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RESEARCH ARTICLE

A "three-sentence proof" of Hansson's theorem

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Abstract We provide a new proof of Hansson's theorem: every preorder has a complete preorder extending it. The proof boils down to showing that the lexicographic order extends the Pareto order.

Keywords Ordering extension theorem · Lexicographic order · Pareto order · Preferences

JEL Classification C65 · D01

1 Introduction

Two extensively studied binary relations in economics are the Pareto order and the lexicographic order. It is a well-known fact that the latter relation is an ordering extension of the former. For instance, in Petri and Voorneveld (2016), an essential ingredient is Lemma 3.1, which roughly speaking requires the order under consideration to be an extension of the Pareto order. The main message of this short note is that some fundamental order extension theorems can be reduced to this basic fact. An advantage of the approach is that it seems less abstract than conventional proofs and hence may offer a pedagogical advantage in terms of exposition. Mandler (2015) gives an elegant proof of Spzilrajn's theorem (1930) that stresses the importance of the lexicographic

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approach in proving ordering extension theorems. He shows that criteria can be built out of the relation $\succ$ on the domain X in such a way that if the criteria are ordered lexicographically the corresponding relation extends $\succ$. At a technical level the proof presented here is quite similar. We use a result from Evren and Ok (2011), that uses criteria defined in a similar way as in Mandler (2015). However, in terms of exposition we argue that our proof is easier. Another difference compared to Mandler (2015) is that he gives a proof of Szpilrajn's theorem (1930), whereas this note presents a proof of Hansson's theorem (1968). Of course given a proof of one of them the other follows quite easily, but we believe that our proof of Hansson's theorem is easier to digest. Colloquially the proof in this note reads:

> "Every preorder is essentially a Pareto order and the lexicographic order extends the Pareto order. Since the lexicographic order is complete every preorder has an ordering extension".

We thus argue that abstract extension theorems of orders can be embodied in this simple principle.

The note is organized as follows. Section 2 contains notation and preliminaries. In Sect. 3, we present our proof of Hansson's theorem. Section 4 concludes. We gather some standard results in set theory in Appendix A.

2 Notation and preliminaries

As usual $\mathbb{R}$ denotes the set of real numbers . A binary relation/order $\succsim$ on a set X, is a subset $\succsim \subset X \times X$. If $(x, y) \in \succsim$ we write this as $x \succsim y$. An order $\succsim$ on X is:

Reflexive If $x \succsim x$ for all $x \in X$.
Transitive If $x \succsim y$, $y \succsim z$ implies $x \succsim z$ for all $x, y, x \in X$.
Complete If $x \succsim y$ or $y \succsim x$ for all $x, y \in X$.
Antisymmetric If $x \succsim y$ and $y \succsim x$ implies $x = y$ for all $x, y \in X$.

A preorder $\succsim$ is a reflexive, transitive order. If a preorder $\succsim$ is complete, then we call $\succsim$ a complete preorder. An order $\succsim$ is a *linear* order if $\succsim$ is antisymmetric, transitive and complete. Given a linear order $\succsim$ on X and a subset A of X, $a \in A$ is a smallest element if $x \succsim a$ for all $x \in A$. A linear order on a set X is a well order if every nonempty subset A of X contains a smallest element. The *well ordering theorem* states that every set X admits a binary relation $\succsim$ such that X with $\succsim$ is well ordered.

If $\succsim$ is an order we define a relation $\succ$ on X by $x \succ y$ if and only if $x \succsim y$ and not $y \succsim x$. We also define an order $\sim$ on X by $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$. A complete preorder $\succsim'$ on X *extends* (or is an ordering extension of) a preorder $\succsim$ on X if for all $x, y \in X$: $x \succsim y$ implies that $x \succsim' y$ and $x \succ y$ implies that $x \succ' y$.

Given a set I and for each $i \in I$ a complete preorder $\succsim_i$ on X the *Pareto* order $\geq$ on X is defined by for all $x, y \in X$: $x \geq y$ if and only if $x \succsim_i y$ for all $i \in I$. We will sometimes refer to $\succsim_i$ as a *coordinate relation*.

Let I be a set with linear order $\leq$ on I and a collection of complete preorders $\succsim_i$ on X for all $i \in I$. Define a *lexicographic* relation $\geq_L$ on X by $x \geq_L y$ if and only if $x \succsim_i y$ for all $i \in I$, or there is a $j \in I$ such that $x \succ_j y$ and $x \sim_i y$ for all $i \in I$

with $i < j$. It is a standard fact in basic set theory that if $\leq$ is a well order then $\geq_L$ is a complete preorder on X. For example, Mandler (2015) alludes to this result, see also Ciesielski (1997). For completeness a proof is presented in Lemma A.1 in the Appendix A.

3 The proof

We now give our proof of Hansson's theorem. A first crucial ingredient in our proof is a corollary to a result in Evren and Ok (2011). It shows that every preorder essentially is a Pareto order. The result has the same universal character as a representation result by Chipman (1960), which shows that every complete preorder essentially is a lexicographic order:

Lemma 3.1 *For every preorder $\succsim$ there is a Pareto order $\geq$ with coordinate relations $\succsim_i$ for all $i \in I$ such that $x \succsim y$ if and only if $x \geq y$ for all $x, y \in X$.*

Proof Let $I = X$ and for each $x \in X$ let $u_x(y) := \mathbf{1}_{\{z \in X | z \succsim x\}}(y)$ for all $y \in X$ (where $\mathbf{1}_A$ denotes the indicator function of a set A). Define a relation $\succsim_x$ by $z \succsim_x y$ if and only if $u_x(z) \geq u_x(y)$. The result follows by Evren and Ok (2011, Proposition 1). □

Another important observation in our proof is that the lexicographic order extends the Pareto order. This observation is recorded as Lemma 3.2.

Lemma 3.2 *Let $\geq$ be a Pareto order on X with coordinate relations $\succsim_i$. Then the lexicographic order $\geq_L$ on X with coordinate relations $\succsim_i$ for all $i \in I$ extends $\geq$.*

Proof Let $x, y \in X$ with $x \geq y$. If $y >_L x$ then there is an $i \in I$ such that $y \succ_i x$, contradicting that $x \succsim_i y$ for all $i \in I$. Let $x, y \in X$ with $x > y$. Then $x \succ_j y$ for some $j \in I$ and $x \succsim_i y$ for all $i \in I$. If $y \geq_L x$ then either $x \sim_i y$ for all $i \in I$ or $y \succ_i x$ for some $i \in I$, a contradiction. □

We are now ready for our proof of Hansson's theorem:

Theorem 3.3 *Let $\succsim$ be a preorder order on X. Then there exists a complete preorder $\succsim'$ on X extending $\succsim$.*

Proof By Lemma 3.1 there is a Pareto relation $\geq$ with coordinate relations $\succsim_i$ such that $x \geq y$ if and only if $x \succsim y$ for all $x, y \in X$. Well order I by $\leq$ and let $\geq_L$ be the lexicographic relation with coordinate relations $\succsim_i$ for all $i \in I$. Then $\geq_L$ is a complete preorder by Lemma A.1 and Lemma 3.2 implies that $\geq_L$ extends $\succsim$. □

4 Concluding remarks

A short "three sentences" proof of Hansson's theorem is given. Some of the advantages of the approach are:

(a) The theorem follows from intuitively plausible and simple principles. Once it is understood that the lexicographic order extends the Pareto order, the rest of the proof follows smoothly.

(b) The use of the well ordering theorem is transparent. We only had to use the well ordering theorem once in the second sentence of Theorem 3.3. Hence if I is some set that is known to be well ordered (like the set of natural numbers or a finite set), we see that the proof of Hansson's theorem follows without invoking the well ordering Theorem 3.3.

Finally, some readers may object to the claim made in the title of this note, namely that the proof of Theorem 3.3 is only three sentences long. To prove the main result we did indeed have to invoke as many as three lemmas. However, it is our belief that the prerequisites are either well known or otherwise useful to know, and given that, the proof of Hansson's theorem is short.

A Appendix: Lemma A.1

Lemma A.1 *Let I be an arbitrary set and $\leq$ a well order on I. For each $i \in I$ let $\succsim_i$ be a complete preorder on X. Then the lexicographic order $\geq_L$ on X, with order $\leq$ on I and relations $\succsim_i$ on X for all $i \in I$, is a complete preorder on X.*

Proof Assume I is well ordered by $\leq$. For all $x, y \in X$, let $N(x, y) := \{i \in I : x \succ_i y\} \cup \{i \in I : y \succ_i x\}$. We show that $\geq_L$ is complete and transitive.

Totality: Let $x, y \in X$. If $x \sim_i y$ for all $i \in I$ then $x \geq_L y$. Otherwise $x \succ_i y$ or $y \succ_i x$ for some $i \in I$ and hence $N(x, y) \neq \emptyset$. Since I is well ordered there is a smallest $j \in I$ such that $x \succ_j y$ or $y \succ_j x$ and hence $x \sim_i y$ for all $i < j$. If $x \succ_j y$ ($y \succ_j x$) it follows that $x \geq_L y$ ($y \geq_L x$).

Transitivity Let $x, y, z \in X$, $x \geq_L y$ and $y \geq_L z$. W.l.o.g. assume that $x >_L y$ and $y >_L z$ (the other cases are either trivial or follows by using similar arguments as in the present case). Then $N(x, y) \neq \emptyset$ and $N(y, z) \neq \emptyset$. Hence there is a smallest $j \in N(x, y)$ and a smallest $j' \in N(y, z)$. If $j = j'$, then we are done. Assume $j < j'$ then $x \succ_j y \sim_j z$ and $x \sim_i y \sim_i z$ for all $i < j$. Thus $x \geq_L z$. If $j' < j$ then $x \sim_{j'} y \succ_{j'} z$ and $x \sim_i y \sim_i z$ for all $i < j'$ and hence $x \geq_L z$. □

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