Afriat and arbitrage

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Abstract

This paper presents a proof of Afriat's (Int Econ Rev 8:67–77) theorem on revealed preference by using the idea that a rational consumer should not be vulnerable to arbitrage. The main mathematical tool is the separating hyperplane theorem.

Keywords Afriat's theorem · Revealed preference · Arbitrage · Separating hyperplane theorem

JEL Classification D11 · G12

1 Introduction

There are a number of proofs of Afriat (1967)'s famous theorem on revealed preference. This paper proposes another one. It is based on the idea that a consumer should not be vulnerable to arbitrage or money pumps if her behavior is internally consistent. The only non-trivial mathematical tool it uses is the separating hyperplane theorem. The current proof has the advantage that it is based on familiar economic ideas and so is perhaps more accessible than some others. The paper also links the proof to arbitrage arguments in other areas of economics.

Echenique et al. (2011) propose a measure of the seriousness of violations of the Generalized Axiom of Revealed Preference (GARP) by using a money pump argument. This idea can be used to motivate the proof in the current paper. Suppose that the consumer chooses x_i at prices p_i and x_j at prices p_j but x_j is affordable at prices p_i and x_i is also affordable at prices p_j . The Weak Axiom of Revealed Preference, and a fortiori GARP, is violated as x_i is revealed preferred to x_j but x_j is also revealed preferred to x_i . Echenique et al. (2011) argue that the consumer is vulnerable to a money pump. An arbitrageur could buy x_i and sell x_j at prices p_j and sell x_i and buy x_j at prices p_i . Each transaction is profitable and the arbitrageur would make

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an overall profit of mp , or equivalently the consumer makes an overall loss of $-mp$ where:

$$-mp = p_i \cdot (x_j - x_i) + p_j \cdot (x_i - x_j) < 0 \quad (1)$$

Echenique et al. (2011) use mp as measure of the seriousness of violations of GARP. Instead of invoking an arbitrageur one could simply regard this as a measure of how much money the consumer could save by consuming x_j at prices at p_i instead of x_i and x_i instead of x_j at prices p_j , which given her cyclic preferences she would be willing to.

Now a consumer does not care about a money pump directly but rather utility and it is possible that (1) holds even if GARP is not violated. GARP only rules it out if preferences are quasi-linear (see Brown and Calsamiglia 2007). One can however show that if GARP holds there exist numbers λ_i and λ_j such that for any pair of observations

$$\lambda_i p_i \cdot (x_j - x_i) + \lambda_j p_j \cdot (x_i - x_j) \geq 0 \quad (2)$$

λ_i and λ_j can be thought of as the value of money, or the marginal utilities of money, in states i and j so that the consumer is not vulnerable to a money pump in utility terms.

One can give an interpretation in terms of asset pricing. In the example above, when GARP is violated the arbitrageur gains at each step of the trade (buying x_j and selling x_i to the consumer at prices p_i and then buying x_i and selling x_j at prices p_j). This is analogous to an asset which is profitable whatever the state (here the states are the observations i and j) and as such is a ‘free lunch’. According to the fundamental theorem of asset pricing [see for example Dybvig and Ross 2008 or Ross (2005, Chapter 1)], if arbitrage is not possible, that is there are no free lunches, then there exist a set of state prices or risk neutral probabilities such that any portfolio with zero cost has non-positive expected return. The λ_i are analogous to these state prices or risk neutral probabilities.

These arguments on arbitrage can also be connected to subjective probability. Nau and McCardle (1991) show that if there does not exist a bet which is always favorable to the buyer, then there must exist subjective probabilities such that every ‘acceptable’ gamble has non-negative expected value. There is also some analogy with the work Cournot (1927) Chapter III on exchange rates—see Ellerman (1984) for discussion of Cournot’s work.

There are a number of proofs of Afriat’s theorem. Afriat (1967)’s original proof was combinatorial in nature and quite complicated. It contained a gap which was filled by Diewert (1973) and Varian (1982). Teo and Vohra (2003) give a simpler combinatorial proof and bring out clearly the connection with finding negative length cycles in graphs, a theme initially explored by Varian (1982).

Shiozawa (2016) further extends the link made by Teo and Vohra (2003) to shortest path problems and the condition that there should exist weights so that there are no negative length cycles. His proofs are combinatorial in nature. The existence of weights so that there are no negative length cycles can be thought of as implying the existence of marginal utilities so that arbitrage cycles cannot exist, so (2) holds. Shiozawa (2016) however concentrates on connections to shortest path problems rather than links to conditions for absence of arbitrage.

The proofs closest to this one are that of Fostel et al. (2004) and that of Geanakoplos (2013). Fostel et al. (2004) give an elegant proof using the duality theorem of linear programming. Geanakoplos (2013) appeals to the minimax theorem in zero-sum games.

This paper instead uses the separating hyperplane theorem. The duality theorem of linear programming, the minimax theorem and the separating hyperplane theorem are of course closely connected—see for example Trustringham (1971) for a concise account. The separating theorem is, however, familiar to economists and so the proof is perhaps more readily accessible than Fostel et al. (2004)'s and Geanakoplos (2013)'s.

The second fundamental theorem of welfare economics can of course be proved using the separating hyperplane theorem. Krashinsky and Yannelis (1994) show that a separation argument can be used to prove the Gale–Nikaido–Debreu lemma and thus the existence of competitive general equilibrium (see also Yannelis 1985). This paper shows that a separation argument can be used to prove another classical result, Afriat's theorem.

The paper proceeds as follows. Section 2 outlines the main result and gives an application to quasi-linear utility. Section 3 shows that the same method used in proving Afriat's theorem also yields the extension of Afriat's theorem to general budget sets given by Forges and Minelli (2009). It also gives an application to homothetic utility functions.

2 Results

Consider a sequence of observations (p_i, x_i) on a consumer, where $i = 1, \dots, n$ and $p_i \in \mathcal{R}_{++}^K$ and $x_i \in \mathcal{R}_+^K$ are vectors of prices and of purchased quantities respectively. Afriat (1967) asks when this sequence is consistent with rational behavior, that is whether there is a locally non-satiated utility function u such that for each i , x_i maximizes u over the budget set $B_i = \{x \in \mathcal{R}_+^K : p_i \cdot x \leq p_i \cdot x_i\}$.

For two distinct observations i and j , $j \neq i$, if $p_i \cdot x_i \geq p_i \cdot x_j$ then x_i is said to be revealed preferred to x_j : x_j was affordable at prices p_i and budget $p_i \cdot x_i$ but was not chosen, so x_i must be weakly preferred. The data satisfies the Generalized Axiom of Revealed Preference if the induced relation is acyclic.

Definition 1 A cycle is an ordered set of distinct observations $\{i, j, k, \dots, r\} \subseteq N = \{1, \dots, n\}$.

For a cycle c if i is an element of the cycle denote the next element in the cycle by $c(i)$. Let

$$a_{ij} = p_i \cdot (x_j - x_i) \quad i, j \in N \quad (3)$$

Definition 2 The observations are said to satisfy the Generalized Axiom of Revealed Preference (GARP) if for any cycle $\{i, j, k, \dots, r\}$ if all the inequalities hold

$$\begin{aligned}
 a_{ij} &\leq 0 \\
 a_{jk} &\leq 0 \\
 &\vdots \\
 a_{ri} &\leq 0
 \end{aligned}$$

then they all hold with equality.

Theorem 1 (Afriat 1967) *The following are equivalent:*

1. *The observations (p_i, x_i) , $i = 1, \dots, n$, can be rationalized by a continuous, increasing, concave utility function.*
2. *The observations satisfy the Generalized Axiom of Revealed Preference*
3. *There exist numbers u_i and λ_i , $i = 1, \dots, n$, such that for all i and j*

$$u_j \leq u_i + \lambda_i p_i \cdot (x_j - x_i) \quad (4)$$

The difficult part of Afriat's theorem is the implication $2) \Rightarrow 3)$. Here this is proven by an arbitrage argument.

For any cycle $c = \{i, j, k, \dots, r\}$ its elements a_{ij} etc. can be thought of as (minus) the returns to an arbitrageur who engages in cyclic trade of selling x_i and buying x_j at prices p_i etc. Each cycle can therefore be thought of as a potential trade or asset. The idea outlined in the introduction is that if revealed preference is satisfied, there must exist prices for money, or marginal utilities, so that the overall gain for a consumer from any cyclic trade is non-negative (and for the arbitrageur non-positive) in utility terms. Given this, 3) follows easily.

Recall the following version of the separating hyperplane theorem [see for example Ok (2007, Chapter G.3, Proposition 5)]:

Lemma 1 *Let C and D be disjoint non-empty convex sets in $\mathcal{R}^L$ with C closed and D compact. There exists $q \in \mathcal{R}^L$ such that*

$$\inf_{c \in C} q \cdot c > \max_{d \in D} q \cdot d \quad (5)$$

The assumption that D is compact guarantees that the inequality in (5) can be taken to be strict.

The following well-known result is a simple consequence of Lemma 1

Lemma 2 *Let A be a convex, closed subset of $\mathcal{R}^L$ with $A \cap \mathcal{R}_-^L = \{0\}$. If A is polyhedral or a cone, there exists $q \gg 0$ such that $q \cdot a \geq 0$ for all $a \in A$.*

Proof The result follows from Theorem B.3.5 in Appendix B of Karlin (1959). A simple proof can be given following Appendix B of Duffie (2010). Note that if A is polyhedral and generated by $a^1, \dots, a^r$ then the cone generated by $a^1, \dots, a^r$ satisfies the assumptions of the lemma, so it is enough to prove the result when A is cone. Let M be the convex hull of the vectors f_l , $l = 1, \dots, L$, where f_l has l th entry -1 and all other entries 0.

$M \subset \mathcal{R}_-^L$. A and M are disjoint and M is compact so by Lemma 1 there exists q strictly separating A and M : $q \cdot a \geq \alpha > \beta \geq q \cdot m$ for all $a \in A$ and $m \in M$, for some α and β . Since A is a cone with $0 \in A$, α can be taken to be 0. Considering the vectors f_i in M shows that $q >> 0$. $\square$

Recall that a set is polyhedral if it is the convex hull of a finite number of vectors. The restriction that A be a cone or polyhedral is used to guarantee that q can be taken to have all components strictly positive.

Lemma 2 underlies the result that the absence of arbitrage opportunities implies the existence of state prices and the existence of an equivalent risk neutral measure in finance: see for example the proof of Theorem 2.7 in Harrison and Pliska (1981) or of Theorem 1.A of Duffie (2010). A general discussion of the so-called fundamental theorem of asset pricing can be found in Dybvig and Ross (2008).

In the context of asset pricing the set A is the set of all possible portfolios. If there are S states, which all occur with positive probability, then a portfolio can be represented as $a = (-b_1, \dots, -b_S, p_a)$, where the final component is the price of the portfolio, so that $L = S + 1$, and the other components represent minus the payoff in each state. The condition that $A \cap \mathcal{R}_-^L = \{0\}$ is the no arbitrage condition: any portfolio with guaranteed positive returns, that is non-negative returns in all states and strictly positive in some, has a strictly positive price (and any portfolio with zero returns in all states has a non-negative price).

Provided short sales are allowed, A is in fact a linear space. One therefore deduces that $q \cdot a = 0$ for all $a \in A$. Normalizing $q_{S+1} = 1$, it follows that for a portfolio a , $p_a = \sum_s q_s b_s$, that is the price of any portfolio equals its expected payoff in terms of risk-neutral probabilities or state prices. If short sales are not possible, so that portfolios can only be traded in non-negative amounts, A is simply a cone not a linear space and from $q \cdot a \geq 0$ one deduces $p_a \geq \sum_s q_s b_s$, that is the price of any portfolio is at least equal to its expected payoff in terms of risk-neutral probabilities or state prices. In particular if a portfolio has zero price it must have non-positive expected payoff. See for example Leroy and Werner (2014, Chapter 7) for further discussion of arbitrage with sales restrictions.

A similar result is used in Theorem 1 of Nau and McCardle (1991) showing the existence of subjective probabilities if arbitrage is not possible. There A is the set of all acceptable gambles and is polyhedral.

Lemma 2 implies that if there is no arbitrage opportunity, that is a gamble with strictly negative payoffs, then there exist subjective probabilities q such that every acceptable gamble has non-negative expected payoffs.

In the present article A is the convex hull of the set of the returns in each state generated by all possible cyclic trades and is polyhedral. Each observation is thought of as a state and a cycle c is thought of as generating return of (minus) $a_{ic(i)} = p_i \cdot (x_{c(i)} - x_i)$ in state i to a hypothetical arbitrageur conducting cyclic trade, where $c(i)$ is the next element of the cycle, if i belongs to the cycle (and return 0 if it does not). Cyclic trades have zero price in this interpretation and cannot be traded in negative amounts so, as in the case of asset-pricing with short sales restrictions, there must exist marginal utilities or state prices so that every cycle has non-positive expected return to the arbitrageur.

The convex hull may be thought of as mixtures of trades if it is possible for the hypothetical arbitrageur to make fractional sales and purchases of bundles. If randomization over trades is possible then it may represent expected returns from randomizing over trades. It may also simply be treated as an artificial construction.

There is of course no actual securities market on which the consumer can trade or indeed an actual arbitrageur who carries out cyclic trades. In particular the consumption bundles observed may not be those which would emerge if the consumer could transfer income between observations.

The interpretation of observations as states and marginal utilities as state prices in a hypothetical market simply provides a useful if imperfect analogy which is helpful in understanding Afriat's theorem.

Lemma 3 *If the observations satisfy GARP then there exist $\lambda_i > 0$, $i = 1, \dots, n$, such that for any cycle c , $\sum_{i \in c} \lambda_i a_{ic(i)} \geq 0$.*

Proof For any cycle c let a^c be the element of $\mathcal{R}^n$ with i -th element $a_{ic(i)}$ if i belongs to the cycle, 0 if i does not belong to the cycle. Denote the set of all cycles by $\mathcal{C}$. Let A be the convex hull of the a^c , that is $a \in A$ if there exist $\mu^c \geq 0$, $\sum_{c \in \mathcal{C}} \mu^c = 1$, such that $a = \sum_{c \in \mathcal{C}} \mu^c a^c$. The null cycle, involving no swaps, has all entries zero, so $0 \in A$. The result follows from Lemma 2 provided it is shown that $A \cap \mathcal{R}_-^n = \{0\}$.

So suppose $a \in A$ and $a \in \mathcal{R}_-^n$ but $a \neq 0$. Now $a = \sum_{c \in \mathcal{C}} \mu^c a^c$, for some μ^c , so if for some i , $a_i < 0$ there exists a^c with $\mu^c > 0$ $a_i^c < 0$. Follow the cycle c from i . If there is no element j of the cycle with $a_j^c > 0$ then GARP is violated, so let j be the first element in the cycle after i with $a_j^c > 0$. Now $a_j \leq 0$, so there must exist another cycle c' with $a_j^{c'} < 0$. Now switch to following the cycle c' from j onwards. If for some subsequent k , $a_k^{c'} > 0$ then switch to following a new cycle c'' with $a_k^{c''} < 0$ and so on. Since the number of observations is finite, at some point indices must repeat and so if l is the first index at which there is a repetition, if we consider the subset of the path we have constructed which joins l to l we have constructed a cycle c^* from l to l with $a_i^{c^*} \leq 0$ for all i and strict inequality for some i . This contradicts GARP. $\square$

The remainder of the proof is standard. Lemma 3 constructs the marginal utilities of money, λ_i . Utilities consistent with these can be constructed by ensuring that if a consumer starts with x_i and then is given enough money to buy x_j at these prices and current marginal utilities then her utility must be at least the utility of consuming x_j , u_j . The marginal utility of money may not be constant but this will still be an over-estimate of u_j if utility is concave.

The proof is based on this idea but also considers indirect ways of giving the consumer enough to buy x_j , for example first giving them enough money to buy x_k at price p_i then enough money to buy x_j at prices p_k , to ensure the estimates are mutually consistent.

Lemma 4 *If the conclusion of Lemma 3 holds then there exist u_i , $i = 1, \dots, n$, such that $u_j \leq u_i + \lambda_i a_{ij}$ for all i, j , $i \neq j$.*

Proof The λ_i are those constructed in Lemma 3. Set $u_1 = 0$. A path π from 1 to j is a sequence of observations π_k , $k = 1, \dots, \pi^*$, with first element 1 and final element j . For $j \neq 1$ let

$$u_j = \min_{\pi \text{ is path from 1 to } j} u_1 + \sum_{k=1}^{\pi^*-1} \lambda_{\pi_k} a_{\pi_k \pi_{k+1}}$$

That is u_j is the least-cost path from 1 to j , where the cost of moving from i to l directly is $\lambda_i a_{il}$. By Lemma 3, any loop has non-negative cost, so this is well-defined. These u_j have the desired property as any path to from 1 to i can be extended to a path to j by moving directly from i to j . $\square$

Lemma 3 and Lemma 4 in combination prove the implication 2) $\Rightarrow$ 3) in Afriat's theorem. The proof of the implication 3) $\Rightarrow$ 1) is standard:

Lemma 5 *There exists a continuous, concave, increasing utility function $u(x)$ which rationalizes the data.*

Proof For all x , let

$$u(x) = \min_i u_i + \lambda_i p_i \cdot (x - x_i) \quad (6)$$

As the infimum of a set of increasing, linear functions u is continuous, concave and increasing.

Lemma 4 implies that for any j , $u(x_j) = u_j$. On the other hand if $p_j \cdot x \leq p_j \cdot x_j$ then $u(x) \leq u_j + \lambda_j p_j \cdot (x - x_j) \leq u(x_j)$, so u rationalizes the data. $\square$

Brown and Calsamiglia (2007) provide conditions under which observations can be rationalized by a quasi-linear utility function. That is there is a utility function $u(x)$ such that for each i , x_i maximizes $u(x) - p_i x$.

Definition 3 The observations are said to satisfy the Quasi-Linear Generalized Axiom of Revealed Preference (QGARP) if for any cycle c , $\sum_{i \in c} a_{ic(i)}^c \geq 0$.

In other words, a money pump in the sense of Echenique et al. (2011) is not possible. This implies immediately that Lemma 3 holds with $\lambda_i = 1$, that is with constant marginal utilities.

Corollary 1 *The following are equivalent:*

1. The observations (p_i, x_i) , $i = 1, \dots, n$, can be rationalized by a continuous, increasing, concave, quasi-linear utility function.
2. The observations satisfy the Quasilinear Generalized Axiom of Revealed Preference
3. There exist numbers u_i , $i = 1, \dots, n$, such that for all i and j

$$u_j \leq u_i + p_i \cdot (x_j - x_i) \quad (7)$$

Proof The proof is almost identical to that of Lemma 1 but simpler as Definition 3 immediately implies that Lemma 3 holds with $\lambda_i = 1$ and hence so does Lemma 4. The only part that requires proof is the quasi-linearity of utility but note that since the utility function constructed in Lemma 5 is $u(x) = \min_i u_i + p_i(x - x_i)$, for any i $u(x) - p_i x \leq u(x_i) - p_i x_i$ as was to be shown. $\square$

This proof presented in this section is standard except that it uses Lemma 2 to prove Lemma 3. Its merit is that it uses a standard tool widely familiar to economists, the separating hyperplane theorem, and brings out connections with arbitrage arguments.

3 Extensions

This section shows that the same method as in the previous section can be used to prove Forges and Minelli (2009)'s generalization of Afriat's theorem. It applies this result to prove Varian (1983)'s result on rationalization by homothetic utility functions.

Forges and Minelli (2009) prove a generalization of Afriat's theorem. They consider a set of observations $(x_k, B_k)_{k=1, \dots, n}$, with the consumer able to choose from the budget set B_k , where $B_k = \{x \in R_+^K | g_k(x) \leq 0\}$ with g_k a continuous, increasing function such that $g_k(x_k) = 0$. They say that the utility function v rationalizes the observations if for all k $v(x_k) \geq v(x)$ for all $x \in B_k$. Afriat's original theorem corresponds to the case $g_k(x) = p_k \cdot (x - x_k)$.

For any i and j let $a_{ij} = g_i(x_j)$. The following is the natural generalization of GARP to this context:

Definition 4 The observations are said to satisfy the Generalized Axiom of Revealed Preference (GARP) if for any cycle $\{i, j, k, \dots, r\}$ if all the inequalities hold

$$\begin{aligned} a_{ij} &\leq 0 \\ a_{jk} &\leq 0 \\ &\vdots \\ a_{ri} &\leq 0 \end{aligned}$$

then they all hold with equality.

Forges and Minelli (2009) prove

Theorem 2 (Forges and Minelli 2009) *The following are equivalent:*

1. The observations (x_i, B_i) , $i = 1, \dots, n$, can be rationalized by a continuous, locally non-satiated utility function.
2. The observations satisfy the Generalized Axiom of Revealed Preference
3. There exist numbers u_i and λ_i , $i = 1, \dots, n$, such that for all i and j

$$u_j \leq u_i + \lambda_i g_i(x_j) \tag{8}$$

Proof The implication $2) \Rightarrow 3)$ follows directly from the proof of Afriat's theorem in the previous section: nothing in the proof there uses the form of a_{ij} and so the proofs of Lemma 3 and 4 carry over immediately to this context. The implication $3) \Rightarrow 1)$ follows as there by setting $g_j(x) = \min_i u_i + \lambda_i g_i(x_j)$. $\square$

Varian (1983) proves a result on rationalization by homothetic utility functions. We revert to the case of linear budget sets but normalize so that $p_k x_k = 1$ for all k . The data $(p_k, x_k)_{k=1, \dots, n}$ is said to satisfy the Homothetic Axiom of Revealed Preference (HARP) if

Definition 5 The observations are said to satisfy the Homothetic Axiom of Revealed Preference (HARP) if for any cycle $\{i, j, k, \dots, r\}$

$$(p_i \cdot x_j)(p_j \cdot x_k) \dots (p_r \cdot x_i) \geq 1$$

Corollary 2 (Varian 1983) *The following are equivalent:*

1. *The observations (x_i, B_i) , $i = 1, \dots, n$, can be rationalized by a continuous, increasing, homothetic utility function.*
2. *The observations satisfy the Homothetic Axiom of Revealed Preference*
3. *There exist numbers $u_i > 0$ such that for all i and j*

$$u_j \leq u_i \times (p_i x_j) \quad (9)$$

Proof This corollary follows from Lemma 2 in a similar way to how Brown and Cal-samiglia (2007) follows from Theorem 1. Set $g_i(x) = \ln p_i \cdot x$. The budget constraint $p_i x \leq 1$ is equivalent to $g_i(x) \leq 0$ and HARP is equivalent to the requirement that for any cycle c , $\sum_{i \in c} a_{ic(i)}^c \geq 0$, which is Definition 3 and one can argue as for Corollary 1 to obtain v_i satisfying $v_j \leq v_i + g_i(x_j)$ for all i and j and corresponding utility function $v(x)$ such that for all i , x_i maximizes $v(x)$ over $g_i(x) = \ln p_i x \leq 0$, that is over $p_i x \leq 1$. Taking $u_i = e^{v_i}$ yields 3). Note that the constructed utility function is $v(x) = \min_i v_i + g_i(x) = \min_i v_i + \ln(p_i \cdot x)$. This is ordinally equivalent to the utility function $u(x) = \min_i u_i \times p_i \cdot x$, which is homogeneous of degree 1 in x and so preferences can indeed be represented by a homothetic utility function. $\square$

4 Conclusion

This paper has presented a simple proof of Afriat's theorem using the separating hyperplane theorem. It has brought out connections with arbitrage.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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References

- Afriat, S.: The construction of utility functions from expenditure data. *Int. Econ. Rev.* **8**, 67–77 (1967)
- Brown, D., Calsamiglia, C.: The nonparametric approach to welfare analysis. *Econ. Theory* **31**, 183–188 (2007)
- Cournot, A.: *Researches into the Mathematical Principles of the Theory of Wealth* by Augustin Cournot 1838. The MacMillan Company, New York, translated by Nathaniel T. Bacon (1927)
- Diewert, E.: Afriat and revealed preference. *Rev. Econ. Stud.* **40**, 419–425 (1973)
- Duffie, D.: *Dynamic Asset Pricing Theory*, 3rd edn. Princeton University Press, Princeton and Oxford (2010)
- Dybvig, P., Ross, S.: Arbitrage. In: Durlauf, S., Blume, L. (eds.) *New Palgrave Dictionary of Economics*, 2nd edn. Palgrave Macmillan, London (2008)
- Echenique, F., Lee, S., Shum, M.: The money pump as a measure of revealed preference violations. *J. Polit. Econ.* **119**, 1201–1223 (2011)
- Ellerman, D.: Arbitrage theory: a mathematical introduction. *SIAM Rev.* **26**, 241–261 (1984)
- Forges, F., Minelli, E.: Afriat's theorem for general budget sets. *J. Econ. Theory* **144**, 135–145 (2009)
- Fostel, A., Scarf, H., Todd, M.: Two new proofs of Afriat's theorem. *Econ. Theory* **24**, 211–219 (2004)
- Geanakoplos, J.: Afriat from maxmin. *Econ Theory Bull* **54**, 443–448 (2013)
- Harrison, J., Pliska, S.: Martingales and stochastic integrals in the theory of continuous trading. *Stoch. Process. Appl.* **11**, 215–260 (1981)
- Karlin, S.: *Mathematical Methods and Theory in Games, Programming and Economics*, vol. one. Addison Wesley, Reading (1959)
- Krasa, S., Yannelis, N.: An elementary proof of the Knaster–Kuratowski–Mazurkiewicz–Shapley theorem. *Econ. Theory* **4**, 467–471 (1994)
- Leroy, S., Werner, J.: *Principles of Financial Economics*, 2nd edn. Cambridge University, Cambridge (2014)
- Nau, R., McCardle, K.: Rationality, arbitrage and equilibrium. *Theory Decis.* **31**, 199–240 (1991)
- Ok, E.: *Real Analysis with Economic Applications*. Princeton University Press, Princeton (2007)
- Ross, S.: *Neoclassical Finance, Princeton Lectures in Finance*, vol. 4. Princeton University Press, Princeton (2005)
- Shiozawa, K.: Revealed preference test and shortest path problem; graph-theoretic structure of the rationalizability test. *J. Math. Econ.* **67**, 38–48 (2016)
- Teo, C., Vohra, R.: Afriat's theorem and negative cycles. Discussion Paper Northwestern University (2003)
- Trustring, K.: *Linear Programming*. Routledge and Kegan Paul, London (1971)
- Varian, H.: The nonparametric approach to demand analysis. *Econometrica* **50**, 946–973 (1982)
- Varian, H.: Non-parametric tests of consumer behaviour. *Rev. Econ. Stud.* **50**, 99–110 (1983)
- Yannelis, N.: On a market equilibrium theorem with an infinite number of commodities. *J. Math. Anal. Appl.* **108**, 595–599 (1985)

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