Using oriented volume to prove Sperner's lemma

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Abstract An elementary proof of Sperner's lemma, using only oriented volumes of simplices, is provided. Several generalizations are indicated.

Keywords Sperner's lemma · Oriented volume

JEL Classification C6

1 Introduction

Sperner's (1928) well-known lemma is used to prove the Brouwer fixed point theorem (and is actually equivalent to it), and has been used frequently, along with various generalizations, to establish fundamental results in game theory and in economics (see e.g. in Kannai 1992; McLennan and Tourky 2008; Scarf 1982; Shapley and Vohra 1991 and in the references quoted there). Motivated by the wish to provide a proof “sufficiently short or intuitively appealing to be attractive in the context of texts on mathematical economics”, a proof which can be “reasonably be assumed to be part of the prior knowledge of students studying mathematical economics, at least at the advanced level” was supplied in McLennan and Tourky (2008).¹ The proof of Sperner's lemma provided in Kannai (1992) is based (via Kannai 1981) on an argument involving volume (as in McLennan and Tourky 2008), but involves “advanced calculus”, unfortunately not too familiar to the present generation of students studying mathematical economics. In

¹ Of course, others might feel that certain well-known proofs might be simpler.

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the present note we provide an elementary proof of Sperner's lemma, inspired both by McLennan and Tourky (2008) and the method of Kannai (1992) and Kannai (1981). This proof uses the concept of oriented volume² rather than the volume itself, but requires no extra parameter for homotopy. Observe that we actually prove a stronger form of Sperner's lemma (the oriented version), due to Brown and Cairns (1961), see Sect. 4.

Sperner's lemma is stated and proved in Sect. 2, where the necessary concepts are recalled for convenience of the reader. (For completeness we follow McLennan and Tourky (2008) and define affine dependence, simplices, and triangulations.) The main technical tool is the Main Lemma, establishing equality (2) of the oriented volume of the image of a simplex with the sum of the oriented volumes of images of the simplices in a triangulation of the simplex under a Sperner labeling. In Sect. 3, a proof along similar lines is provided for generalizations of Sperner's lemma, such as the Sperner–Shapley's lemma, [this generalization is useful, e.g., in proving Scarf's theorem on non-emptiness of the core of a balanced game, see Shapley (1973) and also Kannai (1992)], and for more general forms (used for example in Section 7 of Scarf 1982). In addition, the Main Lemma is generalized to a certain class of pseudomanifolds with boundaries. In Sect. 4, we comment on relations between arguments and results described in this note and well-known ideas.

As stated earlier, the prerequisites needed for understanding Sect. 2 are minimal. In Sect. 3, we use freely standard topological concepts and terminology, such as may be found, e.g., in Seifert and Threlfall (1980) or Spanier (1966).

I am very much indebted to Prof. H. E. Scarf for several very useful comments.

2 Sperner's lemma

An *affine combination* of points $p_0, \dots, p_i$ in R^d is a sum $a_0 p_0 + \dots + a_i p_i$ where $a_0, \dots, a_i$ are real numbers with $a_0 + \dots + a_i = 1$. We say that $p_0, \dots, p_i$ are *affinely independent* if it is not possible to write one of them as an affine combination of the others (equivalently, $p_1 - p_0, \dots, p_i - p_0$ are linearly independent). A subset M of R^d is an *affine space* if it is closed under affine combinations. (It is easy to see that an affine space is just a translation of a linear space.) The dimension of an affine space M is the maximal number of elements in affinely independent subsets of M .

A simplex σ is the convex hull of affinely independent points $p_0, \dots, p_i$. The points $p_0, \dots, p_i$ are the vertices of σ . (If the points are not affinely independent we say that the simplex is *degenerate*.) The dimension of a simplex is the dimension of the smallest affine space that contains it, equivalently the number of its vertices minus one. A *face* of simplex is a (possibly empty) subsimplex. Note that a *proper face* of a simplex σ (i.e., a face of σ other than σ itself) is the intersection of σ with a hyperplane that bounds a halfspace of R^d that contains σ . A (finite) *simplicial complex* is a finite collection T of simplices in R^d such that any face (including $\emptyset$) of an element of T is

² It was pointed out by H. E. Scarf that signed volumes were utilized by P. White in an unpublished Yale dissertation to discuss index theory for generalizations of Sperner's lemma.

an element of T and the intersection of any two elements of T is a (possibly empty) common face. The *underlying space* $|T|$ of T is the union of all simplices in T .

A *triangulation* of a simplex σ is a simplicial complex T such that $|T| = \sigma$. Fix such a T , and let V_T be the set of all vertices in T . For each $x \in \sigma$, let F_x be the minimal face of σ containing x (i.e., the intersection of all faces of σ containing x).

A *Sperner labeling* for T is a function l associating one of the vertices of F_v to each $v \in V_T$. Let T be a triangulation of a d -dimensional simplex σ , and let l be a labeling. A d -dimensional simplex in T is *full* (or *completely labeled*) if the labels of its vertices are all distinct. Note that if l is a Sperner labeling for T and p is a vertex of σ , then $l(p) = p$, hence $l(\sigma) = \sigma$. More generally, the labeling l induces a Sperner labeling for $T \cap \tau$ where τ is any sub-simplex of σ , so that $l(\tau) = \tau$.

A simplex becomes *oriented* when a particular ordering of its vertices is specified. Two orderings which differ by an even permutation determine the same orientation. An ordered simplex is denoted by $(p_0 \cdots p_i)$. (Slightly abusing notations, we do not distinguish between a simplex and the ordered set of its vertices.) Two oriented non-degenerate i -dimensional simplices σ and τ in R^i (or, more generally, in the same i -dimensional affine space) are said to have the same orientation if and only if the linear map of R^i (or of the i -dimensional affine space containing them) mapping σ onto τ preserving the orientation has a positive determinant. If $\sigma = (p_0 \cdots p_i)$ is an ordered simplex and f is any R^d -valued function defined on the set of vertices of σ , then $f(\sigma)$ is the ordered (possibly degenerate) simplex $(f(p_0) \cdots f(p_i))$.

The *oriented (or signed) volume* of a (possibly degenerate) ordered d -simplex $(p_0 \cdots p_d)$ in R^d is the determinant

$$\text{vol}(p_0 \cdots p_d) \doteq \frac{1}{d!} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ p_0 & p_1 & \cdots & p_d \end{bmatrix} = \frac{\det(p_1 - p_0, \dots, p_d - p_0)}{d!}. \quad (1)$$

(Note that the determinant on the left is $d + 1$ dimensional.)

We may now formulate our main result.

Lemma 1 (Main Lemma) *Let T be a triangulation of a d -dimensional simplex Δ in R^d , and let all d -dimensional simplices of T have the same orientation as Δ . Then, for any Sperner labeling f for T , the equality*

$$\text{vol}(\Delta) = \sum_{\tau \in T} \text{vol}(f(\tau)) \quad (2)$$

holds.

Sperner's lemma follows easily from the Main Lemma.

Lemma 2 (Sperner's lemma) *Let T be a triangulation of a d -dimensional simplex Δ . Then, any Sperner labeling for T has an odd number of full simplices.*

Proof that the Main Lemma implies Sperner's lemma Given a triangulation T of Δ , we may assume that all d -dimensional simplices of T have the same orientation as Δ .

Applying (2) to a Sperner labeling i of T , we see that

$$\text{vol}(\Delta) = \sum_{\tau \in T} \text{vol}(i(\tau)). \quad (3)$$

Let $n_+(n_-)$ denote the number of simplices τ such that $\text{vol}(i(\tau))$ has the same (opposite) sign as $\text{vol}(i(\Delta))$. Then, (3) implies that $n_+ - n_- = 1$, while the number of full simplices is equal to $n_+ + n_-$ (a simplex τ in T is full if and only if $\text{vol}(i(\tau)) \neq 0$). Hence, the number of full simplices is odd. $\square$

Proof of the Main Lemma We apply induction on the dimension. If $d = 1$ then there exist real numbers a, b such that $\Delta = (ab)$. We consider the case $a < b$, the case $b < a$ follows similarly. A triangulation T is given by a finite sequence of real numbers $p_1, \dots, p_m$ such that $a < p_1 < \dots < p_m < b$, and the oriented 1-dimensional simplices of T are $(ap_1), (p_1p_2), \dots, (p_mb)$. Then,

$$\text{vol}((ab)) = b - a = f(p_1) - f(a) + \sum_{i=1}^{m-1} (f(p_{i+1}) - f(p_i)) + f(b) - f(p_m), \quad (4)$$

which is the same as (2). For $d > 1$, note that for every oriented simplex $\sigma = (q_0 \dots q_d)$, degenerate or not,

$$\text{vol}(\sigma) = \frac{1}{d!} \begin{vmatrix} 1 & 1 & \dots & 1 \\ q_0 & q_1 & \dots & q_d \end{vmatrix} = \frac{1}{d!} \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & q_0 & q_1 & \dots & q_d \end{vmatrix}. \quad (5)$$

Expanding (5) along the second row, we see that

$$\begin{aligned} \text{vol}(\sigma) &= \sum_{i=0}^d (-1)^i \frac{1}{d!} \begin{vmatrix} 1 & 1 & 1 & \dots & \hat{1} & \dots & 1 \\ \mathbf{0} & q_0 & q_1 & \dots & \hat{q}_i & \dots & q_d \end{vmatrix} \\ &= \sum_{i=0}^d (-1)^i \text{vol}(\mathbf{0}q_0q_1 \dots \hat{q}_i \dots q_d), \end{aligned} \quad (6)$$

where $\mathbf{0}$ denotes the origin in R^d .

Assume that the points $p_0, \dots, p_d$ in R^d are such that $\Delta = (p_0p_1 \dots p_d)$. We may assume without loss of generality that the origin $\mathbf{0}$ is not contained in the affine $(d-1)$ -dimensional subspace H_i containing the $(d-1)$ -dimensional simplex $\Delta_i = (p_0p_1 \dots \hat{p}_i \dots p_d)$, for every index $0 \leq i \leq d$. It follows from (6) that

$$\text{vol}(\Delta) = \sum_{i=0}^d (-1)^i \text{vol}(\mathbf{0}\Delta_i) \quad (7)$$

Let σ, τ be oriented d -dimensional simplices in T such that $\sigma \cap \tau$ is $(d-1)$ -dimensional. Let q_σ (q_τ) be the vertex in $\sigma \setminus \sigma \cap \tau$ ($\tau \setminus \sigma \cap \tau$), and let $\sigma =$

$\varepsilon(q_\sigma q_1 q_2 \cdots q_d)$ where $\varepsilon = \pm 1$. Then, $\tau = -\varepsilon(q_\tau q_1 q_2 \cdots q_d)$ (note that q_σ and q_τ are situated on opposite sides of the hyperplane generated by $\sigma \cap \tau$). It follows that the oriented volumes of $\sigma \cap \tau$ [hence also of $f(\sigma \cap \tau)$] appear in the formula for the oriented volume of $(\mathbf{0}q_1 q_2 \cdots q_d)$ [hence those of $f((\mathbf{0}q_1 q_2 \cdots q_d))$] in opposing signs, depending on whether they are regarded [in the right hand side of (6)] as faces of σ or of τ [respectively of $f(\sigma)$ or of $f(\tau)$]. Hence, we may rewrite the right hand side of (2) as

$$\sum_{\tau \in T} \text{vol}(f(\tau)) = \sum_{i=0}^d (-1)^i \sum_{\tau \in T, \tau \cap \Delta_i \neq \emptyset} \text{vol}(\mathbf{0}f(\tau_i)) = \sum_{i=0}^d (-1)^i \sum_{\tau \in T \cap \Delta_i} \text{vol}(\mathbf{0}f(\tau)), \quad (8)$$

where we denote by τ_i the intersection of τ with Δ_i .

Here, the sum in (8) may be extended on $(d-1)$ -dimensional simplices only [note that in the right hand side of (2) only the d -dimensional simplices τ may contribute a non-zero term].

Let σ and τ be any $(d-1)$ -dimensional simplices in $T \cap \Delta_i$ and let $q_\sigma \sigma$ ($q_\tau \tau$) be the (unique!) d -dimensional simplex in T containing σ (τ). By assumption, $\text{vol}(q_\sigma \sigma)$ and $\text{vol}(q_\tau \tau)$ have the same sign. Hence, $\text{vol}(\sigma)$ and $\text{vol}(\tau)$ have the same sign as well. We apply now the induction hypothesis to Δ_i and infer that

$$\text{vol}(\Delta_i) = \sum_{\tau \in T \cap \Delta_i} \text{vol}(f(\tau)). \quad (9)$$

However, in (9), the sum may be extended over $(d-1)$ -dimensional simplices τ only, and the $(d-1)$ -dimensional volume of the simplex $f(\tau)$ (note that f is a Sperner labeling, so that $f(\tau) \subset f(\Delta_i) = \Delta_i$) equals $\frac{d}{h} \text{vol}(\mathbf{0}f(\tau))$, where h is the distance between the origin $\mathbf{0}$ and H_i . Similarly, the $(d-1)$ -dimensional volume of the simplex Δ_i equals $\frac{d}{h} \text{vol}(\mathbf{0}\Delta_i)$. Hence,

$$\text{vol}(\mathbf{0}\Delta_i) = \sum_{\tau \in T \cap \Delta_i} \text{vol}(\mathbf{0}f(\tau)). \quad (10)$$

Applying (7), (8) and (10), we obtain the relation

$$\begin{aligned} \text{vol}(\Delta) &= \sum_{i=0}^d (-1)^i \text{vol}(\mathbf{0}\Delta_i) \\ &= \sum_{i=0}^d (-1)^i \sum_{\tau \in T \cap \Delta_i} \text{vol}(\mathbf{0}f(\tau)) = \sum_{\tau \in T} \text{vol}(f(\tau)) \end{aligned} \quad (11)$$

□

3 Generalizations

1. Let T be a triangulation of a d -dimensional simplex σ . A vertex map f of T is called a *generalized Sperner labeling* if every vertex $v \in V_T$ is mapped into a point in the smallest affine subspace containing F_v . Note that a labeling f (or, more generally, any vertex map) may be identified with a simplicial map $|f|$ by constructing an appropriate simplicial complex L and a subdivision T' of T such that the map $|f|$ obtained by linear extension of f to the simplices of T induces a simplicial map of T' into L . (For a standard Sperner labeling $L = \Delta$ and there is not need to subdivide T .) The statement and proof of the Main Lemma, suitably modified, continue to hold.

Lemma 3 (Generalized Main Lemma) *Let T be a triangulation of a d -dimensional simplex Δ in R^d , and let all d -dimensional simplices of T have the same orientation as Δ . Let f be a generalized Sperner labeling for T , let T' and L be a subdivision of T and a simplicial complex, respectively, such that the map $|f|$ is a simplicial map of T' into L . Then, the equality*

$$\text{vol}(\Delta) = \sum_{\tau \in T'} \text{vol}(f(\tau)) \quad (12)$$

holds.

The following generalized Sperner's lemma is valid in this case as well. (For a standard Sperner labeling the existence of a completely labeled (full) simplex is equivalent to the assertion that the image $|f|(\Delta)$ contains every point of Δ .)

Lemma 4 (Generalized Sperner's lemma) *If f is a generalized Sperner labeling, then the image $|f|(\Delta)$ contains every point of Δ .*

Proof that the Generalized Main Lemma implies the Generalized Sperner's lemma Suppose, on the contrary that the point $y \in \Delta$ is not assumed by $|f|$. Let h denote the usual deformation of the space $R^d \setminus \{y\}$ onto the boundary of Δ , i.e., $h(x)$ is the point on the intersection of the boundary of Δ and the line joining y with x such that $h(x)$ lies between x and y . Hence, the function $g(x) = h(|f|(x))$ is well defined on Δ assuming values in the boundary of Δ . By the simplicial approximation theorem, there exists a subdivision T'' of T' such that the map g is a simplicial map of T'' into the restriction of L to the boundary of Δ . Hence, the right hand side of (12) vanishes, a contradiction. $\square$

The Generalized Main Lemma is proved by induction on the dimension d , as in the proof of the Main Lemma. (Observe that in the 1-dimensional case we used the assumption $a < p_1 < \dots < p_m < b$ only to obtain a triangulation of the interval (a, b) . This is not required in our case.)

2. Non-emptiness of the core of a game with non-transferable utility may be obtained using a generalization of Sperner's lemma due to Shapley (Kannai 1992; Shapley 1973). Let D denote the set $\{0, 1, \dots, d\}$ and let there be given an array of non-negative numbers $\pi = \{\pi_{S,i} : S \subset D, i \in D, \pi_{S,i} = 0 \text{ for } i \notin S, \sum_{i \in S} \pi_{S,i} >$

0 for $S \neq \emptyset$. The collection $\{S_1, \dots, S_k\}$ of subsets of D is called π -balanced if there exist positive numbers $\lambda_1, \dots, \lambda_k$ such that

$$\sum_{j=1}^k \lambda_j \pi_{S_j, i} = 1, \quad \text{for all } i \in D. \quad (13)$$

A *Shapley–Sperner labeling* for a triangulation T of a simplex σ is a function l associating a subset of the set of the vertices of F_v to each $v \in V_T$.

Lemma 5 (Sperner–Shapley's lemma) *Let T be a triangulation of a d -dimensional simplex Δ , and let f be a Shapley–Sperner labeling for T . Then, there exists a (not necessarily fully dimensional) simplex σ in T such that the collection $\{f(q); q \in \sigma\}$ is π -balanced.*

Proof of Sperner–Shapley's lemma Without loss of generality let Δ be the convex hull of the unit vectors $e_i, i = 0, \dots, d$ in R^d . Set

$$\tilde{f}(v) = \frac{\sum_{i \in f(q)} \pi_{f(q), i} e_i}{\sum_{i \in f(q)} \pi_{f(q), i}}. \quad (14)$$

Then, the map $\tilde{f}$ is a generalized Sperner labeling. Hence, we may apply the Generalized Sperner's lemma to conclude that the image $|\tilde{f}|(\Delta)$ contains every point of Δ . In particular the barycenter $m_\Delta (\doteq (1/(d+1), \dots, 1/(d+1)))$ is contained in $|\tilde{f}|(\Delta)$. This means that there exists a simplex σ in T such that $m_\Delta \in |\tilde{f}|(\sigma)$, i.e., m_Δ is in the convex hull of $\{\tilde{f}(p) : p \in \sigma\}$. Hence, there exist non-negative numbers $\lambda_p, p \in \sigma$ summing to 1, such that

$$\sum_{p \in \sigma} \lambda_p \tilde{f}(p) = \sum_{p \in \sigma} \frac{\sum_{i \in f(p)} \lambda_p \pi_{f(p), i} e_i}{\sum_{i \in f(p)} \pi_{f(p), i}} = (1/(d+1), \dots, 1/(d+1)). \quad (15)$$

Multiplying both sides of (15) by $d+1$ we obtain (13). $\square$

Note that the Lemma remains valid if we replace (13) by

$$\sum_{j=1}^k \lambda_j \pi_{S_j, i} = \alpha_i, \quad \text{for all } i \in D \quad (16)$$

where $\alpha_i \geq 0$ for all $i \in D$.

3. The Generalized Main Lemma may be extended to the case of d -dimensional pseudomanifolds with boundaries contained in R^d . Every sub-simplex of such a pseudomanifold is contained in a unique affine subspace generated by its vertices. Hence, the notion of a *generalized Sperner labeling* is well defined. Let $\sigma = (p_0, \dots, p_d)$ be a simplex, and set $\sigma_i = (p_0, \dots, \hat{p}_i, \dots, p_d)$. Then, the following

generalized form of (7)

$$\text{vol}(f(\sigma)) = \sum_{i=0}^d (-1)^i \text{vol}(\mathbf{0}f(\sigma_i)) \quad (17)$$

continues to hold, and we have the following

Lemma 6 (Generalized Main Lemma bis) *Let T be a d -dimensional pseudomanifold with boundary $\dot{T}$ contained in $\mathbb{R}^d$, and let all d -dimensional simplices of T have the same orientation. Let f be a generalized Sperner labeling for T , let T' and L be a subdivision of T and a simplicial complex, respectively, such that the map $|f|$ is a simplicial map of T' into L . Then, the equality*

$$\text{vol}(|T|) \left[\doteq \sum_{\sigma \in T} \text{vol}(\sigma) \right] = \sum_{\tau \in T'} \text{vol}(f(\tau)) = \sum_{\tau \in \dot{T}'} \text{vol}(\mathbf{0}f(\tau)) \quad (18)$$

holds.

This Lemma implies that there exist a simplex τ in T such that $\text{vol}(f(\tau))$ is positive. Hence, (compare Kannai 1981) there exists no continuous map h of $|T|$ onto $|\dot{T}|$ such that h restricted to $|\dot{T}|$ is the identity map (no retraction theorem).

4 Concluding remarks

1. The various forms of the Main Lemma are inspired by the arguments of Kannai (1981). Gauss divergence theorem (or Stokes theorem) and other analytic tools are replaced by elementary facts about volumes of simplices and simple cancellations of terms generated by interior faces.
2. Note that in our proof of Sperner's lemma, we actually prove that for a Sperner labeling i , the number of d -simplices τ for which $i(\tau)$ has the same orientation as Δ exceeds by one the number of the d -simplices τ for which $i(\tau)$ has the opposite orientation (Brown and Cairns 1961). (This "oriented version" of Sperner's lemma is a simple consequence of degree theory.) This stronger form was re-derived by many authors, starting with Ky Fan, using so-called "Combinatorial Stokes Formulas", see a brief survey in the introduction of Meunier (2008).
3. The proof of Sperner–Shapley's lemma is less elementary than that of the standard Sperner's lemma, involving (even though only a relatively weak form of) the simplicial approximation theorem. Earlier proofs are also less elementary than the proofs of the standard result: Shapley's (1973) original proof involves a genericity argument, and the proof in Kannai (1992) invokes the no-retraction theorem (or Brouwer's fixed point theorem) for continuous maps.
4. Either one of the ratios

$$\frac{\sum_{\tau \in T'} \text{vol}(f(\tau))}{\text{vol}(|T|)} = \frac{\sum_{\tau \in \dot{T}'} \text{vol}(f(\tau))}{\text{vol}(|T|)} \quad (19)$$

may be used to define the degree of the map f . In the case of a Sperner labeling, and more generally in the case of a generalized Sperner labeling, (2) (or (18)) implies that the degree is equal to 1.

5. The proofs of the Main Lemma and of Sperner's lemma are straight-forward and involve no auxiliary constructions (except perhaps introducing the origin in (6) in order to rewrite (5) without using volumes other than d -dimensional ones). In particular, no homotopy is used.
6. With extra work one may generalize the Main Lemma and the no retraction theorem to the case of general orientable pseudomanifolds with boundaries. However, the Main Lemma cannot be generalized to general pseudomanifolds, while the version of Sperner's lemma and the no retraction theorem are still valid (Spanier 1966).

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