setting has a unique (pure strategy) Nash equilibrium with null quantities. As an implication, the provision of perfectly complementary goods might actually be impossible, if the market is not either perfectly competitive or monopolized.

**Keywords** Cournot duopoly · Bertrand duopoly · Perfect complements · Homogeneous products

**JEL Classification** D11 · D43 · D61

## 1 Introduction

Many years ago Hugo Sonnenschein noted that Cournot's duopoly model is dual to his model of "complementary monopoly": see Sonnenschein (1968) and Cournot (1838). According to current terminology (see e.g. Vives (1999): chapters 4 and 5, and Belleflamme and Peitz (2015): chapter 3), that result can be restated by saying that the quantity-setting (Cournot) duopoly with homogeneous product, in which firms take as given the inverse demand system, is dual to the price-setting (Bertrand) duopoly with perfect complements, in which firms take as given the direct demand system. Formally, this is so because the revenue functions share the same formal structure in the two cases, namely they are given respectively by $\widetilde{R}_i(q_1, q_2) = q_i P(q_1 + q_2)$ and $R_i(p_1, p_2) = p_i D(p_1 + p_2)$, $i = 1, 2$, where $P(\cdot)$ and $D(\cdot)$ are the relevant

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inverse and direct demand functions, with $q_1$ and $q_2$ and $p_1$ and $p_2$ the corresponding quantities and prices.

Sonnenschein (1968): p. 317 used this result to extend to Cournot's second model a well-known criticism[1] of his duopoly solution: "each duopolist can obtain a greater revenue by reducing his price a little and selling the quantity that clears the market (provided, of course, the other duopolist does not change his price)". His paper is relatively well known: for instance, it has 185 citations in Google Scholar (on the 6th of February, 2022) and it appears in the references of advanced texts as Vives (1999) and d'Aspremont and Dos Santos Ferreira (2021). Moreover, the Cournot's model of "complementary monopoly" is still the subject of current research: for instance, Amir and Gama (2019) argue, *inter alia*, that the duality stressed by Sonnenschein (1968) breaks down if production costs are different from zero. Interestingly, it is also used to study the case of commodity bundling: see for example Nalebuff (2000).

Anyway, Sonnenschein (1968) did not mention that a duality similar to the one he stressed also holds between (in current terminology) the Bertrand (price-setting) duopoly with homogeneous products and the Cournot (quantity-setting) duopoly with perfect complements. The latter relationship seems to have gone unnoticed until now, as far as we know: the purpose of this note is to illustrate it, and to discuss some of its implications. One of them being that, under general cost conditions, the Cournotian market is unable to provide perfectly complementary goods.

## 2 The setting

Consider the case of two homogeneous goods, and the corresponding direct demand functions $q_i\left(p_i, p_j\right)$, $i, j = 1, 2$ and $i \neq j$. Let $D(p)$ be the overall demand for the aggregate quantity $q = q_1 + q_2$, where $p = p_i = p_j$ is the common price, assumed to be (strictly, when positive) decreasing and continuous. Then, as is well known:

$$q_i\left(p_i, p_j\right) = \begin{cases} = 0 & \text{if } p_i > p_j \\ = \alpha_i\left(p_i\right) D\left(p_i\right) & \text{if } p_i = p_j \\ = D\left(p_i\right) & \text{if } p_i < p_j \end{cases} \tag{1}$$

where $\alpha_i(p)$, $i = 1, 2$ are arbitrary functions such that $0 \leq \alpha_i(p) \leq 1$, $\alpha_1(p) + \alpha_2(p) = 1$, reflecting the fact that by homogeneity of products the demand functions are not uniquely defined when prices are identical. Demand $q_i\left(p_i, p_j\right)$ is illustrated in Fig. 1.

Now consider the case of two perfectly complementary goods, supposing for the sake of simplicity that they must be consumed in a one-to-one ratio. Define $P(\widetilde{q})$ the marginal willingness to pay for the common quantity $\widetilde{q} = q_1 = q_2$, assumed to be

[1] Sonnenschein (1968) attributed this criticism to Edgeworth (1987). However, while Edgeworth (1987): p. 22 clearly maintained that the oligopoly equilibrium was generally indeterminate, he cited Bertrand (1883) and Marshall (1890): p. 485, respectively for the cases of constant (namely, null) and decreasing marginal costs, and his own theory considered only the case of increasing marginal costs, assuming that no firm could supply the whole demand of the market at the "limiting" price (also see Edgeworth (1922)).

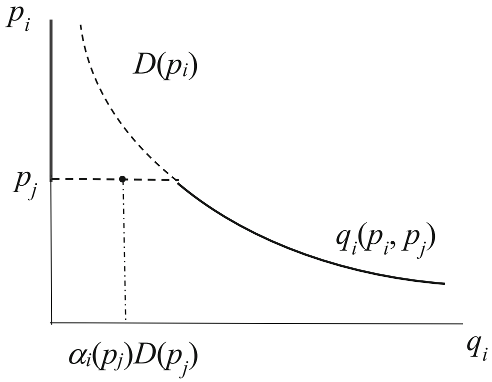


**Fig. 1** Direct demand with homogeneous products

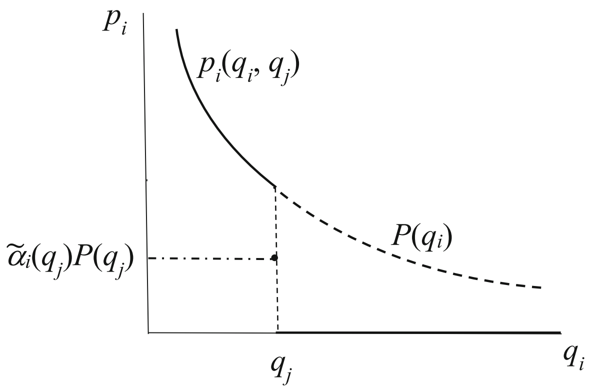


**Fig. 2** Inverse demand with perfect complements

(strictly, when positive) decreasing and continuous. Then the inverse demand system $p_i\left(q_i, q_j\right)$, $i, j = 1, 2$ and $i \neq j$ is given by:

$$p_i\left(q_i, q_j\right) = \begin{cases} = 0 & \text{if } q_i > q_j \\ = \widetilde{\alpha}_i\left(q_i\right) P\left(q_i\right) & \text{if } q_i = q_j \\ = P\left(q_i\right) & \text{if } q_i < q_j \end{cases} \tag{2}$$

where the arbitrary functions $\widetilde{\alpha}_i\left(q_i\right)$, $i = 1, 2$ are such that $0 \leq \widetilde{\alpha}_i\left(\widetilde{q}\right) \leq 1$, $\widetilde{\alpha}_1\left(\widetilde{q}\right) + \widetilde{\alpha}_2\left(\widetilde{q}\right) = 1$, reflecting the fact that by perfect complementarity the inverse demand functions are not uniquely defined when quantities are identical. By definition, the inverse demand system (2) gives the market-clearing prices as a function of quantities: if $q_1 = q_2 = \widetilde{q}$ then $p_1\left(\widetilde{q}, \widetilde{q}\right) + p_2\left(\widetilde{q}, \widetilde{q}\right) = P\left(\widetilde{q}\right)$ but only a null price $p_i$ can induce to demand $q_i > q_j$, while $p_j = P\left(q_j\right)$ in the latter case. Notice that $\widetilde{\alpha}_i\left(\widetilde{q}\right)$ might depend on $\widetilde{q}$ with no consequence for the following discussion. Inverse demand $p_i\left(q_i, q_j\right)$ is illustrated in Fig. 2.

Formally, (2) can be immediately obtained from (1) by exchanging prices with quantities and replacing $D(p)$ and $\alpha(p)$ with $P(\widetilde{q})$ and $\widetilde{\alpha}(\widetilde{q})$. The term "dual" is sometimes used ambiguously in the economic literature: in this note, following Sonnenschein (1968), we just refer to the previous exchange of variables in a formally equivalent demand structure.

### 2.1 Duopoly competition

It is well known that, once completed with the assumption of constant returns to scale (and no capacity constraints), the Bertrand symmetric duopoly model with homogeneous products (and complete information) has a unique Nash equilibrium (in pure strategy) in which prices are equal to the constant marginal cost and profits are null (see e.g. Tirole (1988): chapter 5 on this "Bertrand paradox", and Amir and Evstigneev (2018) for a technical discussion).

The Cournot duopoly model with perfect complements (and complete information) seems to be much less known: however, the following result is easily established.

**Proposition 1** *Suppose that the duopolists' cost functions $C_i(q_i)$ are non-decreasing and differentiable, and that it exists a finite $P(0)$ such that $P(0) > C_i'(0)$, $i = 1, 2$: then in the Cournot duopoly game with perfect complements and simultaneous moves there exists a unique Nash equilibrium (in pure strategy) in which both quantities are null.*

***Proof*** The profit function of duopolist $i$ is given by $\widetilde{\pi}_i(q_i, q_j) = p_i(q_i, q_j)q_i - C_i(q_i)$, with $\widetilde{\pi}_i(0, 0) = -C_i(0)$. Since revenues of duopolist $i$ are null for $q_i > q_j$, and his cost function is not decreasing, $q_i = 0$ is clearly a best reply to $q_j = 0$, thus $\{q_1 = 0, q_2 = 0\}$ is indeed a Nash equilibrium. To establish uniqueness, consider the cases in which $q_i = q_j = \widetilde{q} > 0$: then at least one duopolist (in fact, both if $0 < \widetilde{\alpha}_i(\widetilde{q}) < 1$) can increase his profit by (possibly slightly) decreasing his quantity and obtaining a jump up in his revenue and possibly a cost reduction. If $q_i > q_j > 0$ for some $i, j = 1, 2, i \neq j$, duopolist $i$ can get a profit increase by (possibly slightly) reducing his quantity below $q_j$, again by getting a jump up in his revenue and possibly a cost reduction. Finally, if $q_i > q_j = 0$, then duopolist $j$ can get a profit increase by slightly raising his quantity above zero. □

#### 2.1.1 Discussion

The equilibrium result of a Cournot duopoly with perfect complements appears as strong as that of the dual Bertrand duopoly with homogeneous product,[2] and it is similarly robust to the generalization to $n > 2$ competitors. However, while we leave this for future research, it should also be similarly amendable by relaxing the assumption of a one-shot setting or by introducing some price ceilings (see e.g. Tirole (1988): chapters 5 and 6, on the Bertrand setting).[3] Notice that it is not subject to the criticism mentioned by Sonnenschein (1968).

[2] But note that both Nash equilibria are in weakly dominated strategies.

[3] Of course, also the assumption of complete information should matter: see e.g. Belleflamme and Peitz (2015): section 3.1.2.

That an imperfectly competitive market might find difficult to provide complementary goods was long ago suggested by Spence (1976): pp. 220–221, who noted that in such a case in an equilibrium some good may not be produced at all. And similar results have been proved more recently by using the theory of supermodular games: see Vives (1999): Appendix of chapter 6. The economic content of Proposition 1 is to show that the provision of perfectly complementary goods might actually be impossible, under general cost conditions, if the market is not either perfectly competitive or monopolized. This is illustrated in the following simple example.

### 2.1.2 A quadratic example

Suppose that the representative consumer has quasi-linear preferences with direct utility given by:

$$U(q_0, q_1, q_2) = q_0 + a \min\{q_1, q_2\} - \frac{b}{2}(\min\{q_1, q_2\})^2,$$

where $q_0$ is the quantity of the *numéraire* and $a, b > 0$. Let us assume that the expenditure of the representative consumer is large enough to guarantee a positive consumption of the *numéraire*. Then her marginal willingness to pay for the common quantity $\widetilde{q}$ is given by $P(\widetilde{q}) = \max\{0, a - b\widetilde{q}\}$, and the assumptions of Proposition 1 for the existence of a unique Nash equilibrium (in pure strategies) with $q_i^C = 0$ in a Cournotian duopoly in which producers set quantities are satisfied if we also assume that $C_i(q_i) = cq_i$ with $a > 2c$,[4] $i = 1, 2$.

Since the representative consumer is only interested in a common quantity $\widetilde{q}$, and she has to pay for both $q_1$ and $q_2$, her direct demand system is given by $q_i(p_1, p_2) = \max\left\{0, \frac{a-(p_1+p_2)}{b}\right\}$, $i = 1, 2$. Notice that a perfectly competitive market would provide the Pareto-efficient quantities $q_i^{\circ} = \frac{a-2c}{b}$ at prices $p_i^{\circ} = c$. Suppose that, on the contrary, two duopolists, producing respectively commodity 1 and 2, compete *à la* Bertrand (i.e., by setting prices), as in the Cournot's "complementary monopoly" model discussed by Sonnenschein (1968). Each duopolist's profit function is then given by $\pi_i(p_1, p_2) = (p_i - c)\, q_i(p_1, p_2)$, with "reaction function" $p_i(p_j) = \max\left\{c, \frac{a+c-p_j}{2}\right\}$, $i, j = 1, 2$, $i \neq j$. It follows that in the unique Nash equilibrium (in pure strategies) $p_i^B = \frac{a+c}{3}$ and $q_i^B = q_i\left(\frac{a+c}{3}, \frac{a+c}{3}\right) = \frac{a-2c}{3b}$. But notice that the equilibrium profit is given by $\pi_i^B = \pi_i\left(\frac{a+c}{3}, \frac{a+c}{3}\right) = \frac{(a-2c)^2}{9b}$, and that each duopolist could obtain a larger profit, approximately equal to $\left[P\left(q_i^B\right) - c\right] q_i^B = \frac{2a^2-5ac+2c^2}{9b}$, by reducing slightly his quantity below $q_i^B$ and selling it at the market-clearing price (approximately equal to $P\left(q_i^B\right) = 2\frac{a+c}{3}$), provided that the other duopolist does not change his quantity: see Sonnenschein (1968).

[4] We make this assumption to ensure that the perfectly competitive, monopolistic and Bertrand quantities are on the contrary positive: see below.

Finally, a two-product monopolist with total profit

$$\pi(p_1, p_2) = (p_1 - c)\, q_1(p_1, p_2) + (p_2 - c)\, q_2(p_1, p_2)$$
$$= [(p_1 + p_2) - 2c] \max\left\{0, \frac{a - (p_1 + p_2)}{b}\right\}$$

would instead provide quantities $q_i^m = \frac{a-2c}{2b}$ at prices $p_i^m$ such that $p_1^m + p_2^m = \frac{a+2c}{2}$, with $q_i^\circ > q_i^m > q_i^B > q_i^C = 0$ and $2c = p_1^\circ + p_2^\circ < p_1^m + p_2^m < p_1^B + p_2^B = \frac{2(a+c)}{3} < p_1^C + p_2^C = P(0) = a$.

## 3 Conclusions

Demand complementarity has been a key concept since the dawn of modern economics: according to Samuelson (1974), Edgeworth (1881) was the first to use a positive sign of the utility function cross derivative, but this was identified as a condition of complementarity only by von Auspitz and Lieben (1889), to become later the so-called Edgeworth-Pareto criterion. The introduction of the notion of perfect complements, which is the focus of this note (and of many other papers: see e.g. Solan and Vieille (2006) for a recent example), is instead due to Fisher (1892).

In particular, following Sonnenschein (1968), we have shown that the Cournot duopoly with perfectly complementary goods is dual to the classical Bertrand duopoly with homogeneous goods, and that, under pretty mild technical conditions, it has a unique equilibrium with null quantities. Once again, and perhaps nicely, the contributions of the 19th-century pioneers of duopoly theory appear complementary rather than rival. [5]

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[5] See d'Aspremont and Dos Santos Ferreira (2021) for an interesting reinterpretation of the Cournot-Bertrand debate.

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