Time-variant reliability analysis of a continuous system with strength deterioration based on subset simulation

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Received: 8 November 2018 / Revised: 24 December 2018 / Accepted: 1 April 2019 / Published online: 28 May 2019
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Abstract To conduct a reliability analysis for mechanical components, it is necessary to consider the combined influence of strength deterioration and dynamic loads. An efficient method based on subset simulation is proposed in this paper to analyze time-variant reliability by considering the strength deterioration of mechanical components in a continuous system. A gamma process is used to describe the deterioration of system strength. A model for time-variant reliability considering strength deterioration is constructed for a continuous system. A representative example and tubular cantilever structure are assessed to demonstrate the efficiency and accuracy of the proposed method. The reliability probability examples were analyzed using a first-order reliability method and benchmark results for the proposed method were derived using direct Monte Carlo simulation (MCS). The results of the proposed method and MCS are consistent, indicating that the proposed method is an effective reliability analysis method for evaluating small failure probabilities in a continuous system subjected to strength deterioration and dynamic loads.

Keywords Time-variant reliability · Strength deterioration · Subset simulation (SS) · Continuous system

1 Introduction

The reliability analysis of mechanical components is critical for improving the operational reliability and efficiency of mechanical equipment. Considering the influence of changes in the environment and uneven material surfaces, the structural and performance parameters of mechanical components are often uncertain. Some parameters are time-invariant random variables subjected to a particular distribution (e.g., normal distribution), while others are time-variant random variables that change regularly over time or follow a random process. According to different typologies of uncertainty, analysis methods for reliability can be divided into two main categories: time-variant and time-invariant methods. Over the past few years, several studies on reliability methods have been reported. Such methods have been widely utilized in engineering applications [1–3]. Castaldo et al. [1] proposed a sectional approach based on Monte Carlo simulation (MCS) to estimate the expected lifetime of a deteriorating reinforced concrete bridge pier. Huang et al. [2] proposed a new second-order reliability method with saddle point approximation for accurate, convergent, and computationally efficient estimates of the probability of failure. Zhu et al. [3] incorporated finite element simulations with Latin hypercube sampling to assess the fatigue reliability of high-pressure turbine discs under complex loadings coupled with multi-source uncertainties.

In most cases, mechanical components are subjected to strength deterioration and the failure process is gradual. Several methods have been reported for analyzing

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component deterioration. Mori and Ellingwood [4] developed methods to evaluate the time-dependent reliability of reinforced or pre-stressed concrete structures using structural reliability principles. Li [5] postulated that the time-dependent reliability analysis of a deteriorating structural system was based on the reliability of a given failure sequence for the structure. Ciampoli [6] formulated a probabilistic method based on a stochastic differential equation for the reliability assessment of structural components subjected to deterioration. Li et al. [7] developed an improved method for evaluating the time-dependent reliability of structures to assess the effects of nonstationarity in the load and resistance deterioration processes on safety and serviceability quantitatively.

Mechanical components typically bear dynamic loads during regular system operation. Therefore, it is essential to evaluate the continuous reliability of mechanical components by combining dynamic loads with strength deterioration. Rice [8] presented a novel first-passage probability formula, which has been widely used to evaluate time-variant reliability. Andrieu-Renaud et al. [9] developed a method known as PHI2 based on an out-crossing approach that solved reliability problems using classical time-invariant reliability tools, such as the first-order reliability method (FORM) and second-order reliability method. Zhang et al. [10] proposed an approach to translate time-variant reliability into static reliability by discretizing the stochastic processes of resistance and stress at calculation time. Li and Kiureghian [11] developed a method based on the principles of optimal linear estimation theory for the efficient discretization of random fields to simulate stochastic processes.

Subset simulation (SS) was originally proposed by Au and Beck [12] to estimate the small failure probability of high-dimensional parameter space problems. Small failure probability is expressed as the product of the large conditional probabilities of selected intermediate failure events. This method has been widely used in statistics, applied mathematics, and structural engineering. Au and Beck [13] used SS to effectively simulate procedure for seismic performance assessment of structures in the context of modern performance-based design. Vahdatirad et al. [14] estimated low-event probabilities using SS to analyze the first natural frequency of a turbine supported by a surface footing. Norouzi and Nikolaidis [15] re-evaluated the reliability of a dynamic system subjected to stationary Gaussian stochastic excitation for various load spectra by reweighting the results of a single SS. Song et al. [16] analyzed the reliability sensitivity of failure probabilities by transforming the distribution parameters of basic variables into a set of conditional failure probabilities using SS. Bourinet et al. [17] presented an approach referred as 2SMART for estimating small failure probabilities by

considering SS from the perspective of support vector machine classification. Zuev et al. [18] presented a modified Metropolis algorithm based on Markov chain MCS and used SS for sampling from conditional distributions to compute small failure probabilities in general high-dimensional reliability problems.

Recently, SS has been widely used to evaluate dynamic structural reliability [19–22]. Li et al. [19] developed a generalized SS approach to estimate the failure probabilities of multiple stochastic responses. Wang et al. [20] presented an improved SS method with a splitting approach to estimate the time-dependent reliability of systems with random parameters excited by stochastic processes. Yu and Wang [21] proposed a novel time-variant reliability analysis method using failure process decomposition to transform time-variant reliability problems into time-invariant problems for dynamic structures with uncertainty. Yu et al. [22] proposed a novel time-variant reliability analysis method for multiple failure modes and temporal parameters based on a combination of the extreme value moment method and improved maximum entropy method. However, time-variant reliability problems with strength deterioration for mechanical components have rarely been evaluated using SS. In this paper, a novel approach is proposed for the continuous system reliability analysis of mechanical components using SS. Expressions for the performance functions of a continuous system are constructed using a model for time-variant reliability with strength deterioration and the failure probabilities of mechanical components are derived.

The remainder of this paper is organized as follows. Section 2 describes a model for deterioration processes and presents a performance function for a continuous system, as well as a model of time-variant reliability with strength deterioration. Section 3 describes an approach for evaluating continuous system reliability using SS. Section 4 presents two numerical examples to demonstrate the efficiency and accuracy of the proposed approach. Conclusions are provided in Section 5.

2 Model of time-variant reliability with strength deterioration

The gamma process proposed by Abdel-Hameed [23] is the most appropriate process for describing the deterioration of strength. Excluding cyclic loads, a given system is largely influenced by random environmental factors, such as random vibrations in the external environment, stochastic variation in ambient temperature, or corrosion. These factors lead to a monotonic decrease in strength and stochastic process evolution over time. Furthermore, the gamma process is a continuous-time stochastic process with

independent, nonnegative increments (e.g., variation in tubular cantilever structural strength). Therefore, it is suitable for modeling gradual damage that accumulates monotonically over time in a sequence of small increments, such as wear, fatigue, corrosion, creep, and a degrading health index. Additionally, relatively straightforward mathematical calculations are a major advantage of modeling deterioration processes as gamma processes.

Van Noortwijk [24] described the statistical properties of gamma processes as a deterioration model. The symbol $Y(t)$ is used to denote the deterioration at a time t , where $t \geq 0$. Therefore, according to the definition of a nonstationary gamma process, the probability density function (PDF) of $Y(t)$ can be expressed as

$$f_{Y(t)}(y) = G(y|v(t), u) \\ = \frac{1}{\Gamma(v(t))u^{v(t)}} y^{v(t)-1} \exp(-y/u) I_{(0,\infty)}(y), \quad (1)$$

where $I_{(0,\infty)}(y) = 1$ for $y \in (0, \infty)$, $I_{(0,\infty)}(y) = 0$ for $y \in (0, \infty)$, and $\Gamma(v) = \int_0^\infty y^{v-1} e^{-y} dy$ is the gamma function for a positive shape parameter v . Furthermore, the gamma process is a stochastic process with a shape function $v(t)$, which is non-decreasing and right-continuous, and positive scale parameter u . Therefore, the expected value can be expressed as

$$E(Y(t)) = \int_0^\infty y f_{Y(t)}(y) dy = uv(t). \quad (2)$$

The variance can be written as

$$\text{var}(Y(t)) = \int_0^\infty (y - E(Y(t)))^2 f_{Y(t)}(y) dy \\ = u^2 v(t). \quad (3)$$

Experimental studies have confirmed that the expectation of deterioration at a time t is generally proportional to the following power law

$$E(Y(t)) = uct^b \\ = at^b \propto t^b, \quad (4)$$

where a , b , and c are positive physical constants.

For applying the gamma process model to engineering problems, statistical methods are required to estimate the parameters of the shape function $v(t) = ct^b$ and scale parameter u . Through various calculations, practical knowledge [25–28] can be obtained regarding the shape function in terms of the parameter b in Eq.(4). Consequently, the value of b is assumed to be known, but the values of c and u are unknown. The estimators of c and u can be obtained using the most common method of parameter estimation, namely the maximum likelihood.

Cinlar et al. [26] provided a method for transforming a nonstationary gamma process into a stationary gamma process. The corresponding observations of cumulative deterioration $\{y_{i_t} | i_t = 1, 2, \dots, n_t\}$, where $y_0 < y_1 < y_2 < \dots < y_{n_t}$ and $y_0 = 0$, are obtained at inspection time points $\{t_{i_t} | i_t = 1, 2, \dots, n_t\}$, where $t_0 < t_1 < t_2 < \dots < t_{n_t}$ and $t_0 = 0$. The likelihood function for the observed deterioration increments $\{\delta_{i_t} = y_{i_t} - y_{i_t-1} | i_t = 1, 2, \dots, n_t\}$ is used to estimate the values of $\hat{c}$ and $\hat{u}$.

Based on the above analysis, the residual strength $S(t)$, when formulated as a stochastic process, is equal to the initial strength S_0 , and is assumed to decrease monotonically based on a deterioration function $Y(t)$. Therefore, the general form of assumed residual strength can be expressed as

$$S(t) = S_0 - Y(t), \quad (5)$$

where S_0 is a random variable with a normal distribution, which is closely related to component elements, internal defects, heat treatment, and work-hardening of materials. $Y(t)$ is a stochastic process depending on the conditions of suffered stress and operating environment. Therefore, S_0 and $Y(t)$ are uncorrelated.

Based on the above definitions and the stress-strength interference theory, the performance function for time-variant reliability with strength deterioration for a continuous system can be expressed as

$$g(\mathbf{X}, t) = S(t) - \max(\sigma(\mathbf{Z}, t)), \quad (6)$$

where the stochastic process $\mathbf{X}(t)$ is composed of the initial strength S_0 , deterioration function $Y(t)$, and external loads $\mathbf{Z}(t)$ which include the static and dynamic loads at the time t . Therefore, $\max(\sigma(\mathbf{Z}, t))$ denotes the maximum dynamic stress magnitude imposed by external loading during the time interval.

Consequently, the failure probability of a continuous system with strength deterioration can be calculated as

$$P_F(t) = P\{g(\mathbf{X}, t) < 0\} \\ = P\{S(t) - \max(\sigma(\mathbf{Z}, t)) < 0\}. \quad (7)$$

The model of time-variant reliability can be expressed as

$$R(t) = 1 - P_F(t). \quad (8)$$

MCS is generally a feasible method to estimate the probability of failure once a system performance function is formulated. However, MCS is not appropriate for obtaining small probabilities of failure (e.g., $P_F < 10^{-3}$) because it suffers from a crucial lack of efficiency to achieve the required accuracy. Therefore, in this study, a Markov chain Monte Carlo (MCMC) method based on the Metropolis algorithm was used to efficiently compute small failure probabilities in a continuous system bearing

dynamic loads and suffering from gradual failure caused by strength deterioration.

3 SS-based time-variant reliability analysis with strength deterioration

The core idea of the SS method proposed by Au and Beck [12] is to estimate the frequency of a rare event based on the frequencies of more common events in a sequence of intermediate failure events. SS has been widely used to analyze reliability effectively. However, solving the problem of time-variant reliability analysis with strength deterioration using SS has received little attention. Based on the performance function described in Section 2, SS is expanded in this section to handle the problem of time-variant reliability analysis with strength deterioration for a continuous system.

3.1 SS for a continuous system

Regardless of whether or not the stochastic process $\mathbf{X}(t) = (S_0, Y(t), \mathbf{Z}(t))$ is defined with respect to continuous-time or discrete-time variables, only the values for a process at a discrete number of time instances are generated. For the sake of convenient calculation, this time grid is defined by $0, t_1, t_2, \dots, t_{n_t-1}, t_{n_t}$, where $t_i = (i/n_t)t$ for $t > 0$ and $i_t = 0, 1, \dots, n$. Then, $\mathbf{X}(t)$ can be expressed as vectors of random variables $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{n_t-1}, \mathbf{X}_{n_t}$ (i.e., $\{\mathbf{X}_{i_t} | i_t = 1, 2, \dots, n_t\}$).

The symbols $\{F_{i_t} | i_t = 1, 2, \dots, n_t\}$ define the continuous system failure events at the time instances t_{i_t} . Furthermore, $F_{i_t,i} = \{g(\mathbf{X}_{i_t}) < b_i\}$, $i = 1, 2, \dots, m$, where $F_{i_t,1} \supset F_{i_t,2} \supset \dots \supset F_{i_t,m} = F_{i_t}$, denotes a set of intermediate conditional failure events with a decreasing sequence of intermediate threshold values $b_1 > b_2 > \dots > b_{m-1} > b_m = 0$. Consequently, the probability of failure at a time t_{i_t} can be expressed as

$$\begin{aligned} P_{F,i_t} &= P(g(\mathbf{X}_{i_t}) < 0) = P(F_{i_t}) = P(F_{i_t,m}) \\ &= P(F_{i_t,m} | F_{i_t,m-1}) P(F_{i_t,m-1}) = \dots \\ &= P(F_{i_t,1}) \prod_{i=2}^m P(F_{i_t,i} | F_{i_t,i-1}), \end{aligned} \quad (9)$$

where $\mathbf{X}_{i_t} = [X_{i_t,1}, X_{i_t,2}, \dots, X_{i_t,n}]$ is a vector of random variables with a PDF $q(\mathbf{X}_{i_t}) = \prod_{j=1}^n q_j(X_{i_t}(j))$, where n is the number of variables. The symbol $q_j(\cdot)$ represents the univariate PDF for each component of $\mathbf{X}_{i_t}$.

Equation (9) calculates the product of a sequence of intermediate conditional probabilities $\{P(F_{i_t,i} | F_{i_t,i-1}) | i = 2, 3, \dots, m\}$ and the first level $P(F_{i_t,1})$. To make the

conditional probabilities in Eq.(9) sufficiently large to be estimated efficiently, it is important to select intermediate failure events appropriately. Particularly, the prior determination of b_i is not a trivial task. Therefore, in this study, b_i was selected such that the estimated conditional probabilities were equal to a fixed value $p_0 \in (0, 1)$.

This was accomplished by setting the intermediate threshold values $\{b_i | i = 1, 2, \dots, m-1\}$ equal to the $(p_0 N_1 + 1)$ th values of $\{g(\mathbf{X}_{i_t,k_1}^{(i-1)}) | k_1 = 1, 2, \dots, N_1\}$, sorted in an ascending order. The symbols $\{\mathbf{X}_{i_t,k_1}^{(i-1)} | i = 2, 3, \dots, m-1\}$ denote the samples generated at the $(i-1)$ th conditional level. $\mathbf{X}_{i_t,k_1}^{(0)}$ are the independent and identically distributed (i.i.d.) samples simulated by the original MCS according to the PDF $q(\mathbf{X}_{i_t,k_1}^{(0)})$. N_1 is the number of samples at each level. In this study, the value of p_0 was set to 0.1 for the sake of efficiency.

Additionally, the probabilities $P(F_{i_t,1})$ and $\{P(F_{i_t,i} | F_{i_t,i-1}) | i = 2, 3, \dots, m\}$ are required to obtain P_{F,i_t} from Eq.(9). MCS can be used to estimate $P(F_{i_t,1})$ as

$$P(F_{i_t,1}) = \frac{1}{N_1} \sum_{k_1=1}^{N_1} I_{F_{i_t,1}}(\mathbf{X}_{i_t,k_1}^{(0)}), \quad (10)$$

where $I_{F_{i_t,1}}(\cdot)$ is an indicator function. If $\mathbf{X}_{i_t,k_1}^{(0)} \in F_{i_t,1}$, $I_{F_{i_t,1}}(\mathbf{X}_{i_t,k_1}^{(0)}) = 1$. Otherwise $I_{F_{i_t,1}}(\mathbf{X}_{i_t,k_1}^{(0)}) = 0$. Similarly, $P(F_{i_t,i} | F_{i_t,i-1})$ can be computed using a formula similar to Eq.(10). However, to obtain samples according to the conditional distributions $q(\mathbf{X}_{i_t,i} | F_{i_t,i-1}) = q(\mathbf{X}_{i_t,i}) I_{F_{i_t,i-1}}(\mathbf{X}_{i_t,k_1}) / P(F_{i_t,i-1})$ efficiently, simulation should be carried out using Markov chain MCS based on the modified Metropolis algorithm proposed by Papaioannou et al. [29], rather than traditional MCS.

3.2 Implementation procedure

In general, SS for a continuous system involves six main steps, which are detailed below.

- (i) Discretize the duration of continuous system operation T using an appropriate sampling interval Δt to obtain a number of time instances $n_t = T/\Delta t$ and discrete time instances $\{t_{i_t} = i_t \Delta t | i_t = 0, 1, \dots, n_t\}$.
- (ii) At time instance t_1 , generate N_1 i.i.d. samples $\mathbf{X}_1^{(0)}$ with a size of n using MCS as the first level of SS.
- (iii) Substitute these N_1 samples into the performance function and sort $\{g(\mathbf{X}_{1,k_1}^{(0)}) | k_1 = 1, 2, \dots, N_1\}$ in ascending order. To make $P(F_{1,1}) =$

- $P\{g(X_1) < b_1\}$ equal to p_0 , assign the $(p_0 N_1 + 1)$ th value to b_1 .
- (iv) For every $i = 2, 3, \dots$, generate $(1 - p_0)N_1$ extra conditional samples based on the $p_0 N_1$ seed samples in the failure region $F_{1,i-1} = \{g(X_1) < b_{i-1}\}$ using the modified Metropolis algorithm [29, 30] to obtain a total of N_1 samples (i.e., $X_{1,k_1}^{(i-1)}$) in $F_{1,i-1} = \{g(X_1) < b_{i-1}\}$. Similarly, calculate the values of the performance function using these N_1 samples and sort $\{g(X_{1,k_1}^{(i-1)}) | k_1 = 1, 2, \dots, N_1\}$ in ascending order again. Assuming that $P(F_i | F_{i-1}) = P\{g(X_1) < b_{i-1} | g(X_1) < b_i\}$ are equal to p_0 , take the $(p_0 N_1 + 1)$ th values of the sequence as b_i to obtain $F_{1,i} = \{g(X_1) < b_i\}$. If $b_i \leq 0$, define the final level failure region as $F_{1,m} = \{g(X_1) < b_m\}$, where $m = i$ and $b_m = 0$.
- (v) Let N_f denote the number of samples at the final level of SS. Then, the conditional probability of the final level can be calculated as
- $$P(F_{1,m} | F_{1,m-1}) = N_f / N_1. \quad (11)$$
- Substitute $P(F_{1,1}) = p_0$, $\{P(F_{1,i} | F_{1,i-1}) = p_0 | i = 2, 3, \dots, m-1\}$, and Eq.(11) into Eq.(9), the failure probability at time instance t_1 can be denoted as
- $$P_{F,t_1} = p_0^{m-1} \frac{N_f}{N_1}. \quad (12)$$
- (vi) Repeat steps (ii)–(v) for the time instances $t_{i_1}, i_1 = 2, 3, \dots, n_t$. Then, the continuous system failure probability can be denoted as $P_F = \{P_{F,i_1} | i_1 = 1, 2, \dots, n_t\}$.

In this study, SS was used to calculate the time-variant reliability for a continuous system with strength deterioration. The small failure probability is transformed into the product of larger probabilities, thereby achieving a reduction in the number of required calculations and improving calculation efficiency while maintaining high precision. Figure 1 presents the implementation process of the proposed method concisely.

4 Example applications

In this section, two representative examples of a high-dimensional stochastic problem and continuous engineering system are used to demonstrate the accuracy and efficiency of the proposed method. MCS is the most accurate method for solving the problem of time-variant reliability analysis. Considering the nonlinearity of the performance function,

the FORM [31] was adopted based on its excellent performance and simplicity. The results obtained using the proposed method were compared with those obtained using the FORM and MCS. To achieve the desired accuracy, MCS requires $10^{k+2} - 10^{k+3}$ samples if the failure probability is on the order of 10^{-k} [32]. The variance values for the random parameters can be obtained through reliability testing or statistical analysis of experimental data, or can be estimated based on the following principles if there is no relevant data. The variance can be estimated based on a variation coefficient. In general, for mechanical performance parameters related to metal materials, this coefficient is set to 0.05 [33]. If the system is subjected to the influence of a large number of independent factors (no dominant factor), random variables commonly follow a normal distribution [34].

4.1 High-dimensional example

This example was chosen to assess the performance of the proposed method for handling relatively high numbers of random variables. It is known that increasing the dimensionality of the random variable space introduces various challenges (e.g., curse of dimensionality [35]). We propose analyzing the academic example of one failure mode of a bevel gear, where z_{12} is the diameter of the small gear and z_{13} is the length of the contact line at the midpoint of the tooth surface. $z_i (i = 1, 2, \dots, 11, 14, \dots, 17)$ is the coefficient of the bevel gear and F is the nominal tangential force on a reference circle at the midpoint of the tooth width. The statistical distributions and parameters for the random variables are listed in Table 1. According to international standards [36], the contact stress of a bevel gear $\sigma(Z, t)$ is expressed by the load on the bevel gear transmission. The deterioration process for the contact fatigue strength of the tooth surface $S(t)$ can be described by a gamma process. The performance function is written as

$$\begin{aligned} g(X, t) &= g(S_0, Y, F, z_1, \dots, z_{17}, t) \\ &= S(t) - \max(\sigma(Z, t)) \\ &= (S_0 - Y(t)) \prod_{i=1}^6 z_i - \prod_{i=7}^{11} z_i \sqrt{\frac{F(t)}{z_{12} z_{13}}} \frac{u+1}{u} \prod_{i=14}^{17} z_i, \end{aligned} \quad (13)$$

where $\{z_i | i = 1, 2, \dots, 17\}$, and F and S_0 are all independent. Here, $u = 1.611$. Based on the $S-N$ curves, the parameters for the deterioration process of system strength are $u = 1.3822 \times 10^{-8}$, $b = 0.05$, and $c = 4.9751 \times 10^{10}$.

According to the performance function, the failure probabilities of the continuous system P_F were evaluated using the proposed method, FORM, and MCS. The results

Fig. 1 Flow chart for the analysis method for the time-variant reliability of continuous systems with strength deterioration based on SS

are listed in Table 2. Figure 2 presents the cumulative probability distributions of the continuous system as calculated by the proposed method and MCS at each time instance on logarithmic scales. One can see a clear connection between the cumulative probability, value of the performance function, and time of service. As shown in Table 2 and Fig. 2, the failure probabilities and cumulative probabilities of the continuous system estimated by the

proposed method agree well with those acquired from MCS. Compared to the results of the proposed method, the failure probabilities for the continuous system obtained by the FORM are significantly higher or lower than those evaluated by MCS.

To illustrate the high efficiency of the proposed method relative to MCS, Figs. 3 and 4 compare the coefficient of variation (COV) of the failure probabilities δ_{t_i} [12, 37] and

Table 1 Probability distributions of the basic random variables

Random variable	Distribution	Mean	Standard deviation	Autocorrelation coefficient function
$S_0/(\text{N}\cdot\text{mm}^{-2})$	Normal	1 350	162	–
$F(t)/\text{N}$	Gaussian process	1 072.61	105.59	$\exp\left(-\left(0.02\tau ^2\right)\right)$
z_1	Normal	0.818 5	0.027 01	–
z_2	Normal	0.923 3	0.030 47	–
z_3	Normal	1.065	0.035 15	–
z_4	Normal	1.014	0.033 46	–
z_5	Normal	1	0.033	–
z_6	Normal	1	0.033	–
z_7	Normal	1	0.033	–
z_8	Normal	2.468	0.012 34	–
z_9	Normal	189.8	9.49	–
z_{10}	Normal	0.752 4	0.003 762	–
z_{11}	Normal	0.993 5	0.004 967 5	–
z_{12}/mm	Normal	140	0.7	–
z_{13}/mm	Normal	191.489	0.957 4	–
z_{14}	Normal	1.375	0.045 38	–
z_{15}	Normal	1.156	0.038 15	–
z_{16}	Normal	1.307	0.043 13	–
z_{17}	Normal	1.046 9	0.034 55	–

Note: τ is the time interval

Table 2 Failure probabilities at different time instances

Service time t_i/a	MCS $P_{F,i}$ ($\times 10^{-3}$)	FORM $P_{F,i}$ ($\times 10^{-3}$)	SS $P_{F,i}$ ($\times 10^{-3}$)
1	1.56	1.56	1.53
2	2.66	2.50	2.66
3	3.32	3.28	3.34
4	3.68	3.98	3.83
5	4.52	4.62	4.49
6	5.30	5.22	5.36
7	5.46	5.78	5.52
8	6.04	6.31	6.04
9	6.74	6.82	6.74
10	7.72	7.31	7.83

computational times t_s acquired by the proposed method and MCS. For the same computing environment, Fig. 3 presents the variation in the COVs with an increasing failure probability. Furthermore, all values of δ_{i_i} meet the condition of being less than 30% to provide adequate accuracy. Based on these conditions, compared to the computational time required for MCS, the time required for the proposed method is much shorter, as shown in Fig. 4. Consider the service time instance of $t = 3$ a as an example. The number of samples required for MCS was 10^6 , but the

proposed method required only 10^3 initial samples. The computational time required by the proposed method ($t_s = 6.920$ 1 s) is only 3.64% of the time required for the direct MCS ($t_s = 190.111$ 5 s). These results demonstrate that the proposed method for estimating the failure probabilities of time-variant reliability for a high-dimensional continuous system with strength deterioration is relatively accurate and efficient.

4.2 Tubular cantilever structure

The tubular cantilever structure presented in Fig. 5 was considered as a second example. This example was studied by Madsen et al. [38], and Du and Chen [39] as an engineering design problem. The cantilever structure was analyzed based on its resistance to yielding caused by bending and sheering stress. Seven variables were involved in this example: two stochastic process variables $F_1(t)$ and $T(t)$, and five random variables S_0 , F_2 , F_3 , d , and h . S_0 was the initial strength of the tubular cantilever structure and strength deterioration was modeled as a gamma process. $F_1(t)$, F_2 , F_3 , and $T(t)$ are the three external forces and torque imposed on the cantilever structure, respectively, and d and h are the dimensions of the cross section. Their statistical distributions and parameters are listed in Table 3 and are all independent. Additionally, the locations of the force points and angles relative to vertical of the forces

Fig. 2 Cumulative PDF curves of performance functions for a high-dimensional continuous system

Fig. 3 Variations in the failure probabilities and COVs of a high-dimensional continuous system

Fig. 4 Variations in the failure probabilities and computational times for a high-dimensional continuous system

$F_1(t)$ and F_2 were $L_1 = 60$ mm, $L_2 = 120$ mm, $\theta_1 = 5^\circ$, and $\theta_2 = 10^\circ$.

As shown in Fig. 5, $F_1(t)$, F_2 , F_3 , and $T(t)$ cause the cantilever structure to experience bending and torsion.

Fig. 5 Geometry of the tubular cantilever structure example

Based on analysis of mechanical model, the bending and sheering stresses can be calculated as follows

$$\sigma_x(t) = \frac{F_3 + F_2 \sin \theta_1 + F_1(t) \sin \theta_2}{\frac{\pi}{4} (d^2 - (d - 2h)^2)} + \frac{\frac{d}{2} (F_2 L_1 \cos \theta_1 + F_1(t) L_2 \cos \theta_2)}{\frac{\pi}{64} (d^4 - (d - 2h)^4)}, \quad (14)$$

$$\tau_{xz}(t) = \frac{T(t)d}{4 \frac{\pi}{64} (d^4 - (d - 2h)^4)}. \quad (15)$$

According to Eqs. (14) and (15), the maximum stress can be expressed as

$$\sigma_{\max}(t) = \sqrt{\sigma_x^2(t) + 3\tau_{xz}^2(t)}. \quad (16)$$

Based on the deterioration of strength, the performance function takes on the following form

$$g(X, t) = S(t) - \max(\sigma(Z, t)) = S_0 - Y(t) - \sigma_{\max}(t), \quad (17)$$

where $Y(t)$ is defined by parameters $u = 1.4863 \times 10^{-6}$, $b = 0.2$, and $c = 2.8349 \times 10^7$.

The failure probability results are listed in Table 4. These results were obtained by the proposed method, FORM, and MCS based on Eq.(17). A comparison of the cumulative probability distributions of the tubular cantilever structure failure system as calculated by the proposed method and MCS at each service time instance is presented in Fig. 6. Figures 7 and 8 indicate that the computational time required by the proposed method is still much shorter than that required by MCS in the case of $\delta_{it} < 30\%$.

For the instance of $t = 1$, the number of samples required for MCS was 10^6 , but the proposed method required only 10^3 initial samples. The computational time required by the proposed method ($t_s = 6.9519$ s) is

Table 3 Probability distributions of the basic random variables

Random variable	Distribution	Mean	Standard deviation	Autocorrelation coefficient function
S_0/MPa	Normal	560	56	—
$F_1(t)/\text{N}$	Gaussian process	1 800	180	$\sin(0.3\tau)/ 0.3\tau $
F_2/N	Normal	1 800	180	—
F_3/N	Gumbel	1 000	100	—
$T(t)/(\text{N}\cdot\text{mm})^{-1}$	Gaussian process	420 000	42 000	$\exp(- 0.1\tau)$
d/mm	Normal	42	4.2	—
h/mm	Normal	5	0.5	—

Note: τ is the time interval

Table 4 Failure probabilities at each time instance

Service time t_i/a	MCS $P_{F,i}$ ($\times 10^{-3}$)	FORM $P_{F,i}$ ($\times 10^{-3}$)	SS $P_{F,i}$ ($\times 10^{-3}$)
1	0.38	0.261	0.381
2	0.80	0.561	0.700
3	1.16	0.947	1.220
4	1.84	1.420	1.820
5	2.58	1.990	2.580
6	3.56	2.670	3.780
7	4.26	3.440	4.540
8	5.84	4.320	6.030
9	7.02	5.310	7.230
10	7.72	6.410	7.610

Fig. 6 Cumulative PDF curves of the performance function for the tubular cantilever structure

significantly shorter than that required for MCS ($t_s = 338.288$ s). And the $P_{F,1}$ value estimated by the proposed method is 3.81×10^{-4} , which agrees well with the result obtained from MCS, 3.80×10^{-4} . However, the $P_{F,1}$ value calculated by the FORM is 9.87×10^{-6} , which differs significantly from the MCS result. This discrepancy

Fig. 7 Variations in the failure probabilities and COVs for the tubular cantilever structure **Fig. 8** Variations in the failure probabilities and computational times for the tubular cantilever structure

is largely based on the errors induced by the nonlinearity of the performance function for the tubular cantilever structure system and variety of the distributions of random variables. Additionally, the failure probabilities of the events were all small. These results further demonstrate that SS modified for a continuous system is an accurate and

efficient method for estimating time-variant reliability in engineering design problems with strength deterioration and small failure probabilities.

5 Conclusions

A novel reliability analysis method was proposed in this paper. The proposed method combines the advantages of SS with the discretization of continuous systems. Continuous service times are divided into appropriate time intervals to determine the variation in failure probabilities for continuous systems at each time instance using SS. This method evaluates the small failure probabilities of time-variant reliability with strength deterioration efficiently and accurately. The proposed model for time-variant reliability with strength deterioration performs well on reliability analysis problems for continuous systems subjected to strength deterioration and dynamic loads.

Two examples demonstrated that the proposed method estimates $P_{F,i}$ values reasonably accurately compared to common reliability analysis methods (e.g., the FORM), particularly if the failure probabilities are as low as 10^{-4} . Furthermore, the computational time required for the proposed method is considerably shorter than that required for MCS. The proposed method can be utilized for continuous systems with small failure probabilities as an effective reliability analysis method. Additionally, it provides a new perspective for time-variant reliability problems with strength deterioration characterized by small failure probabilities, multiple random variables, and nonlinearity of performance functions.

Acknowledgements The authors gratefully acknowledge the support of the National Natural Science Foundation of China (Grant Nos. U1708254 and 51575094).

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