Comment on “Collateral premia and risk sharing under limited commitment” [Econ. Theory 46, 475–501 (2011)]

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Abstract In a recent issue of Economic Theory, Kilenthong (Econ Theory 46:475–501, 2011) studies the problem of a social planner who can redistribute future consumption by changing agents’ asset portfolios subject to individual collateral constraints. One of the findings is that aggregate consumption good endowment and collateral asset payoffs together imply a minimum level of aggregate collateral beyond which optimal allocations exhibit full risk sharing. However, this result is incorrect. In the present note it is shown by means of a counterexample that: (1) when such a minimum level exists, it depends on the welfare weights, (2) such a minimum level need not exist.

Keywords Collateralized contracts · Limited commitment

JEL Classification D53

Kilenthong (2011) considers a setting with S states of the world with probabilities π_s , one consumption good per state, H agents with von Neumann-Morgenstern expected utility functions, one collateral asset with payoffs $R \in \mathbb{R}_{++}^S$ in positive net supply, one non-collateral asset with identical payoffs R , and J collateralized contracts in zero net supply. These contracts are assets with payoffs $D_j \in \mathbb{R}_+^S$ with $D_{sj} \in \{0, R_s\} \forall s$ and a collateral constraint for the portfolios of all agents: for each unit of contract issued, one unit of the collateral asset needs to be held. For each agent h , denote by $c^h \in \mathbb{R}_+^{S+1}$ and $e^h \in \mathbb{R}_+^{S+3}$ his consumption and endowments as well as by $k_{\text{coll}}^h \in \mathbb{R}_+$, $k_{\text{ncoll}}^h \in \mathbb{R}_+$, and $\theta^h \in \mathbb{R}^J$ his holdings of the collateral asset, non-collateral asset, and contracts respectively. The functions $u^h : \mathbb{R}_+ \rightarrow \mathbb{R}$ are assumed continuous, differentiable,

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concave, and strictly increasing. Given welfare weights $\lambda^h \geq 0$ with $\sum_h \lambda^h = 1$, the planner's problem is

$$\max_{\substack{c_0^h, k_{\text{coll}}^h \\ \theta_j, k_{\text{ncoll}}^h}} \sum_h \lambda^h \left(u^h(c_0^h) + \beta \sum_s \pi_s u^h(c_s^h) \right) \quad (1)$$

subject to the feasibility constraints

$$c_s^h = e_s^h + (k_{\text{coll}}^h + k_{\text{ncoll}}^h) R_s + \sum_j \theta_j^h D_{sj} \quad \forall h \quad \forall s > 0 \quad (2)$$

$$\sum_h c_0^h = \sum_h e_0^h \quad (3)$$

$$\sum_h k_i^h = \sum_h e_i^h, \quad i = \text{coll}, \text{ncoll} \quad (4)$$

$$\sum_h \theta_j^h = 0 \quad \forall j \quad (5)$$

and the collateral constraints

$$k_{\text{coll}}^h \geq - \sum_j (\theta_j^h)^- \quad \forall h \quad (6)$$

The asset payoff matrix $(R \ D)$ is assumed to have full rank. If not a single collateral constraint binds and one assumes an interior solution, the first-order conditions of the planner's problem imply for any two agents

$$\lambda^1 \partial_c u^1(c_s^1) = \lambda^2 \partial_c u^2(c_s^2) \quad \forall s \quad (7)$$

In this case, the marginal rates of substitution $\text{MRS}_s^h = \beta \pi_s \frac{\partial_c u^h(c_s^h)}{\partial_c u^h(c_0^h)}$ are equalized across agents and full risk sharing is attained. The following claim made in [Kilenthong \(2011\)](#) states a sufficient condition for full risk sharing. Let $\bar{e} = \sum_h e^h$ denote the vector of aggregate endowment.

Claim [[Kilenthong \(2011\)](#), Lemma 4] If $\bar{e}_{\text{coll}} \geq \sum_s \frac{\bar{e}_s}{R_s}$, an optimal allocation exhibits full risk sharing.

The problem with this claim is that the stated condition is not linked to the objective function of the planner. By a suitable choice of utility functions and welfare weights, I construct a counterexample which illustrates two issues. First, the condition in the claim may understate the minimum level of aggregate collateral needed to attain full risk sharing. Second, such a minimum level and therefore an optimal solution with full risk sharing need not exist.

Counterexample Consider a two state, two agent setting, i.e. $S = 2$, $H = 2$, with collateral asset payoffs $R = (1, 2)$ and one type of collateralized contract with payoffs

$D = (1, 0)$. This is a minimal spanning set as defined in [Kilenthong \(2011\)](#). Probabilities of the two states are $\pi = (\frac{1}{5}, \frac{4}{5})$. The two agents have identical preferences $u^h(c) = \log(c)$, $h = 1, 2$ with time discount factor $\beta = 1$ and identical endowments $e_0^h = 60$, $e_1^h = 1$, $e_2^h = 0$, $e_{\text{coll}}^h = 1$, $e_{\text{ncoll}}^h = 0$, $h = 1, 2$ but differ in their welfare weights $\lambda^1 = \frac{7}{8}$, $\lambda^2 = \frac{1}{8}$.

The resulting maximizers of the planner's problem are $c_0^1 = 105$, $k_{\text{coll}}^1 = \frac{7}{4}$, $\theta^1 = \frac{1}{4}$, $k_{\text{ncoll}}^1 = 0$ and $c_0^2 = 15$, $k_{\text{coll}}^2 = \frac{1}{4}$, $\theta^2 = -\frac{1}{4}$, $k_{\text{ncoll}}^2 = 0$. Even though the condition in the claim reads $2 \geq 2$ in this example and is thus satisfied, agent 2 has a binding collateral constraint. Therefore, the optimal allocation of date 1 consumption $c_1^1 = \frac{12}{4}$, $c_2^1 = \frac{14}{4}$ and $c_1^2 = \frac{4}{4}$, $c_2^2 = \frac{2}{4}$ leads to different marginal rates of substitution $\text{MRS}^1 = (7, 24)$ and $\text{MRS}^2 = (3, 24)$. $\square$

The crucial aspect of the counterexample are the imbalanced welfare weights. The two additional assumptions on preferences $u^h(c) = \log(c) \forall h$ and endowments $e_1^h > 0 \forall h$, $\bar{e}_2 = 0$, $\bar{e}_{\text{ncoll}} = 0$ for $D = (R_1, 0)$ are made for the sake of tractability. Under the first, Eq. 7 sets the optimal percentage of aggregate consumption in each state for each agent equal to his welfare weight. Under the second, the required asset holdings for attaining this full risk sharing allocation become $\hat{k}_{\text{coll}}^h = \lambda^h \bar{e}_{\text{coll}}$ and $\hat{\theta}^h = \frac{\lambda^h \bar{e}_1 - e_1^h}{R_1}$. The transfer in state 2 can be achieved through a pro rata allocation of the collateral asset. The transfer of the positive good endowments in state 1 requires the issuance of contracts. However, in the example the welfare weight of agent 2 is small relative to his state 1 endowment. As a consequence, he is allotted a share of aggregate collateral too little to be able to issue enough contracts. In such a case, the full risk sharing asset holdings violate the collateral constraint.

Note that the full risk sharing collateral allocation $\hat{k}_{\text{coll}}^h$ is increasing in $\bar{e}_{\text{coll}}$ while the full risk sharing contract allocation $\hat{\theta}^h$ is independent of $\bar{e}_{\text{coll}}$. Thus, if one increases the aggregate collateral, then at some point the collateral constraint will become slack in the optimum and full risk sharing is attained in the example. However, this need not be the case in general. To see this, simply change the utility function of agent 1 in the example to $u^1(c) = c$ and consider a variation of aggregate collateral. Given $\bar{e}_{\text{coll}} \geq \frac{\lambda^2}{\lambda^1 R_2}$, any increase in $\bar{e}_{\text{coll}}$ is fully allocated to agent 1 while the optimal allocations of agent 2 are fixed at $c_0^2 = \frac{1}{7}$, $k_{\text{coll}}^2 = \frac{1}{14}$ and $\theta^2 = -\frac{1}{14}$. Thus, one can raise aggregate collateral arbitrarily high and agent 2 still has a binding constraints, the marginal rates of substitution $\text{MRS}^1 = (\frac{1}{5}, \frac{4}{5})$ and $\text{MRS}^2 = (\frac{1}{35}, \frac{4}{5})$ are not equal, and risk sharing is incomplete. The claim cannot hold.

This also affects Proposition 4 in [Kilenthong \(2011\)](#) which relies on Lemma 4.

Reference

- Kilenthong, W.T.: Collateral premia and risk sharing under limited commitment. *Econ. Theory* **46**, 475–501 (2011)