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A discussion of stability analysis for systems of differential equations with multiple and distributed delays

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Abstract
In this article, we consider a class of systems of multiple delay differential equations (MDDEs). We first define a characteristic matrix equation that can be used to analyze the stability of the equilibrium of a system of MDDEs. Then we construct a matrix based on the coefficients of the characteristic matrix equation and use the spectrum of this matrix to derive necessary and sufficient conditions for the system to be stable. Next we discuss a comparison of the stability equivalency between a system of delay differential equations (DDEs) to the system of MDDEs and relate our results to distributed delay systems (DDSs). Numerical examples are given to justify our theory.

Keywords Delay differential equations · Systems · Multiple delays · Distributed · Numerical methods

1 Introduction

Studying the stability of delay differential equations (DDEs) has become increasingly important in recent times; see [1–6] and the growing body of literature in the field. In the past few years, there has been a significant amount of research focused on MDDEs; see [3, 7–12], as well as their applications in various fields such as the dynamics of electrical power systems, macroeconomic models, electricity market models, and more, as outlined in [8, 13–19]. In addition to these areas, there has been an increasing interest in studying other types of systems that exhibit memory effects, including systems of fractional differential and difference equations, as well as systems of fractional nabla difference equations, as highlighted in [20–22]. Conventional approaches for the stability analysis of DDEs are based on Lyapunov functional method (LFM) and techniques that require the solution of a linear matrix inequality (LMI) problem; see [23–25]. The complexity to construct the Lyapunov function and the heavy computational burden to solve the LMI problem limit the application of LFMs on engineering fields. Moreover, as LFMs provide only sufficient but not necessary conditions for system stability, they tend to be conservative. There also exist a variety of frequency-domain approaches to solve the stability of DDEs, [2, 4, 26–30]. Most of these techniques are based on the solution of an eigenvalue problem. This consists in estimating the dominant modes of the DDEs through the solution of the characteristic equation of the system. In [6, 10, 16, 17], a general eigenvalue analysis approach is developed to solve the stability of large system described by a set of delay differential algebraic equations (DDAEs). Compared to LFMs, eigenvalue-based approaches are less computationally intensive and provide a more accurate stability analysis. For this reason, we consider an eigenvalue-based approach also in this paper.

In this article, we will firstly provide stability criteria for a class of systems of DDEs with multiple delays based on eigenvalue analysis. Additionally, we explore the relationship between the stability of a DDE system with one delay and a system of MDDEs. This discussion offers a fresh perspective and novel insights that may inspire future research in this field. We are interested in the evaluation of the small signal stability of a nonlinear system of MDDEs in the following form:

$$\dot{Y} = f(Y, Y(t-\tau_1), Y(t-\tau_2), \ldots, Y(t-\tau_n)),$$

where $\tau_i > 0$ is constant time delay, $Y \in \mathbb{R}^m$ are the state variables, $f : \mathbb{R}^{(n+1)\times m} \mapsto \mathbb{R}^m$ are the differential equa-

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tions and can be implicit with its partial derivatives at an equilibrium point to be singular matrices, i.e., $\det\left(\frac{\partial f}{\partial K}\right) = 0$, $\forall K = Y, Y(t-\tau_1), Y(t-\tau_2), \dots, Y(t-\tau_n)$ and $\dot{Y}$ not to be zero columns.

We consider only *small disturbances*, e.g., disturbances whose effects on the stability of a given equilibrium point can be studied through the linearized set of the equations that model the system. If we consider small disturbances, e.g., disturbances whose effects on the stability of a given equilibrium can be studied through the linearized set of the equations that model the system. The linearized systems of MDDEs have the form:

$$\dot{Y} = f_Y \delta Y + f_{Y_{d_1}} \delta Y_{d_1} + \dots + f_{Y_{d_n}} \delta Y_{d_n}.$$

where $\delta Y = Y - Y_{eq}$, Y_{eq} is equilibrium and $f_{\dot{Y}}$ full rank at an equilibrium. The characteristic equation is then given by

$$\det\left(\lambda f_{\dot{Y}} - f_Y - \sum_{k=1}^{n} e^{-\lambda\tau_k} f_{Y_{d_k}}\right) = 0,$$

and its characteristic roots will provide the necessary information for small signal stability of the system of MDDE. To sum up the small signal stability of the nonlinear MMDE at a given equilibrium can be studied from the following linear system MDDE:

$$\dot{Y}(t) = A_0 Y(t) + \sum_{k=1}^{n} A_k Y(t-\tau_k). \tag{1}$$

with characteristic equation

$$\det\left(\lambda I_m - A_0 - \sum_{k=1}^{n} e^{-\lambda\tau_k} A_k\right) = 0. \tag{2}$$

where

$$A_0 = f_Y, \quad A_k = f_{Y_{d_k}}.$$

A necessary and sufficient condition for the equilibrium solution to be asymptotically stable is that the roots of the characteristic equation all have negative real parts; see [3].

In the remainder of the paper I_m denotes the identity matrix $m \times m$, $0_{i,j}$ the zero matrix of i rows, j columns, and $\|\cdot\|$ a natural norm. The remainder of this paper is organized as follows. In Sect. 2, we present a theorem that establishes a stability criterion for MDDE systems in the form of (1). In Sect. 3, we explore the stability equivalence between systems of DDEs and systems of MDDEs and discuss the relevance of our findings for DDSs. Section 4 is devoted to numerical examples, which demonstrate the effectiveness of our approach. Finally, we summarize our key findings in the Conclusions section.

2 Stability analysis of multiple delay systems

In this section, we present our main results. We will use the following definition:

Definition 2.1 A square matrix A is called *stable* if every eigenvalue of A has strictly negative real part.

Initially we consider $\tau_k = k\tau$ in (1) and provide the following Theorem:

Theorem 2.1 *We consider system (1) with $\tau_k = k\tau$. Then the following conditions must hold for delay-independent stability of (1):*

1. *The matrix A_0 is stable;*
2. *The matrix $A_0 + \sum_{k=1}^{n} A_k$ is stable;*
3. *The spectral radius of A_ω is less than 1, $\forall \omega > 0$, where*

$$A_\omega = \begin{bmatrix} (j\omega I_m - A_0)^{-1}A_1 & (j\omega I_m - A_0)^{-1}A_2 & \dots & (j\omega I_m - A_0)^{-1}A_{n-1} & (j\omega I_m - A_0)^{-1}A_n \\ I_m & 0_{m,m} & \dots & 0_{m,m} & 0_{m,m} \\ 0_{m,m} & I_m & \dots & 0_{m,m} & 0_{m,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{m,m} & 0_{m,m} & \dots & I_m & 0_{m,m} \end{bmatrix}.$$

Proof If $\tau \to \infty$ then (2) takes the form

$$\det(\lambda I_m - A_0) = 0.$$

Hence, the matrix A_0 has to be stable in order to have stability for (1) at the equilibrium state. If $\tau = 0$ then (2) takes the form

$$\det(\lambda I_m - A_0 - A_1 - \dots - A_n) = 0,$$

which means that the matrix $A_0 + \sum_{k=1}^{n} A_k$ has to be stable in order to have stability for (1) at the equilibrium state. By applying the Fourier transform $\mathcal{F}(Y) = \Psi(\omega)$ into (1), we get:

$$
\begin{aligned}
j\omega\Psi(\omega) &= A_0\Psi(\omega) + A_1 e^{-j\omega\tau}\Psi(\omega)\\
&\quad + A_1 e^{-j\omega 2\tau}\Psi(\omega) + \cdots + A_n e^{-j\omega n\tau}\Psi(\omega),
\end{aligned}
$$

or, equivalently,

$$
[j\omega I_m - A_0 - \sum_{k=1}^{n} A_k e^{-j\omega k\tau}]\Psi(\omega) = 0_{m,1}.
$$

Then $\det(j\omega I_m - A_0 - \sum_{k=1}^{n} A_k e^{-j\omega k\tau}) = 0$ is the characteristic equation of (1). We adopt the following notation:

$$
\begin{aligned}
&x_1(t) = Y(t),\\
&x_2(t) = Y(t-\tau),\\
&x_3(t) = Y(t-2\tau),\\
&\ldots,\\
&x_{n-1}(t) = Y(t-(n-2)\tau),\\
&x_n(t) = Y(t-(n-1)\tau).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&x_1(t-\tau) = Y(t-\tau),\\
&x_2(t-\tau) = Y(t-2\tau),\\
&x_3(t-\tau) = Y(t-3\tau),\\
&\vdots\\
&x_{n-1}(t) = Y(t-(n-1)\tau)\\
&A_n x_n(t-\tau) = A_n Y(t-n\tau) = \dot{Y} - \sum_{k=0}^{n-1} A_k Y(t-k\tau),
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&x_1(t-\tau) = x_2(t),\\
&x_2(t-\tau) = x_3(t),\\
&x_3(t-\tau) = x_4(t),\\
&\vdots\\
&x_{n-1}(t-\tau) = x_n(t)\\
&A_n x_n(t-\tau) = x_1'(t) - \sum_{k=0}^{n-1} A_k x_{k+1}(t),
\end{aligned}
$$

or, equivalently, in matrix form

$$
GX(t-\tau) = F_1 X'(t) + F_2 X(t),
$$

where

$$
X(t) = \begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\\ \vdots\\ x_n(t) \end{bmatrix},
$$

$$
G = \begin{bmatrix}
I_m & 0_{m,m} & \ldots & 0_{m,m} & 0_{m,m}\\
0_{m,m} & I_m & \ldots & 0_{m,m} & 0_{m,m}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0_{m,m} & 0_{m,m} & \ldots & I_m & 0_{m,m}\\
0_{m,m} & 0_{m,m} & \ldots & 0_{m,m} & A_n
\end{bmatrix},
$$

and F_1, F_2 are given by

$$
F_1 = \begin{bmatrix}
0_{m,m} & 0_{m,m} & 0_{m,m} & \ldots & 0_{m,m}\\
0_{m,m} & 0_{m,m} & 0_{m,m} & \ldots & 0_{m,m}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0_{m,m} & 0_{m,m} & 0_{m,m} & \ldots & 0_{m,m}\\
I_m & 0_{m,m} & 0_{m,m} & \ldots & 0_{m,m}
\end{bmatrix},
$$

$$
F_2 = \begin{bmatrix}
0_{m,m} & I_m & 0_{m,m} & \ldots & 0_{m,m}\\
0_{m,m} & 0_{m,m} & I_m & \ldots & 0_{m,m}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0_{m,m} & 0_{m,m} & 0_{m,m} & \ldots & I_m\\
-A_0 & -A_1 & -A_2 & \ldots & -A_{n-1}
\end{bmatrix}.
$$

It is worth noting that the system under consideration is a set of DDAEs. Therefore, since we have established the equivalence of the two systems, we can conclude that the characteristic equation of (1) is

$$
\det(j\omega F_1 + F_2 - e^{-j\omega\tau}G) = 0.
$$

Let $F_\omega := F(j\omega) = j\omega F_1 + F_2$. We have that:

$$
F_\omega = \begin{bmatrix}
0_{m,m} & I_m & 0_{m,m} & \ldots & 0_{m,m}\\
0_{m,m} & 0_{m,m} & I_m & \ldots & 0_{m,m}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0_{m,m} & 0_{m,m} & 0_{m,m} & \ldots & I_m\\
j\omega I_m - A_0 & -A_1 & -A_2 & \ldots & -A_{n-1}
\end{bmatrix}.
$$

Then the characteristic equation of the DDAEs can be written as:

$$
\det(F_\omega - e^{-j\omega\tau}G) = 0.
$$

Let $\rho(\cdot)$ be spectral radius of a matrix. Using Theorem 2.1 in [3] we get that $\forall\omega > 0$, if $\rho(F_\omega^{-1}[e^{-j\omega\tau}G]) < 1$, then the set of DDAEs is stable independent of delay, where

$$F_\omega^{-1} = \begin{bmatrix} (j\omega I_m - A_0)^{-1}A_1 & (j\omega I_m - A_0)^{-1}A_2 & \dots & (j\omega I_m - A_0)^{-1}A_{n-1} & (j\omega I_m - A_0)^{-1} \\ I_m & 0_{m,m} & \dots & 0_{m,m} & 0_{m,m} \\ 0_{m,m} & I_m & \dots & 0_{m,m} & 0_{m,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{m,m} & 0_{m,m} & \dots & I_m & 0_{m,m} \end{bmatrix},$$

and

$$F_\omega^{-1}G = \begin{bmatrix} (j\omega I_m - A_0)^{-1}A_1 & (j\omega I_m - A_0)^{-1}A_2 & \dots & (j\omega I_m - A_0)^{-1}A_{n-1} & (j\omega I_m - A_0)^{-1}A_n \\ I_m & 0_{m,m} & \dots & 0_{m,m} & 0_{m,m} \\ 0_{m,m} & I_m & \dots & 0_{m,m} & 0_{m,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{m,m} & 0_{m,m} & \dots & I_m & 0_{m,m} \end{bmatrix}.$$

Note that $\forall \omega > 0$:

$$\rho(F_\omega^{-1}[e^{-j\omega\tau}G]) = \rho(e^{-j\omega\tau}F_\omega^{-1}G) = |e^{-j\omega\tau}|\rho(F_\omega^{-1}G) \\ = \rho(F_\omega^{-1}G).$$

Let $A_\omega = F_\omega^{-1}G$. Since the set of DDAEs is equivalent to (1), we have that $\forall\omega > 0$ if $\rho(A_\omega) < 1$ holds then (1) is stable independent of delay. □

3 Discussion on the equivalency of MDDEs to a system of DDEs and DDSs

We consider now the system of DDEs:

$$\dot{Y}(t) = A_0Y(t) + AY(t-\tau). \tag{3}$$

where $A, A_i \in \mathbb{R}^{m\times m}$, $i =, 1, \dots, n$, $Y : [0, +\infty] \to \mathbb{R}^{m\times 1}$, and $A_0 \in \mathbb{R}^{m\times m}$ is stable matrix. In the following discussion we will attempt to relate systems (1), (3) for small disturbances, i.e., $\left\|\sum_{k=1}^n A_kY(s-\tau_k) - A\hat{Y}(s-\tau)\right\| < \epsilon$, and investigate their stability such that if (3) is asymptotically stable, then (1) is also asymptotically stable. An implicit solution of system (1) is given by:

$$Y(t) = e^{A_0t}c + \int_0^t e^{A_0(t-s)}\sum_{k=1}^n A_kY(s-\tau_k)\mathrm{d}s,$$

or, equivalently, if we apply the Weierstrass canonical form; see [22],

$$Y(t) = Pe^{Jt}Qc + \int_0^t Pe^{J(t-s)}Q\sum_{k=1}^n A_kY(s-\tau_k)\mathrm{d}s,$$

Similarly, an implicit solution of (3) is given by:

$$\hat{Y}(t) = e^{A_0t}c + \int_0^t e^{A_0(t-s)}A\hat{Y}(s-\tau)\mathrm{d}s,$$

whereby applying the Weierstrass canonical form we get:

$$\hat{Y}(t) = Pe^{Jt}Qc + \int_0^t Pe^{J(t-s)}QA\hat{Y}(s-\tau)\mathrm{d}s.$$

The matrices P, Q have as columns the left, and right respectively linear independent eigenvectors of A_0, while J is the Jordan matrix of the eigenvalues of A_0. By subtracting these two solutions we get:

$$Y(t) - \hat{Y}(t) \\ = \int_0^t Pe^{J(t-s)}Q\sum_{k=1}^n A_kY(s-\tau_k)\mathrm{d}s \\ - \int_0^t Pe^{J(t-s)}QA\hat{Y}(s-\tau)\mathrm{d}s,$$

or, equivalently,

$$Y(t) - \hat{Y}(t) \\ = \int_0^t Pe^{J(t-s)}Q\left[\sum_{k=1}^n A_kY(s-\tau_k) - A\hat{Y}(s-\tau)\right]\mathrm{d}s.$$

By applying a natural norm we get:

$$\|Y(t) - \hat{Y}(t)\| \\ = \left\|\int_0^t Pe^{J(t-s)}Q\left[\sum_{k=1}^n A_kY(s-\tau_k) - A\hat{Y}(s-\tau)\right]\mathrm{d}s\right\|,$$

whereby using the property of the norm we have that:

$$\|Y(t) - \hat{Y}(t)\| \leq \epsilon \|P\| \|Q\| \int_0^t \|e^{J(t-s)}\| \mathrm{d}s. \tag{4}$$

If λ_i is an eigenvalue of A_0 with algebraic multiplicity p_i, the Jordan matrix has the form:

$$J := J_{p_1}(\lambda_1) \oplus \cdots \oplus J_{p_\nu}(\lambda_\nu),$$

where

$$J_{p_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & \dots & 0 & 0 \\ 0 & \lambda_i & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix} \in \mathbb{C}^{p_i \times p_i}, \quad i = 1, 2, ..., \nu.$$

In addition:

$$e^{Jt} := e^{J_{p_1}(\lambda_1)t} \oplus \cdots \oplus e^{J_{p_\nu}(\lambda_\nu)t},$$

where

$$e^{J_{p_i}(\lambda_i t)} = \begin{bmatrix} e^{\lambda_i t} & e^{\lambda_i t} t & e^{\lambda_i t}\frac{t^2}{2!} & \dots & e^{\lambda_i t}\frac{t_i^p}{p_i!} \\ 0 & e^{\lambda_i t} & e^{\lambda_i t} t & \dots & e^{\lambda_i t}\frac{t^{p_i-1}}{(p_i-1)!} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\lambda_i t} & e^{\lambda_i t} t \\ 0 & 0 & \dots & 0 & e^{\lambda_i t} \end{bmatrix} \in \mathbb{C}^{p_i \times p_i},$$
$$i = 1, 2, \dots, \nu.$$

By taking the norm $\|\cdot\|_1$ of e^{Jt} we have

$$\left\|e^{Jt}\right\|_1 = \max_{1\leq i\leq \nu} \left\|e^{J_{p_i}(\lambda_i t)}\right\|_1 = \max_{1\leq i\leq \nu} \sum_{l=0}^{p_i} e^{\lambda_i t}\frac{t^l}{l!},$$

Hence

$$\begin{aligned} \left\|e^{J(t-s)}\right\|_1 &= \max_{1\leq i\leq \nu} \left\|e^{J_{p_i}(\lambda_i (t-s))}\right\|_1 \\ &= \max_{1\leq i\leq \nu} \sum_{l=0}^{p_i} e^{\lambda_i (t-s)} \frac{(t-s)^l}{l!}. \end{aligned} \tag{5}$$

Let

$$\epsilon \|P\| \|Q\| = M, \quad \forall 0 \leq t. \tag{6}$$

By using (5), (6) into (4) we get:

$$\|Y(t) - \hat{Y}(t)\| \leq M \max_{1\leq i\leq \nu} \sum_{l=0}^{p_i} \int_0^t e^{\lambda_i t} \frac{t^l}{l!} \mathrm{d}s,$$

and hence for $\lambda_i < 0$ we get:

$$\|Y(t) - \hat{Y}(t)\| \to 0, \quad \text{for} \quad t \to +\infty.$$

In the following remark we discuss the possibility of $A = g(A_i)$ for certain cases.

Remark 3.1 We discussed the stability equivalency between a system of DDEs with one delay and a system of MDDEs. The significance of this aspect was to devise a new concept that can provide novel perspectives for researchers. By exploring this idea in our discussion, we aim to pave the way for future research to advance the concept of stability equivalence. We considered only *small disturbances*. Let in (1), (3), $\tau_k = \tau \pm \tilde{\epsilon}_k$, $k = 1, 2, \dots, n$, with $0 < \tau, \tilde{\epsilon}_k << 1$. Then $s\tau_k \cong s\tau \pm \epsilon_k, k = 1, 2, \dots, n, 0 < \epsilon_k \ll 1$, and $e^{-s\tau_k} \cong c_k e^{-s\tau}$, $k = 1, 2, \dots, n$, $1 < c_k \ll 2$. Hence in this special case by applying the Laplace transform into (1), (3) we can observe that the real parts of the rightmost eigenvalues of (3) should also converge to that of (1) if $A = \sum_{k=1}^n c_k A_k$. Consequently, one of the practical options for A is $A = g(A_k) = c \sum_{k=0}^n A_k$, where $c \in \mathbb{R}^+$. The proof of Theorem 2.5 in [3] provides the *necessary and sufficient condition* to ensure that for the special case that $A = g(A_k) = \sum_{k=0}^n A_k$, there exists a $\tau \leq \bar{\tau}$ such that the systems (1), (3) have the same stability assertion. Hence, for the case that $A = g(A_k) = c \sum_{k=0}^n A_k$, particularly $c = 1$ is an appropriate selection to apply the idea described above, and one can obtain that the MDDE system (1) is asymptotically stable if the single delay system (3) with $A = \sum_{k=0}^n A_k$ and $\tau \leq \bar{\tau} = \dfrac{\sum_{k=1}^n \tau_k}{n}$ is asymptotically stable.

The results discussed and obtained in Sect. 3 can also be used for the stability analysis of DDSs in the following form:

$$\dot{Y}(t) = A_0 Y(t) + A_1 \int_{\tau_{\min}}^{\tau_{\max}} \pi(\xi) Y(t-\xi) \mathrm{d}\xi, \tag{7}$$

where $\pi(\cdot)$ is the probability distribution of ξ that satisfies the following property:

$$\frac{1}{\tau_{\max} - \tau_{\min}} \int_{\tau_{\min}}^{\tau_{\max}} \pi(\xi) \mathrm{d}\xi = 1,$$

and $\pi(\xi)$ is a non-negative function.

From [28, 31, 32], we know that the DDS in the form of (7) has the same spectrum as the comparison system:

$$\dot{Y}(t) = A_0 Y(t) + A_1 \kappa h \lim_{z_m \to \infty} \sum_{z=0}^{z_m} \pi(\Xi_z) Y(t - \Xi_z),$$

where

$$h = \frac{\tau_{\max} - \tau_{\min}}{z_m},$$
$$\Xi_z = \tau_{\min} + z\,h,$$

and κ is a weighting parameter decided by the interpolation method used.

With z_m truncated at a finite value, one obtains the following system:

$$\dot{Y}(t) = A_0 Y(t) + A_1 \kappa\, h \sum_{z=0}^{z_m} \pi(\Xi_z) Y(t - \Xi_z)\ , \tag{8}$$

which is a system of MDDEs in the form of (1), and its spectrum can be studied according to Theorem 2.1, and the discussions in this section. Hence, with a proper interpolation method and a fixed z_m, the critical eigenvalues of the DDS (7) converge to those of system (8).

4 Numerical examples

In this section, we provide numerical examples to illustrate the theory presented in Sects. 2 and 3.

Example 4.1 We consider the system of MDDEs (1) for $n = 2$ and

$$A_0 = \begin{bmatrix} -200 & 0 \\ 0 & -100 \end{bmatrix},$$
$$A_i = i \cdot 10^{-3} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad i = 1, 2.$$

To utilize Theorem 2.1, the following steps should be followed. Step 1 we compute the matrix $j\omega - A_0$:

$$j\omega - A_0 = \begin{bmatrix} j\omega + 200 & 0 \\ 0 & j\omega + 100 \end{bmatrix}.$$

Step 2 we compute the matrices $(j\omega - A_0)^{-1} A_1$, $(j\omega - A_0)^{-1} A_2$:

$$(j\omega - A_0)^{-1} A_1 = \begin{bmatrix} -\frac{0.0001}{j\omega+200} & \frac{0.0001}{j\omega+200} \\ 0 & -\frac{0.0001}{j\omega+100} \end{bmatrix},$$

and

$$(j\omega - A_0)^{-1} A_2 = \begin{bmatrix} -\frac{0.0002}{j\omega+200} & \frac{0.0002}{j\omega+200} \\ 0 & -\frac{0.0002}{j\omega+100} \end{bmatrix}.$$

Table 1 The stability assertions of (9) according to Theorem 2.1

Scenario	a_0	a_1	a_2	a_3	$\rho(A_\omega)$
S1	-0.1	-5	-2	-1	49.6
S2	-1	-5	-2	-1	4.6135
S3	-10	-5	-2	-1	0.5
S4	-10	-5	-2	-5	0.8977
S5	-10	-5	-2	-10	1.1194

Step 3 we form the matrix A_ω:

$$A_\omega = \begin{bmatrix} -\frac{0.0001}{j\omega+200} & \frac{0.0001}{j\omega+200} & -\frac{0.0002}{j\omega+200} & \frac{0.0002}{j\omega+200} \\ 0 & -\frac{0.0001}{j\omega+100} & 0 & -\frac{0.0002}{j\omega+100} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Obviously the matrices A_0, $A_0 + \sum_{k=1}^{2} A_k$ are both stable since all their eigenvalues real and negative; in addition, $\rho(A_\omega) = 0.0014 < 1$ and hence from Theorem 2.1 the system of MDDEs is delay independent stable.

Example 4.2 We consider now the DDS (7) with:

$$\tau_{\max} = \tilde{\tau}, \quad \tau_{\min} = 0, \quad \pi(\xi) = \frac{2\xi}{\tilde{\tau}}.$$

Then if $z_m = 2$,, for the comparison system (8) we have:

$$h = \frac{\tilde{\tau}}{2}, \quad \Xi_z = z\,\frac{\tilde{\tau}}{2}, \quad \pi(\Xi_z) = z.$$

Hence by setting $\kappa = \frac{2}{\tilde{\tau}}$ the DDS (8) takes the form:

$$\dot{Y}(t) = A_0 Y(t) + A_1 \frac{2}{\tilde{\tau}} \frac{\tilde{\tau}}{2} \sum_{z=0}^{2} z Y\left(t - z\frac{\tilde{\tau}}{2}\right),$$

or, equivalently, by setting $\tau = \frac{\tilde{\tau}}{2}$:

$$\dot{Y}(t) = A_0 Y(t) + \sum_{i=0}^{2} i A_1 Y(t - i\tau),$$

which is the system of MDDEs (1) for $n = 2$ and $A_i = i A_1$. Let

$$A_0 = \begin{bmatrix} -200 & 0 \\ 0 & -100 \end{bmatrix}, \quad A_1 = \cdot 10^{-3} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad i = 1, 2.$$

Then the matrices A_0, $A_0 + \sum_{k=1}^{2} A_k$ are both stable since all their eigenvalues real and negative, and in addition $\rho(A_\omega) = 0.0014 < 1$. Thus, by applying Theorem 2.1, it can be concluded that the system of MDDEs is delay independent stable.

Table 2 The rightmost eigenvalues of the different scenarios, where λ_k are the rightmost eigenvalues of (9); λ are the rightmost eigenvalues of (10)

Scenario	a_1	a_3	c	τ	λ_k	λ
S1	− 1	− 5	1.15	0.1893	$0.4170 \pm j6.4632$	$0.4170 \pm j8.4958$
S2	1	− 5	1.55	0.1882	$0.4377 \pm j5.5037$	$0.4377 \pm j8.5581$
S3	− 1	5	0.65	0.1794	1.1564	1.1564
S4	− 5	1	1.0	0.0846615	−9.2201	$-9.2201 \pm j9.2008$
S5	− 5	− 1	1.0	0.1339	$-1.9848 \pm j8.5376$	$-1.9848 \pm j10.2259$
S6	− 2	− 2	1.0	0.15922	$-2.1677 \pm j7.1038$	$-2.1676 \pm j8.1640$
S7	2	− 2	1.0	0.186735	$-1.2480 \pm j6.5457$	$-1.2480 \pm j7.4538$
S8	− 2	2	0.8	0.08073	−1.7415	−1.7415

Since the DSS is interconnected with the system of MDDEs, it also inherits the same property of delay independence stability.

Example 4.3 We consider the following MDDE:

$$\dot{Y}(t) = a_0 y(t) + a_1 y(t-\tau_1) + a_2 y(t-\tau_2) + a_3 y(t-\tau_3) \quad (9)$$

1. Let $\tau_1 = 0.1$, $\tau_2 = 0.2$ and $\tau_3 = 0.3$. Table 1 considers several scenarios that satisfy conditions (i) and (ii) of Theorem 2.1.
   S3 and S4 in Table 1 are delay independent stable according to Theorem 2.1.
2. We consider now system (9) with $a_0 = 0.1$, $a_2 = -2$, $\tau_1 = 0.1$, $\tau_2 = 0.2$ and $\tau_3 = 0.3$. According to the discussion in the previous section, there exist a single delay system:

   $$\dot{Y}(t) = a_0 y(t) + a y(t-\tau), \quad (10)$$

   where $a = c\sum_{k=1}^{n} a_k$. Table 2 shows several examples & scenarios, and the results are based on the discussion in Sect. 3.

5 Conclusions

In this article, we derived simple and practical conditions for the small signal stability analysis of the MDDE system (1). Furthermore, we explored the connection between the stability analysis of MDDEs and that of DDEs. In addition to MDDEs, we discussed how the stability criterion can be extended to DDSs. We demonstrated the applicability of our approach with numerical examples, which showed that the proposed method is effective. In summary, this article provided a comprehensive approach for small signal stability analysis of MDDEs and DDSs. Our findings have significant practical implications for the design and analysis of these types of systems.

As a future direction, we plan to expand the scope of our research to encompass other types of systems that exhibit memory effects, such as systems of fractional differential and difference equations, see [21, 22]. Additionally, we intend to explore promising applications where delays are significant, such as in the dynamics of electrical power systems, macroeconomic models, and electricity market models, among others; see [8, 13, 15, 18]. Furthermore, we aim to extend our theoretical findings to systems that employ forward operators. Specifically, we plan to investigate systems where instead of a vector with delay in the form $Y(t-\tau)$, a vector with delay in the form $Y(t+\tau)$ is used, as discussed in [20, 33]. There is already some research in progress in these areas, and we hope to contribute to these efforts with our work.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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