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On fractional diffusion equation with noise perturbation

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Abstract
The stochastic time-fractional diffusion equation can be accounted for a logical description of models with subdiffusion. This work is dedicated to the study of existence and uniqueness of the solution of stochastic time-fractional diffusion equation perturbed with a nonlinear source term. The method of Faedo–Galerkin approximations is employed in order to arrive at the estimate and to establish existence of solution by assuming that the noise coefficient and the nonlinear source term satisfy the required assumptions like Lipschitz continuity and linear growth condition.

Keywords Stochastic fractional differential equations · Existence of solutions · Time-fractional PDE · Diffusion equation

Mathematics Subject Classification 35A01 · 35A02 · 35D30 · 35R11 · 60H15

1 Introduction

Fractional calculus is a wide spread theory having applications in varied areas like mechanics, engineering, biochemistry and even in medicine. Many models that involve the characterization of memory effects into them cannot be modeled efficiently using integer-order equations. Fractional differential equations can help in such circumstances by incorporating the effects due to the memory into the system.

In the case of diffusion process, the mean square displacement is a quantity entitled to measure the dispersion of random particles and rate at which they diffuse. In classical models, they exhibit a linear relation with time, that is, the larger the time, the particles diffuse more faster. But in various processes, like diffusion on fractals, it can be observed that the mean square displacement develops logarithmically for large times. Since the classical integer order diffusion equation works well only in homogeneous medium, its time-fractional counterpart is more advantageous in modeling of phenomena with subdiffusion.

Fractional operators are highly efficient to model anomalous diffusion which is exhibited by systems in which displacement and time have a nonlinear relationship between them. In [1], the efficiency of fractional approach in modeling subdiffusion is validated due to natural exhibition of unusual dynamics by the system in the case of behaviors like slow dispersion, slow approach to the stationary state and memory effects. There are ample applications of such systems like in modeling growth of tumor cells which has memory effect. In [2], analysis of a system of coupled partial differential equations, which models tumor growth under the influence of subdiffusion is done. The study of diffusion of fluids in porous media with memory using time-fractional models involving subdiffusion is done in [3]. The existence, uniqueness and regularity of a mild solution of time-fractional Fokker-Planck equation is proved in [4] under the assumption of sufficient regularity for initial data. For more related works on existence, one can refer to [5–7] and so on. Apart from the analysis on the existence of solutions for fractional partial differential equations, a quite good research on numerical analysis is being carried out recently. For mathematical model describing Belousov–Zhabotinsky reaction, [8] discusses consequences of generalizing the model within the fractional order and also studies the boundedness, stability, existence, and other dynamical conditions.

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The time-fractional diffusion equation is given by

$$\partial_t \phi = \mathrm{div}(m \partial_t^{1-\alpha} \nabla \phi) + f(t, \phi), \tag{1.1}$$

with initial condition $\phi(0) = \phi_0$ and homogeneous Neumann condition at the boundary. In (1.1), ϕ denotes the concentration, $m > 0$ is the diffusion coefficient and f is a nonlinear source term. The derivative on right hand side is Riemann–Liouville fractional derivative and on taking convolution, the equation can be equivalently written using Caputo fractional derivative on the left hand side. The form with Riemann–Liouville derivative can simplify the estimates as done in [6]. Though in recent years there are many fractional operators being introduced, we prefer using these classical operators since either the new operators lack mathematical reasoning or they are just an extension of these classical operators [9].

The inclusion of the concept of uncertainty into models is proved to provide better approximations to their real life physical phenomenon. These types of situations arise in modeling the population of species with relative memory of distribution of resources. The basic idea is to model the random disturbances created in the environment using a stochastic term denoted by random noise. Based on the property of disturbances affecting the system, the noise term is chosen accordingly. On considering the Gaussian noise, we bring the effects of continuous disturbances into account. Some works involving the analysis of stochastic partial differential equations perturbed by Gaussian noise include [10, 11]. The stochastic counterpart of fractional differential equations help in efficiently elucidating complex dynamics exhibited due to hereditary effects of systems in areas like visco-elasticity and signal processing. The study of stochastic fractional equation remains not much explored with few works done on the existence of solutions as in [12–14]. In [15], the comparison between two stochastic models of European option pricing, one with time derivative replaced with fractional derivative and the other with noise term given by fractional Brownian motion, is made. The model with fractional derivative was proved to be efficient than the one with fractional Brownian motion.

In this work, we consider the time-fractional diffusion equation perturbed by Brownian-type noise which results in a stochastic version of (1.1). The novelty of this equation is that it models subdiffusion and therefore it is evident to realize the necessity to provide an analytic proof for existence of its solution. Using Galerkin approximations, the problem is projected on to a finite-dimensional space and in turn gaining an approximate solution for the projected equation. An a-priori estimate of this solution paves the way to compute the solution of our original problem.

The aim of this work is to establish the existence and uniqueness of solution of the stochastic time-fractional diffusion equation. The flow of this paper is as follows. In the next section, we introduce the basic mathematical concepts, inequalities and assumptions required for the proof. The last section is dedicated for the proof of our main result by initially establishing the energy estimates and establishing the convergence of approximations to the original solution by use of fractional Gronwall–Bellman-type inequality.

2 Mathematical model

Let $\mathcal{O} \subset \mathbb{R}^2$. The stochastic time-fractional diffusion equation perturbed by a parameter $\varepsilon > 0$ is given by

$$\begin{aligned} \partial_t \phi^\varepsilon &= \mathrm{div}(m \partial_t^{1-\alpha} \nabla \phi^\varepsilon) + f(t, \phi^\varepsilon) \\ &\quad + \sqrt{\varepsilon}\sigma(t, \phi^\varepsilon)\partial_t W(t), \\ \text{with } \phi^\varepsilon(0) &= \phi_0 \quad \text{and } \frac{\partial \phi^\varepsilon}{\partial \nu} = 0. \end{aligned} \tag{2.1}$$

Here

- ν : outward normal to boundary $\partial\mathcal{O}$,
- $W(t)$: independent Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$,
- $\sigma(t, \phi^\varepsilon)$: noise coefficient satisfying conditions stated later.

The corresponding deterministic equation is given in (1.1). Here, the fractional order α satisfies $0 < \alpha < 1$. For a function F, we denote

$$\partial_t^{1-\alpha} F = \partial_t (g_\alpha * F),$$

where g_α is defined as $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ (also refer [16]). The operator $*$ denotes the convolution with respect tot the time variable is denoted by I^α,

$$(g_\alpha * \varphi)(t) = \int_0^t g_\alpha(t-s)\varphi(s)\mathrm{d}s, \text{ for some } \varphi \in \mathbb{L}^1(0, T).$$

We now introduce the function spaces. We define the spaces V and H in the Gelfand triple $V \hookrightarrow H \hookrightarrow V'$ as

- The Lebesgue space $H = \mathbb{L}^2(\mathcal{O})$ with the norm $\|\cdot\|_H$ defined by

$$(\phi, \psi) = \int_{\mathcal{O}} \phi(x)\psi(x)\mathrm{d}x, \qquad \|\phi\|_H = \sqrt{(\phi, \phi)}.$$

- The Sobolev space $V = \mathbb{H}^1(\mathcal{O})$ with norm $\|\cdot\|_V$ defined by

$$\|\phi\|_V^2 = \int_{\mathcal{O}} |\phi(x)|^2 \mathrm{d}x + \int_{\mathcal{O}} |\nabla \phi(x)|^2 \mathrm{d}x.$$

Here, H and V are Hilbert spaces which naturally occur and guarantee the existence of orthonormal normal basis required for Galerkin approximations. For a Banach space X, the Bochner space [17] is defined

$$\mathbb{L}^p(0,T;X) = \Big\{ \phi : (0,T) \to X : \phi \text{ is strongly measurable}, \|\phi\|^p_{\mathbb{L}^p(0,T;X)} = \int_0^T \|\phi(t)\|_X^p \mathrm{d}s < \infty \Big\}.$$

For $p = \infty$, we define $\mathbb{L}^\infty(0,T;X)$ with $\|\phi\|_{\mathbb{L}^\infty(0,T;X)} = \text{ess}\sup_{t\in(0,T)} \|\phi(t)\|_X$. The fractional Sobolev–Bochner space for a Banach space X is defined as

$$\mathbb{W}^{\alpha,p}(0,T;X) = \{\phi \in \mathbb{L}^p(0,T;X) : \partial_t^{1-\alpha}\phi \in \mathbb{L}^p(0,T;X)\}.$$

For the case $p = 2$, $\mathbb{W}^{\alpha,2}(0,T;X) = \mathbb{H}^\alpha(0,T;X)$. Let Q be the covariance operator of H-valued Wiener process $W(t)$ such that it is strictly positive, symmetric and trace class operator on H. Define $H_0 = Q^{1/2}H$. Then, H_0 is a Hilbert space with scalar product

$$(\phi,\psi)_0 = (Q^{-1/2}\phi, Q^{-1/2}\psi), \quad \text{for } \phi,\psi \in H_0.$$

Let $\mathcal{L}_Q$ be the space of linear operators S such that $SQ^{1/2}$ is a Hilbert–Schmidt operator from H to H with the norm $\|S\|_{\mathcal{L}_Q} = \text{trace}(SQS^*)$. The following assumptions made, especially on the noise coefficient $\sigma(t,\phi)$ and the nonlinear source term $f(t,\phi)$ will be helpful for the proof of existence and uniqueness and in computing energy estimates.

Assumption 2.1 We assume the following conditions on the nonlinear term f and initial concentration ϕ_0.

(i) $\mathcal{O} \subset \mathbb{R}^2$ is a bounded-Lipschitz domain and $T > 0$ is finite.
(ii) $f \in \mathbb{L}^\infty(0,T;H)$ is Lipschitz and the initial concentration $\phi_0 \in V$.

Assumption 2.2 The function $\sigma \in C([0,T]\times V; \mathcal{L}_Q(H_0;H))$ satisfy $\forall\, t \in [0,T]$, $\exists\, K_1 > 0$ and $K_2 > 0$ such that

(A1) $$\|\sigma(t,\phi)\|^2_{\mathcal{L}_Q} \le K_1(1 + \|\nabla\phi\|_H^2),$$

for all $\phi \in V$.

(A2) $\|\sigma(t,\phi) - \sigma(t,\psi)\|^2_{\mathcal{L}_Q} \le K_2\|\nabla(\phi-\psi)\|_H^2$,

for $\phi,\psi \in V$.

3 Existence results

The results on existence and uniqueness of the solution for (2.1) are discussed in this section. The process $\phi^\varepsilon(t,\omega)$ is said to be a weak solution of (2.1) if it satisfies the initial condition ϕ_0 and for test function ψ of required regularity,

$$(\phi^\varepsilon(t),\psi) - (\phi_0,\psi) = \int_0^t \Big[\Big(\text{div}(m\partial_t^{1-\alpha}\nabla\phi^\varepsilon) + f(t,\phi^\varepsilon), \psi\Big)\Big]\mathrm{d}s + \sqrt{\varepsilon}\int_0^t (\sigma(s,\phi^\varepsilon(s))\mathrm{d}W, \psi).$$

Some results required for the proof of existence are stated initially. It is well known that not all rules that apply for integer order are applicable to fractional calculus too. The chain rule of the integer order calculus cannot be considered for the fractional derivatives. The ensuing proposition gives a counterpart for the chain rule for semiconvex functions in fractional setting. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be semiconvex if for some $\lambda \in \mathbb{R}$, the function $x \to f(x) - \frac{\lambda}{2}|x|^2$ is convex.

Proposition 3.1 *Let $\mathcal{V}$ be a Banach space such that $\mathcal{V} \hookrightarrow \mathbb{L}^2(\mathcal{O}) \hookrightarrow \mathcal{V}'$ forms a Gelfand triple. Let $u \in \mathbb{H}^\alpha(0,T;\mathcal{V}')\cap \mathbb{L}^\infty(0,T;\mathcal{V})$ with $u_0 \in \mathbb{L}^2(\mathcal{O})$ and $E \in C^1(\mathbb{R})$, a λ-convex function with $\lambda \in \mathbb{R}$. If $E'(u) \in \mathbb{L}^2(0,T;\mathcal{V})$, then we have for all $t \in (0,T)$*

$$\int_0^t \Big(\langle\partial_t^\alpha u, E'(u)\rangle_{\mathcal{V}} - \lambda\langle\partial_t^\alpha u, u\rangle_{\mathcal{V}}\Big)ds \ge g_{1-\alpha} * \int_{\mathcal{O}} \big[E(u) - E(u_0)\big]dx + \frac{\lambda}{2} g_{1-\alpha} * \big(\|u\|_H^2 - \|u_0\|_H^2\big), \tag{3.1}$$

$$g_\alpha * \langle\partial_t^\alpha u, E'(u)\rangle_{\mathcal{V}} - \lambda g * \langle\partial_t^\alpha u, u\rangle_{\mathcal{V}} \ge \int_{\mathcal{O}} \big[E(u) - E(u_0)\big]dx + \frac{\lambda}{2}\big(\|u\|_H^2 - \|u_0\|_H^2\big). \tag{3.2}$$

The proof of the above proposition is given in [18]. For a special case of $E(\cdot) = \frac{1}{2}|\cdot|^2$ in Hilbert space, one can refer [19]. The following lemma is a corollary of fractional Gronwall–Bellman-type inequality which is required in the proof of existence.

Lemma 3.1 *Let $u, v \in \mathbb{L}^1(0,T;\mathbb{R}_{\ge 0})$ and $a, b > 0$. If u and v satisfy*

$$u(t) + g_\alpha * v(t) \le a + b(g_\alpha * u)(t) \ a.e\ t \in (0,T),$$

then we have

$$u(t) + v(t) \le a \cdot C(\alpha,b,T) \ a.e\ t \in (0,T). \tag{3.3}$$

For proof, refer [18]. We now state the Itô formula.

Theorem 3.1 (Itô formula) [20] *Assume that Φ is an $\mathcal{L}_Q$-valued process stochastically integrable in $[0, T]$, ϕ is an H-valued predictable process Bochner integrable on $[0, T]$, $\mathbb{P}$-a. s., and $X(0)$ is an $\mathcal{F}_0$-measurable H-valued random variable. Then, the following process*

$$X(t) = X(0) + \int_0^t \phi(s)ds + \int_0^t \Phi(s)dW(s), t \in [0, T],$$

is well defined. Assume that a function $F : [0, T] \times H \to \mathbb{R}$. Then, for all $t \in [0, T]$,

$$\begin{aligned} F(t, X(t)) &= F(0, X(0)) \\ &+ \int_0^t [F_s(s, X(s)) + (F_x(s, X(s)), \phi(s))]ds \\ &+ \int_0^t (F_x(s, X(s)), \Phi(s)dW(s)) \\ &+ \frac{1}{2}\int_0^t Tr[F_{xx}(s, X(s))(\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^*]ds. \end{aligned} \tag{3.4}$$

The Burkholder–Davis–Gundy inequality is used to break down the stochastic integral and produces a simplified integral term. Let $M = (M(t), t \geq 0)$ be a Brownian integral with the drift of the form

$$M(t) = \int_0^t F^j(s)\mathrm{d}W^j(s),$$

where each $F^j \in \mathbb{L}^2[0, T]$ for all $t \geq 0$, $1 \leq j \leq d$. Let the quadratic variation process, denoted as $([M, M](t), t \geq 0)$, be defined by

$$[M, M](t) = \sum_{j=1}^d \int_0^t F^j(s)^2\mathrm{d}s.$$

Lemma 3.2 *(Burkholder–Davis–Gundy Inequality) For every $p \geq 1$, there is a constant $C_p \in (0, \infty)$ such that for any real-valued square integrable cádlág martingale M with $M(0) = 0$, and for any $T \geq 0$,*

$$C_p^{-1}\mathbb{E}[M, M]_T^{p/2} \leq \mathbb{E}\sup_{0\leq t\leq T}|M|^p \leq C_p\mathbb{E}[M, M]_T^{p/2}.$$

The following theorem is the main result of this work which states the existence and uniqueness of the solution for (2.1).

Theorem 3.2 *Let Assumptions 2.1 and 2.2 hold. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in [0, \varepsilon_0]$ there exists a pathwise unique weak solution ϕ^ε for the stochastic fractional diffusion equation (2.1) such that*

$$\phi^\varepsilon \in \mathbb{H}^\alpha(0, T; V') \cap \mathbb{L}^\infty(0, T; V),$$

satisfying the energy inequality

$$\mathbb{E}\left(\|\phi^\varepsilon\|^2_{\mathbb{L}^\infty(0,T;V)}\right) \leq \varepsilon C(\mathbb{E}\|\phi_0\|_V^2 + \|f\|^2_{\mathbb{L}^\infty([0,T],H)}), \tag{3.5}$$

where C is an appropriate constant.

From here, $\|\cdot\|$ denotes the norm in H unless specifically mentioned. We use Faedo–Galerkin method of approximation to establish the existence and uniqueness. This is an efficient tool employed to prove existence results since it reduces the problem to finite dimension and use orthonormal basis for approximating their solutions. Let $\{\varphi_n\}_{n\geq 1}$: complete ONB of H corresponding to the Laplacian operator with Neumann boundary condition and $H_n = span(\varphi_1, \cdots, \varphi_n)$ and $P_n : H \to H_n$ be an orthogonal projection onto H_n. Let $W_n = P_n W$ and $\sigma_n = P_n\sigma$. Then for $\psi \in H_n$, consider the equation in H_n,

$$\begin{aligned} \left(\partial_t\phi_n^\varepsilon, \psi\right) &= \left(\mathrm{div}(m\partial_t^{1-\alpha}\nabla\phi_n^\varepsilon) + f(t, \phi_n^\varepsilon), \psi\right) \\ &+ \sqrt{\varepsilon}\left(\sigma_n(t, \phi_n^\varepsilon)\,\partial_t W_n, \psi\right), \end{aligned} \tag{3.6}$$

with $\phi_n^\varepsilon(0) = P_n\phi(0)$. Here we observe that as $n \to \infty$, $\phi_n^\varepsilon(0) = P_n\phi(0) \to \phi_0$ in V. If $\psi \in V$, then $f \in \mathbb{L}^\infty(0, T; H)$ implies that by Lipschitz condition satisfied by the coefficients, from theory of fractional ODES, as in [6], we have a solution to Eq. (3.6) on $[0, T_n]$ such that

$$\phi_n^\varepsilon \in \mathbb{H}^\alpha(0, T_n; H_n) \cap \mathbb{L}^\infty(0, T_n; H_n).$$

It implies that there is a stopping time $T_n \leq T$ such that for $t < T_n$, (3.6) holds and for $t \uparrow T_n < T$, $|\phi_n^\varepsilon(t)| \to \infty$. We now prove $T_n = T$ and estimate ϕ_n^ε for all n and $\varepsilon \in [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$. For $N > 0$, take

$$\tau_N = \inf\{t : |\phi_n^\varepsilon(t)| \geq N\} \wedge T.$$

Proposition 3.2 *Under Assumptions 2.1 and 2.2, there exists $\varepsilon \geq 0$, such that $T_n = T$ and there exists a unique solution $\phi_n^\varepsilon \in \mathbb{H}^\alpha(0, T; H_n) \cap \mathbb{L}^\infty(0, T; H_n)$ satisfying (3.5) for an appropriate constant C.*

Proof We first prove the estimate (3.5) for ϕ_n^ε. Applying Itô's formula for $\|\phi_n^\varepsilon\|^2$, we get

$$\begin{aligned} \|\phi_n^\varepsilon(t)\|^2 &= \|\phi_n^\varepsilon(0)\|^2 \\ &+ 2\int_0^t \left(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi_n^\varepsilon(s)) + f(s, \phi_n^\varepsilon(s)), \phi_n^\varepsilon(s)\right)\mathrm{d}s \\ &+ \varepsilon\int_0^t \|\sigma_n(s, \phi_n^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s \\ &+ 2\sqrt{\varepsilon}\int_0^t \left(\phi_n^\varepsilon(s), \sigma_n(s, \phi_n^\varepsilon(s))\,\mathrm{d}W_n(s)\right). \end{aligned}$$

$$\|\phi_n^\varepsilon(t)\|^2 + 2\int_0^t \left(m\partial_s^{1-\alpha}\nabla\phi_n^\varepsilon(s), \nabla\phi_n^\varepsilon\right)\mathrm{d}s$$
$$= \|\phi_n^\varepsilon(0)\|^2 + 2\int_0^t \left(f(s,\phi_n^\varepsilon(s)), \phi_n^\varepsilon(s)\right)\mathrm{d}s$$
$$+ \varepsilon\int_0^t \|\sigma_n(s,\phi_n^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s$$
$$+ 2\sqrt{\varepsilon}\int_0^t \left(\phi_n^\varepsilon(s), \sigma_n(s,\phi_n^\varepsilon(s))\,\mathrm{d}W_n(s)\right).$$

Using (3.1) in Proposition 3.1, for $E(u) = \frac{|u|^2}{2}$, $\alpha = 1-\alpha$ and $\lambda = 0$, we get

$$2\int_0^t \left(m\partial_s^{1-\alpha}\nabla\phi_n^\varepsilon(s), \nabla\phi_n^\varepsilon(s)\right)\mathrm{d}s \geq mg_\alpha$$
$$* \left(\|\nabla\phi_n^\varepsilon(t)\|^2 - \|\nabla\phi_n^\varepsilon(0)\|^2\right).$$

Substituting the above estimate, we have

$$\|\phi_n^\varepsilon(t)\|^2 + mg_\alpha * \|\nabla\phi_n^\varepsilon(t)\|^2$$
$$\leq \|\phi_n^\varepsilon(0)\|^2 + mg_\alpha * \|(\nabla\phi_n^\varepsilon)(0)\|^2$$
$$+ 2\int_0^t \left(f(s,\phi_n^\varepsilon(s)), \phi_n^\varepsilon(s)\right)\mathrm{d}s$$
$$+ \varepsilon\int_0^t \|\sigma_n(s,\phi_n^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s$$
$$+ 2\sqrt{\varepsilon}\int_0^t \left(\phi_n^\varepsilon(s), \sigma_n(s,\phi_n^\varepsilon(s))\,\mathrm{d}W_n(s)\right).$$

Using Assumptions and Young's inequality,

$$\|\phi_n^\varepsilon(t)\|^2 + mg_\alpha * \|\nabla\phi_n^\varepsilon(t)\|^2$$
$$\leq \|\phi_n^\varepsilon(0)\|^2 + mg_\alpha * \|(\nabla\phi_n^\varepsilon)(0)\|^2$$
$$+ \int_0^t [\|f(s,\phi_n^\varepsilon(s))\|^2 + \|\phi_n^\varepsilon(s)\|^2]\mathrm{d}s$$
$$+ \varepsilon K_1\int_0^t (1 + \|\nabla\phi_n^\varepsilon(s)\|^2)\mathrm{d}s$$
$$+ 2\sqrt{\varepsilon}\int_0^t \left(\phi_n^\varepsilon(s), \sigma_n(s,\phi_n^\varepsilon(s))\,\mathrm{d}W_n(s)\right).$$

Taking supremum over time and then taking expectation,

$$\mathbb{E}\sup_{0\leq t\leq T\wedge\tau_N}\left\{\|\phi_n^\varepsilon(t)\|^2 + mg_\alpha * \|\nabla\phi_n^\varepsilon(t)\|^2\right\}$$
$$\leq \mathbb{E}\left\{\|\phi_n^\varepsilon(0)\|^2 + mg_\alpha * \|(\nabla\phi_n^\varepsilon)(0)\|^2\right\}$$
$$+ \mathbb{E}\int_0^{T\wedge\tau_N} [\|f(s,\phi_n^\varepsilon(s))\|^2 + \|\phi_n^\varepsilon(s)\|^2]\mathrm{d}s$$
$$+ \varepsilon K_1\mathbb{E}\int_0^{T\wedge\tau_N}(1 + \|\nabla\phi_n^\varepsilon(s)\|^2)\mathrm{d}s$$
$$+ 2\sqrt{\varepsilon}\mathbb{E}\sup_{0\leq t\leq T\wedge\tau_N}\int_0^t \left(\phi_n^\varepsilon(s), \sigma_n(s,\phi_n^\varepsilon(s))\,\mathrm{d}W_n(s)\right).$$

For the stochastic integral term, using Burkholder–Davis–Gundy inequality, Young's inequality and (A1) gives,

$$2\sqrt{\varepsilon}\mathbb{E}\sup_{0\leq t\leq T\wedge\tau_N}\int_0^t \left(\sigma_n(\phi_n^\varepsilon(s))\mathrm{d}W_n, \phi_n^\varepsilon(s)\right)$$
$$\leq 2\sqrt{\varepsilon}C_1\mathbb{E}\left\{\int_0^{T\wedge\tau_N} \|\sigma_n(s,\phi_n^\varepsilon(s))\|^2_{\mathcal{L}_Q}\|\phi_n^\varepsilon(s)\|^2\mathrm{d}s\right\}^{1/2}$$
$$\leq 2\sqrt{\varepsilon}C_1\mathbb{E}\left\{\sup_{0\leq t\leq T\wedge\tau_N}\|\phi_n^\varepsilon(t)\|^2\int_0^{T\wedge\tau_N} \|\sigma_n(s,\phi_n^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s\right\}^{1/2}$$
$$\leq \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T\wedge\tau_N}\|\phi_n^\varepsilon(t)\|^2 + \varepsilon C^2K_1\mathbb{E}\int_0^{T\wedge\tau_N}(1 + \|\nabla\phi_n^\varepsilon(s)\|^2)\mathrm{d}s.$$

Combining, we get

$$\mathbb{E}\sup_{0\leq t\leq T\wedge\tau_N}\left\{\|\phi_n^\varepsilon(t)\|^2 + 2mg_\alpha * \|\nabla\phi_n^\varepsilon(t)\|^2\right\}$$
$$\leq 2\mathbb{E}\left\{\|\phi_n^\varepsilon(0)\|^2 + mg_\alpha * \|(\nabla\phi_n^\varepsilon)(0)\|^2\right\}$$
$$+ 2\mathbb{E}\int_0^{T\wedge\tau_N} [\|f(s,\phi_n^\varepsilon(s))\|^2 + \|\phi_n^\varepsilon(s)\|^2]\mathrm{d}s$$
$$+ 2\varepsilon K_1(1 + C^2)\mathbb{E}\int_0^{T\wedge\tau_N}(1 + \|\nabla\phi_n^\varepsilon(s)\|^2)\mathrm{d}s.$$

Using (3.3) from Lemma 3.1, we get

$$\mathbb{E}\sup_{0\leq t\leq T\wedge\tau_N}\left\{\|\phi_n^\varepsilon(t)\|^2 + 2m\|\nabla\phi_n^\varepsilon(t)\|^2\right\}$$
$$\leq C(T)\mathbb{E}\Bigg\{\|\phi_n^\varepsilon(0)\|^2 + m\|(\nabla\phi_n^\varepsilon)(0)\|^2$$
$$+ \int_0^{T\wedge\tau_N} \|f(s,\phi_n^\varepsilon(s))\|^2\mathrm{d}s + \varepsilon K_1(1 + C^2)T\Bigg\}.$$

Then, for $\varepsilon \geq 0$,

$$\mathbb{E}\left(\|\phi_n^\varepsilon(t)\|^2_{\mathbb{L}^\infty(0,T;V)}\right)$$
$$\leq \varepsilon C\cdot\mathbb{E}\left(\|\phi_n^\varepsilon(0)\|^2_V + \|f\|^2_{\mathbb{L}^\infty([0,T];H)}\right).$$

Here $\tau_N \to T_n$ as $N \to \infty$ and for $\{T_n < T\}$, $\sup_{0\leq s\leq\tau_N}|\phi_n^\varepsilon(s)| \to \infty$. Hence $\mathbb{P}(T_n < T) = 0$ and so for large N, $\tau_N = T$ and $\phi_n^\varepsilon \in \mathbb{H}^\alpha(0,T;H_n)\cap\mathbb{L}^\infty(0,T;H_n)$. Hence the proof. □

Proof of Theorem 3.2 Let $\mathcal{O}_T = [0,T]\times\mathcal{O}$. The theorem is proved by splitting it into several steps.

Step 1 From energy estimate obtained in Proposition 3.2, for ϕ_n^ε, there exist a subsequence also denoted by $\{\phi_n^\varepsilon\}_{n\geq 0}$ and processes $\phi^\varepsilon \in \mathbb{H}^\alpha(0,T;V')\cap\mathbb{L}^p(0,T;H)\cap\mathbb{L}^\infty([0,T],V)$, $F^\varepsilon \in \mathbb{L}^2(0,T;V')$ and $S^\varepsilon \in \mathbb{L}^2(0,T;\mathcal{L}_Q)$ such that

(i) $\phi_n^\varepsilon \to \phi^\varepsilon$ strongly in $\mathbb{L}^p(0,T;H)$,
(ii) $\partial_t^{1-\alpha}\nabla\phi_n^\varepsilon \to \partial_t^{1-\alpha}\nabla\phi^\varepsilon$ weakly in $\mathbb{L}^2(0,T;V')$,
(iii) ϕ_n^ε is weak*-converging to ϕ^ε in $\mathbb{L}^\infty(0,T;V)$,
(iv) $f(t,\phi_n^\varepsilon) \to F^\varepsilon$ in $\mathbb{L}^2(\mathcal{O}_T,V')$,
(v) $\sigma_n(t,\phi_n^\varepsilon) \to S^\varepsilon$ in $\mathbb{L}^2(\mathcal{O}_T,\mathcal{L}_Q)$.

As a consequence of Proposition 3.2, we get (i)-(iii). To prove that the limit ϕ^ε satisfies the weak formulation of (2.1), we integrate (3.6) and then decompose their terms using known inequalities. Using Young's inequality,

$$\int_0^T \langle f(s,\phi_n^\varepsilon(s)),\psi\rangle \mathrm{d}s \le \int_0^T [\|f(s,\phi_n^\varepsilon)\|^2+\|\psi\|^2]\mathrm{d}s < \infty.$$

The above estimate with (i) proves (iv). From Assumption 2.2,

$$\mathbb{E}\int_0^T \|\sigma_n(s,\phi_n^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s \le K_1\mathbb{E} \int_0^T (1+\|\nabla\phi_n^\varepsilon(s)\|^2)\mathrm{d}s < \infty.$$

This implies (v). Since as $n\to\infty$, $P_n\phi_0=\phi_n^\varepsilon(0)\to\phi_0$ in H, we have ϕ^ε satisfies

$$\phi^\varepsilon(T)=\phi_0+\int_0^T \mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi^\varepsilon(s))\mathrm{d}s+\int_0^T F^\varepsilon(s)\mathrm{d}s + \sqrt{\varepsilon}\int_0^T S^\varepsilon(s)\mathrm{d}W(s). \tag{3.7}$$

Step 2

It now remains to prove that $F^\varepsilon(s)=f(s,\phi^\varepsilon(s))$ and $S^\varepsilon(s)=\sigma(s,\phi^\varepsilon(s))$. By Fatou's lemma,

$$\mathbb{E}\{\|\phi^\varepsilon(T)\|^2\} \le \liminf_n \mathbb{E}\{\|\phi_n^\varepsilon(T)\|^2\}. \tag{3.8}$$

Applying Itô's formula to (3.6),

$$\begin{aligned}\|\phi_n^\varepsilon(T)\|^2 &= \|\phi_n^\varepsilon(0)\|^2 + 2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi_n^\varepsilon(s)),\phi_n^\varepsilon(s)\big)\mathrm{d}s \\ &+ 2\int_0^T \big(f(s,\phi_n^\varepsilon(s)),\phi_n^\varepsilon(s)\big)\mathrm{d}s \\ &+ \varepsilon\int_0^T \|\sigma_n(s,\phi_n^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s + I(t),\end{aligned}$$

where

$$I(t)=2\sqrt{\varepsilon}\int_0^T \big(\sigma_n(s,\phi_n^\varepsilon(s))\mathrm{d}W_n(s),\phi_n^\varepsilon(s)\big).$$

Here since $I(t)$ is a local martingale with zero average,

$$\mathbb{E}\{I(t)\}=2\sqrt{\varepsilon}\mathbb{E}\int_0^T \big(\sigma_n(s,\phi_n^\varepsilon(s))\mathrm{d}W_n(s),\phi_n^\varepsilon(s)\big)=0.$$

Therefore, we get,

$$\begin{aligned}\mathbb{E}\big(\|\phi_n^\varepsilon(T)\|^2\big) &= \mathbb{E}\|\phi_n^\varepsilon(0)\|^2 + 2\mathbb{E}\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi_n^\varepsilon(s)),\phi_n^\varepsilon(s)\big)\mathrm{d}s \\ &+ 2\mathbb{E}\int_0^T \big(f(s,\phi_n^\varepsilon(s)),\phi_n^\varepsilon(s)\big)\mathrm{d}s + \varepsilon\mathbb{E} \int_0^T \|\sigma_n(s,\phi_n^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s.\end{aligned}$$

Similarly from (3.7),

$$\begin{aligned}\mathbb{E}\|\phi^\varepsilon(T)\|^2 &\le \mathbb{E}\|\phi^\varepsilon(0)\|^2 + 2\mathbb{E}\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi^\varepsilon(s)),\phi^\varepsilon(s)\big)\mathrm{d}s \\ &+ 2\mathbb{E}\int_0^T \big(F^\varepsilon(s),\phi^\varepsilon(s)\big)\mathrm{d}s + \varepsilon\mathbb{E}\int_0^T \|S^\varepsilon(s)\|^2_{\mathcal{L}_Q}\mathrm{d}s.\end{aligned}$$

Using the above two estimates in (3.8),

$$\begin{aligned}&2\mathbb{E}\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi^\varepsilon(s)),\phi^\varepsilon(s)\big)\mathrm{d}s + 2\mathbb{E}\int_0^T \big(F^\varepsilon(s),\phi^\varepsilon(s)\big)\mathrm{d}s + \varepsilon\mathbb{E}\int_0^T \|S^\varepsilon(s)\|^2_{\mathcal{L}_Q}\mathrm{d}s \\ &\le \liminf_n \mathbb{E}\Big\{2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi_n^\varepsilon(s)),\phi_n^\varepsilon(s)\big)\mathrm{d}s + 2\int_0^T \big(f(s,\phi_n^\varepsilon(s)),\phi_n^\varepsilon(s)\big)\mathrm{d}s \\ &\quad + \varepsilon\int_0^T \|\sigma_n(s,\phi_n^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s\Big\}.\end{aligned} \tag{3.9}$$

For a corresponding test function ψ, we have

$$\begin{aligned}&\Big\{2\mathbb{E}\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla(\phi_n^\varepsilon(s)-\psi(s))),\phi_n^\varepsilon(s)-\psi(s)\big)\mathrm{d}s \\ &+ \varepsilon\mathbb{E}\int_0^T \|\sigma_n(s,\phi_n^\varepsilon(s))-\sigma_n(s,\psi(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s \\ &+ 2\mathbb{E}\int_0^T \big(f(s,\phi_n^\varepsilon(s))-f(s,\psi(s)),\phi_n^\varepsilon(s)-\psi(s)\big)\mathrm{d}s\Big\} \le 0.\end{aligned} \tag{3.10}$$

By subtracting (3.10) from right hand side of (3.9), we get

$$2\mathbb{E}\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi^\varepsilon(s)),\phi^\varepsilon(s)\big)\mathrm{d}s$$

$$
\begin{aligned}
&+2\mathbb{E}\int_0^T \big(F^\varepsilon(s), \phi^\varepsilon(s)\big)\mathrm{d}s + \varepsilon\mathbb{E}\int_0^T \|S^\varepsilon(s)\|^2_{\mathcal{L}_Q}\mathrm{d}s\\
&\le \liminf_n \mathbb{E}\Big\{2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi_n^\varepsilon(s)), \psi(s)\big)\mathrm{d}s\\
&+2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\psi(s)), \phi_n^\varepsilon(s)\big)\mathrm{d}s\\
&-2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\psi(s)), \psi(s)\big)\mathrm{d}s\\
&+2\int_0^T \big(f(s,\psi(s)), \phi_n^\varepsilon(s)-\psi(s)\big)\mathrm{d}s\\
&+2\int_0^T \big(f(s,\phi_n^\varepsilon(s)), \psi(s)\big)\mathrm{d}s\\
&+\varepsilon\int_0^T \big(2\sigma_n(s,\phi_n^\varepsilon(s)) - \sigma_n(s,\psi(s)), \sigma_n(s,\psi(s))\big)\mathrm{d}s\Big\}.
\end{aligned}
$$

Applying limit as $n\to\infty$,

$$
\begin{aligned}
2\mathbb{E}&\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi^\varepsilon(s)), \phi^\varepsilon(s)\big)\mathrm{d}s\\
&+2\mathbb{E}\int_0^T \big(F^\varepsilon(s), \phi^\varepsilon(s)\big)\mathrm{d}s + \varepsilon\mathbb{E}\int_0^T \|S^\varepsilon(s)\|^2_{\mathcal{L}_Q}\mathrm{d}s\\
&\le \liminf_n \mathbb{E}\Big\{2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi^\varepsilon(s)), \psi(s)\big)\mathrm{d}s\\
&+2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\psi(s)), \phi^\varepsilon(s)\big)\mathrm{d}s\\
&-2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\psi(s)), \psi(s)\big)\mathrm{d}s\\
&+2\int_0^T \big(f(s,\psi(s)), \phi^\varepsilon(s)-\psi(s)\big)\mathrm{d}s\\
&+2\int_0^T \big(F^\varepsilon(s), \psi(s)\big)\mathrm{d}s\\
&+\varepsilon\int_0^T \big(2S^\varepsilon(s) - \sigma(s,\psi(s)), \sigma(s,\psi(s))\big)\mathrm{d}s\Big\}.
\end{aligned}
$$

and rearranging,

$$
\begin{aligned}
\mathbb{E}\Big\{&2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi^\varepsilon(s)), \phi^\varepsilon(s)-\psi(s)\big)\mathrm{d}s\\
&-2\int_0^T \big(\mathrm{div}(m\partial_s^{1-\alpha}\nabla\psi(s)), \phi^\varepsilon(s)-\psi(s)\big)\mathrm{d}s\\
&+2\int_0^T \big(F^\varepsilon(s)-f(s,\psi(s)), \phi^\varepsilon(s)-\psi(s)\big)\mathrm{d}s\\
&+\varepsilon\int_0^T \|S^\varepsilon(s)-\sigma(s,\psi(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s\Big\}\le 0.
\end{aligned}
$$

Taking $\psi=\phi^\varepsilon$ in the above inequality, we get $S^\varepsilon(t)=\sigma(t,\phi^\varepsilon(t))$. Now we get

$$
2\mathbb{E}\Big[\int_0^T \big(F^\varepsilon(s)-f(s,\psi(s)), \phi^\varepsilon(s)-\psi(s)\big)\mathrm{d}s\Big]\le 0. \tag{3.11}
$$

Let $\psi=\phi^\varepsilon-\mu\tilde\psi$, for $\mu>0$. From Assumption 2.1, since f is Lipschitz,

$$
\big(f(s,\psi(s))-f(s,\phi^\varepsilon(s)), \mu\tilde\psi(s)\big)\le \mu^2 C\|\tilde\psi(s)\|^2.
$$

Then from (3.11),

$$
\mathbb{E}\Big[\int_0^T \Big\{2\mu\langle F^\varepsilon(s)-f(s,\phi^\varepsilon(s)), \tilde\psi(s)\rangle\Big\}\mathrm{d}s\Big]\le 0.
$$

Since $\tilde\psi$ is arbitrary, $F^\varepsilon(t)=f(t,\phi^\varepsilon(t))$. Hence, from the convergence we get,

$$
\begin{aligned}
\phi^\varepsilon(T) &= \phi_0 + \int_0^T \mathrm{div}(m\partial_s^{1-\alpha}\nabla\phi^\varepsilon(s))\mathrm{d}s\\
&\quad+\int_0^T f(s,\phi^\varepsilon(s))\mathrm{d}s + \sqrt{\varepsilon}\int_0^T \sigma(s,\phi^\varepsilon(s))\mathrm{d}W(s).\\
\mathbb{E}&\Big(\|\phi^\varepsilon\|^2_{\mathbb{L}^\infty(0,T;V)}\Big)\le \varepsilon C(\mathbb{E}\|\phi_0\|^2_V + \|f\|_{\mathbb{L}^\infty([0,T],H)}).
\end{aligned}
$$

Hence the existence of solution is proved.

Step 3

In order to prove uniqueness, consider $\psi^\varepsilon\in\mathbb{H}^\alpha(0,T;V')\cap\mathbb{L}^\infty([0,T],V)$ be another solution of (2.1). Then, $\vartheta=\phi^\varepsilon-\psi^\varepsilon$ satisfies

$$
\begin{aligned}
\mathrm{d}\vartheta(t) &= \mathrm{div}(m\partial_t^{1-\alpha}\nabla\vartheta(t)) + [f(t,\phi^\varepsilon(t))\\
&\quad- f(t,\psi^\varepsilon(t))]\mathrm{d}t + \sqrt{\varepsilon}[\sigma(t,\phi^\varepsilon)-\sigma(t,\psi^\varepsilon)\mathrm{d}W].
\end{aligned}
$$

Applying Itô's formula, using (3.3) and Assumptions 2.1 and 2.2,

$$
\begin{aligned}
&\|\vartheta(t)\|^2 + mg_\alpha * \|\nabla\vartheta(t)\|^2\\
&\le \|\vartheta(0)\|^2 + mg_\alpha * \|\nabla\vartheta(0)\|^2\\
&\quad+2\int_0^t \Big(f(s,\phi^\varepsilon(s))-f(s,\psi^\varepsilon(s)), \phi^\varepsilon(s)-\psi^\varepsilon(s)\Big)\mathrm{d}s\\
&\quad+\varepsilon\int_0^t \|\sigma(s,\phi^\varepsilon(s))-\sigma(s,\psi^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s\\
&\quad+2\sqrt{\varepsilon}\int_0^t \big(\vartheta(s), [\sigma(s,\phi^\varepsilon(s))-\sigma(s,\psi^\varepsilon(s))]\mathrm{d}W(s)\big).
\end{aligned}
$$

Using Assumptions and Young's inequality,

$$
\begin{aligned}
&\|\vartheta(t)\|^2 + mg_\alpha * \|\nabla\vartheta(t)\|^2\\
&\le \|\vartheta(0)\|^2 + mg_\alpha * \|(\nabla\vartheta)(0)\|^2\\
&\quad+\int_0^t \Big[\|f(s,\phi^\varepsilon(s))-f(s,\psi^\varepsilon(s))\|^2 + \|\vartheta(s)\|^2\Big]\mathrm{d}s\\
&\quad+\varepsilon K_2\int_0^t \|\nabla\vartheta(s)\|^2\mathrm{d}s\\
&\quad+2\sqrt{\varepsilon}\int_0^t \big(\vartheta(s), \big[\sigma(s,\phi^\varepsilon(s))-\sigma(s,\psi^\varepsilon(s))\big]\mathrm{d}W(s)\big).
\end{aligned}
$$

Taking supremum over time T and then taking expectation,

$$\mathbb{E}\sup_{0\le t\le T}\left\{\|\vartheta(t)\|^2+mg_\alpha*\|\nabla\vartheta(t)\|^2\right\}$$
$$\le\mathbb{E}\bigg\{\|\vartheta(0)\|^2+mg_\alpha*\|(\nabla\vartheta)(0)\|^2$$
$$+\int_0^T\left[\|f(s,\phi^\varepsilon(s))-f(s,\psi^\varepsilon(s))\|^2+\|\vartheta(s)\|^2\right]\mathrm{d}s\bigg\}$$
$$+\varepsilon K_2\mathbb{E}\int_0^T\|\nabla\vartheta(s)\|^2\mathrm{d}s+2\sqrt{\varepsilon}\mathbb{E}\sup_{0\le t\le T}$$
$$\int_0^t\big(\vartheta(s),\big[\sigma(s,\phi^\varepsilon(s))-\sigma(s,\psi^\varepsilon(s))\big]\mathrm{d}W(s)\big).$$

Using Burkholder–Davis–Gundy inequality and simplifying,

$$2\sqrt{\varepsilon}\mathbb{E}\sup_{0\le t\le T}\int_0^t\Big(\big[\sigma(s,\phi^\varepsilon(s))-\sigma(s,\psi^\varepsilon(s))\big]\mathrm{d}W(s),\vartheta(s)\Big)$$
$$\le 2\sqrt{\varepsilon}C_1\mathbb{E}$$
$$\left\{\sup_{0\le t\le T}\|\vartheta(t)\|^2\int_0^T\|\sigma(s,\phi^\varepsilon(s))-\sigma(s,\psi^\varepsilon(s))\|^2_{\mathcal{L}_Q}\mathrm{d}s\right\}^{1/2}$$
$$\le\frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\|\vartheta(t)\|^2+\varepsilon C^2K_2\mathbb{E}\int_0^T\|\nabla\vartheta(s)\|^2\mathrm{d}s.$$

Combining, we get

$$\mathbb{E}\sup_{0\le t\le T}\left\{\|\vartheta(t)\|^2+2m\|\nabla\vartheta(t)\|^2\right\}\le C(T)\mathbb{E}\|\vartheta(0)\|_V^2.$$

Hence $\|\vartheta(t)\|^2=0$ for all $t\in[0,T]$ since $\vartheta(0)=0$. □

Thus, the existence and uniqueness is proved in $\mathbb{H}^\alpha(0,T;V')\cap\mathbb{L}^\infty(0,T;V)$.

4 Conclusion

A stochastic time-fractional equation that models subdiffusion process is considered, and a study on existence and uniqueness of its solution is carried out. For this purpose, the nonlinear source term (f) and the noise coefficient (σ) are assumed to essentially satisfy Lipschitz continuity. A series of inequalities are used to arrive at our results. Burkholder–Davis–Gundy inequality is formally used to reduce the stochastic integral, and fractional Gronwall–Bellman-type inequality is used in the place of integer order Gronwall lemma. Having established the existence of solution, analysis of many further concepts is open. This leads to numerical computation of solution of this stochastic fractional equation. Further, one can study large and moderate deviation principles for this stochastic equation which analyses the behavior of the system for larger time.

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Conflict of interest The authors declare that they have no potential competing interests that are relevant to the content of this article.

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