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Discrete relaxation equations of arbitrary order with periodic boundary conditions

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Abstract
Two simple nabla fractional relaxation equations with related periodic boundary conditions are addressed in this article. Firstly, we construct the corresponding Green's functions and obtain some of their properties. Through relevant fixed-point theorems with adequate restrictions, we provide sufficient conditions for the existence of solutions to the problems under consideration. To further illustrate how applicable previous findings are, we also offer a few examples.

Keywords Nabla fractional difference · Relaxation equation · Periodic boundary condition · Green's function · Fixed point · Existence of a solution

Mathematics Subject Classification 39A12 · 39A27 · 39A60

1 Introduction

The concept of fractional derivative [1, 2] is a generalization of the classical derivative to an arbitrary noninteger order. Fractional differential equations are applicable in various fields of science and engineering, such as mechanics, economics, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, oscillation theory, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, and many other allied areas. In particular, problems concerning the qualitative analysis of linear and nonlinear fractional differential equations have received the attention of many authors; see [3–8] and the references therein.

On the other hand, nabla fractional calculus is an integrated theory of arbitrary order sums and differences in the backward sense. The concept of nabla fractional difference traces back to the works of many famous researchers in the last 2 decades. For a detailed introduction, we refer to the recent monographs [9–11] and the references therein.

Since 2010, there has been an increasing interest in analyzing nabla fractional boundary value problems. To name a few notable works, we refer to [12–19]. In this line, we investigate two simple nabla fractional periodic boundary value problems. Specifically, we shall consider the following nabla fractional relaxation equations associated with periodic boundary conditions:

$$\begin{cases}\left(\nabla^{\delta}_{\rho(0)}y\right)(x) = \vartheta\, y(x) + f(x, y(\rho(x))), \quad x \in \mathbb{N}_1^T,\\ y(0) = y(T),\end{cases} \tag{1.1}$$

and

$$\begin{cases}\left(\nabla^{\delta}_{0*}y\right)(x) = \vartheta\, y(x) + f(x, y(\rho(x))), \quad x \in \mathbb{N}_1^T,\\ y(0) = y(T),\end{cases} \tag{1.2}$$

where $0 < \delta < 1; -1 < \vartheta < 0; T \in \mathbb{N}_2; f : \mathbb{N}_1^T \times \mathbb{R} \to \mathbb{R}$ is a continuous function; $\nabla^{\delta}_{\rho(0)}y$ and $\nabla^{\delta}_{0*}y$ denote the Riemann–Liouville and the Caputo nabla fractional differences of y of order δ, respectively.

The structure of the current article is as follows: Preliminaries for discrete fractional calculus are found in Sect. 2. In Sect. 3, we construct the corresponding Green's functions

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and obtain some of their properties. Using relevant fixed-point theorems and some appropriate restrictions on ϑ and f, we derive sufficient conditions for the existence of solutions to (1.1) and (1.2) in Sect. 4. Additionally, in Sect. 5, we give a few examples to show how the findings in Sect. 4 can be used.

2 Preliminaries

Represent by $\mathbb{N}_\mu = \{\mu, \mu+1, \mu+2, \ldots\}$ and $\mathbb{N}_\mu^\nu = \{\mu, \mu+1, \mu+2, \ldots, \nu\}$ for any $\mu, \nu \in \mathbb{R}$ such that $\nu - \mu \in \mathbb{N}_1$.

Definition 2.1 [10, 20] For $x, k \in \mathbb{R}$ and $t \in \mathbb{R}\backslash\mathbb{Z}^-$, define

$$H_t(x,k) = \frac{(x-k)^{\overline{t}}}{\Gamma(t+1)} = \frac{\Gamma(x-k+t)}{\Gamma(x-k)\Gamma(t+1)},$$

provided the RHS is well defined. Here, $\Gamma(.)$ denotes the Gamma function.

Definition 2.2 [10] For $y : \mathbb{N}_{k+1} \to \mathbb{R}$ and $\delta > 0$, the δ^{th}-order nabla fractional sum of y based at k is given by

$$\left(\nabla_k^{-\delta} y\right)(x) = \sum_{\xi=k+1}^{x} H_{\delta-1}(x, \rho(\xi)) y(\xi), \quad x \in \mathbb{N}_k,$$

where $\rho(\xi) = \xi - 1$.

Definition 2.3 [10] Let $y : \mathbb{N}_{k+1} \to \mathbb{R}$, $\delta > 0$, $n \in \mathbb{N}_1$ with $n-1 < \delta \le n$. The δ^{th}-order Riemann–Liouville nabla fractional difference of y based at k is given by

$$\left(\nabla_k^{\delta} y\right)(x) = \left(\nabla^n \left(\nabla_k^{-(n-\delta)} y\right)\right)(x), \quad x \in \mathbb{N}_{k+n}.$$

Definition 2.4 [10] Let $y : \mathbb{N}_{k-n+1} \to \mathbb{R}$ and $\delta > 0$. The δth-order Caputo nabla fractional difference of y based at k is given by

$$\left(\nabla_{k*}^{\delta} y\right)(x) = \left(\nabla_k^{-(n-\delta)} \left(\nabla^n y\right)\right)(x), \quad x \in \mathbb{N}_{k+1},$$

where $n = \lceil \delta \rceil$.

Next is the composition rule of nabla fractional sum, which will be applicable in the following section.

Theorem 2.1 [10] *Assume* $\delta, \gamma > 0$, $y : \mathbb{N}_{k+1} \to \mathbb{R}$, $n \in \mathbb{N}_1$ *with* $n-1 < \delta \le n$. *Then,*

(1) $\left(\nabla_k^{-\delta}\left(\nabla_k^{-\gamma} y\right)\right)(x) = \left(\nabla_k^{-(\delta+\gamma)} y\right)(x), \quad x \in \mathbb{N}_k.$
(2) $\left(\nabla_k^{\delta}\left(\nabla_k^{-\gamma} y\right)\right)(x) = \left(\nabla_k^{\delta-\gamma} y\right)(x), \quad x \in \mathbb{N}_{k+n}.$

Definition 2.5 [10] Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $-1 < \vartheta < 1$. The discrete Mittag–Leffler function is defined by

$$e_{\vartheta,\alpha,\beta}(x,k) = \sum_{j=0}^{\infty} \vartheta^j H_{\alpha j+\beta}(x,k), \quad x \in \mathbb{N}_k.$$

Clearly,

$$e_{\vartheta,\alpha,\beta}(x, \rho(x)) = \frac{1}{1-\vartheta}, \quad x \in \mathbb{N}_k.$$

Theorem 2.2 [10] *Assume* $0 < \delta < 1$ *and* $-1 < \vartheta < 1$. *The homogeneous difference equation*

$$\left(\nabla_{\rho(k)}^{\delta} y\right)(x) = \vartheta y(x), \quad x \in \mathbb{N}_{k+1}, \tag{2.1}$$

has a general solution

$$y(x) = C e_{\vartheta,\delta,\delta-1}(x, \rho(k)), \quad x \in \mathbb{N}_k. \tag{2.2}$$

Here, C is an arbitrary constant.

Theorem 2.3 [21] *Assume* $0 < \delta < 1$ *and* $-1 < \vartheta < 1$. *The homogeneous difference equation*

$$\left(\nabla_{k*}^{\delta} y\right)(x) = \vartheta y(x), \quad x \in \mathbb{N}_{k+1}, \tag{2.3}$$

is given by

$$y(x) = C e_{\vartheta,\delta,0}(x, k), \quad x \in \mathbb{N}_k. \tag{2.4}$$

Here C is an arbitrary constant.

Theorem 2.4 *Assume* $0 < \delta < 1$, $|\vartheta| < 1$ *and h is a real-valued function defined on* $\mathbb{N}_{k+1}$. *Then, the nonhomogeneous difference equation*

$$\left(\nabla_{\rho(k)}^{\delta} y\right)(x) = \vartheta y(x) + h(x), \quad x \in \mathbb{N}_{k+1}, \tag{2.5}$$

has a general solution

$$y(x) = C e_{\vartheta,\delta,\delta-1}(x, \rho(k)) + \sum_{\xi=k+1}^{x} e_{\vartheta,\delta,\delta-1}(x, \rho(\xi)) h(\xi), \quad x \in \mathbb{N}_k, \tag{2.6}$$

where C is an arbitrary constant.

Proof Denote by

$$w(x) = \sum_{\xi=k+1}^{x} e_{\vartheta,\delta,\delta-1}(x, \rho(\xi)) h(\xi), \quad x \in \mathbb{N}_k.$$

We show that w satisfies (2.5), that is,

$$\left(\nabla_{\rho(k)}^{\delta} w\right)(x) = \vartheta w(x) + h(x), \quad x \in \mathbb{N}_{k+1}. \tag{2.7}$$

To see this, for $x \in \mathbb{N}_k$, consider

$$
\begin{aligned}
w(x) &= \sum_{\xi=k+1}^{x} e_{\vartheta,\delta,\delta-1}(x,\rho(\xi))h(\xi) \\
&= \sum_{\xi=k+1}^{x}\left[\sum_{j=0}^{\infty}\vartheta^j H_{\delta j+\delta-1}(x,\rho(\xi))\right]h(\xi) \\
&= \sum_{j=0}^{\infty}\vartheta^j\left[\sum_{\xi=k+1}^{x} H_{\delta j+\delta-1}(x,\rho(\xi))h(\xi)\right] \\
&= \sum_{j=0}^{\infty}\vartheta^j\left[\sum_{\xi=k}^{x} H_{\delta j+\delta-1}(x,\rho(\xi))h(\xi)\right. \\
&\quad \left.-H_{\delta j+\delta-1}(x,\rho(k))h(k)\right] \\
&= \sum_{j=0}^{\infty}\vartheta^j\left[\left(\nabla_{\rho(k)}^{-(\delta j+\delta)}h\right)(x)\right] \\
&\quad - h(k)\sum_{j=0}^{\infty}\vartheta^j H_{\delta j+\delta-1}(x,\rho(k)) \\
&= \sum_{j=0}^{\infty}\vartheta^j\left[\left(\nabla_{\rho(k)}^{-(\delta j+\delta)}h\right)(x)\right] - h(k)e_{\vartheta,\delta,\delta-1}(x,\rho(k)).
\end{aligned} \tag{2.8}
$$

Now, for $x \in \mathbb{N}_{k+1}$, consider

$$
\begin{aligned}
&\left(\nabla_{\rho(k)}^{\delta}w\right)(x) \\
&= \nabla_{\rho(k)}^{\delta}\left[\sum_{j=0}^{\infty}\vartheta^j\left[\left(\nabla_{\rho(k)}^{-(\delta j+\delta)}h\right)(x)\right]\right. \\
&\quad \left.- h(k)e_{\vartheta,\delta,\delta-1}(x,\rho(k))\right] \quad \text{(By Using (2.8))} \\
&= \sum_{j=0}^{\infty}\vartheta^j\left[\left(\nabla_{\rho(k)}^{\delta}\left(\nabla_{\rho(k)}^{-(\delta j+\delta)}h\right)\right)(x)\right] \\
&\quad - h(k)\left[\nabla_{\rho(k)}^{\delta}e_{\vartheta,\delta,\delta-1}(x,\rho(k))\right] \\
&= \sum_{j=0}^{\infty}\vartheta^j\left[\left(\nabla_{\rho(k)}^{-\delta j}h\right)(x)\right] \\
&\quad - h(k)\left[\vartheta e_{\vartheta,\delta,\delta-1}(x,\rho(k))\right] \quad \text{(By Using Theorems 2.1 \& 2.2)} \\
&= h(x) + \sum_{j=1}^{\infty}\vartheta^j\left[\left(\nabla_{\rho(k)}^{-\delta j}h\right)(x)\right] - \vartheta e_{\vartheta,\delta,\delta-1}(x,\rho(k))h(k) \\
&= h(x) + \vartheta\left[\sum_{j=0}^{\infty}\vartheta^j\left[\left(\nabla_{\rho(k)}^{-(\delta j+\delta)}h\right)(x)\right]\right. \\
&\quad \left.-e_{\vartheta,\delta,\delta-1}(x,\rho(k))h(k)\right] \\
&= \vartheta w(x) + h(x). \quad \text{(By Using (2.8))}
\end{aligned}
$$

The proof is complete. □

Theorem 2.5 *Assume $0 < \delta < 1$, $|\vartheta| < 1$ and h is a real-valued function defined on $\mathbb{N}_{k+1}$. Then, the nonhomogeneous difference equation*

$$
\left(\nabla_{k*}^{\delta}y\right)(x) = \vartheta y(x) + h(x), \quad x \in \mathbb{N}_{k+1}, \tag{2.9}
$$

has a general solution

$$
y(x) = Ce_{\vartheta,\delta,0}(x,k) + \sum_{\xi=k+1}^{x} e_{\vartheta,\delta,\delta-1}(x,\rho(\xi))h(\xi), \quad x \in \mathbb{N}_k, \tag{2.10}
$$

where C is an arbitrary constant.

Proof Denote by

$$
w(x) = \sum_{\xi=k+1}^{x} e_{\vartheta,\delta,\delta-1}(x,\rho(\xi))h(\xi), \quad x \in \mathbb{N}_k.
$$

We show that w satisfies (2.9), that is,

$$
\left(\nabla_{k*}^{\delta}w\right)(x) = \vartheta w(x) + h(x), \quad x \in \mathbb{N}_{k+1}. \tag{2.11}
$$

For $x \in \mathbb{N}_k$, consider

$$
\begin{aligned}
w(x) &= \sum_{\xi=k+1}^{x} e_{\vartheta,\delta,\delta-1}(x,\rho(\xi))h(\xi) \\
&= \sum_{\xi=k+1}^{x}\left[\sum_{j=0}^{\infty}\vartheta^j H_{\delta j+\delta-1}(x,\rho(\xi))\right]h(\xi) \\
&= \sum_{j=0}^{\infty}\vartheta^j\left[\sum_{\xi=k+1}^{x} H_{\delta j+\delta-1}(x,\rho(\xi))h(\xi)\right] \\
&= \sum_{j=0}^{\infty}\vartheta^j\left(\nabla_k^{-(\delta j+\delta)}h\right)(x).
\end{aligned} \tag{2.12}
$$

Now, for $x \in \mathbb{N}_{k+1}$, consider

$$
\begin{aligned}
\left(\nabla_{k*}^{\delta}w\right)(x) &= \left(\nabla_k^{-(1-\delta)}(\nabla w)\right)(x) \\
&= \nabla_k^{-(1-\delta)}\nabla\left[\sum_{j=0}^{\infty}\vartheta^j\left(\nabla_k^{-(\delta j+\delta)}h\right)(x)\right] \quad \text{(By Using (2.12))} \\
&= \nabla_k^{-(1-\delta)}\left[\sum_{j=0}^{\infty}\vartheta^j\left(\nabla\left(\nabla_k^{-(\delta j+\delta)}h\right)\right)(x)\right] \\
&= \nabla_k^{-(1-\delta)}\left[\sum_{j=0}^{\infty}\vartheta^j\left(\nabla_k^{(1-\delta j-\delta)}h\right)(x)\right] \quad \text{(By Using Theorem 2.1)} \\
&= \sum_{j=0}^{\infty}\vartheta^j\left(\nabla_k^{-(1-\delta)}\left(\nabla_k^{-(\delta j+\delta-1)}h\right)\right)(x) \\
&= \sum_{j=0}^{\infty}\vartheta^j\left(\nabla_k^{-\delta j}h\right)(x) \quad \text{(By Using Theorem 2.1)} \\
&= h(x) + \sum_{j=1}^{\infty}\vartheta^j\left(\nabla_k^{-\delta j}h\right)(x)
\end{aligned}
$$

$$= h(x) + \vartheta \left[\sum_{j=0}^{\infty} \vartheta^j \left(\nabla_k^{-(\delta j+\delta)} h \right)(x) \right]$$
$$= h(x) + \vartheta w(x).$$

The proof is complete. □

Lemma 2.6 [21–25] *Let* $0 < \delta < 1$ *and* $-1 < \vartheta < 0$. *Then,*

(1) $0 < e_{\vartheta,\delta,\delta-1}(x, \rho(k)) \leq 1$;
(2) $0 < e_{\vartheta,\delta,0}(x, k) \leq 1$,

for $x \in \mathbb{N}_k$.

3 Green's functions and their properties

In this section, we construct the Green's functions for the following linear boundary value problems

$$\begin{cases} \left(\nabla^{\delta}_{\rho(0)} y\right)(x) = \vartheta y(x) + h(x), & x \in \mathbb{N}_1^T, \\ y(0) = y(T), \end{cases} \tag{3.1}$$

and

$$\begin{cases} \left(\nabla^{\delta}_{0*} y\right)(x) = \vartheta y(x) + h(x), & x \in \mathbb{N}_1^T, \\ y(0) = y(T), \end{cases} \tag{3.2}$$

corresponding to (1.1) and (1.2), respectively, and deduce their properties. Here $h : \mathbb{N}_1^T \to \mathbb{R}$.

Theorem 3.1 *The boundary value problem* (3.1) *has a unique solution*

$$y(x) = \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) h(\xi), \quad x \in \mathbb{N}_0^T, \tag{3.3}$$

where

$$\mathcal{G}_{RL}(x, \xi) = \begin{cases} \mathcal{G}_{RL_1}(x, \xi), & x \in \mathbb{N}_0^{\rho(\xi)}, \\ \mathcal{G}_{RL_2}(x, \xi), & x \in \mathbb{N}_\xi^T. \end{cases} \tag{3.4}$$

Here

$$\mathcal{G}_{RL_1}(x, \xi) = \frac{e_{\vartheta,\delta,\delta-1}(x, \rho(0))}{\left[\frac{1}{1-\vartheta} - e_{\vartheta,\delta,\delta-1}(T, \rho(0))\right]} e_{\vartheta,\delta,\delta-1}(T, \rho(\xi)), \tag{3.5}$$

and

$$\mathcal{G}_{RL_2}(x, \xi) = \mathcal{G}_{RL_1}(x, \xi) + e_{\vartheta,\delta,\delta-1}(x, \rho(\xi)). \tag{3.6}$$

Proof From Theorem 2.4, the nonhomogeneous difference equation in (3.1) has a general solution

$$y(x) = C e_{\vartheta,\delta,\delta-1}(x, \rho(0)) + \sum_{\xi=1}^{x} e_{\vartheta,\delta,\delta-1}(x, \rho(\xi)) h(\xi), \quad x \in \mathbb{N}_0. \tag{3.7}$$

Using the boundary condition $y(0) = y(T)$ in (3.7) and rearranging the terms, we get

$$C = \frac{1}{\left[\frac{1}{1-\vartheta} - e_{\vartheta,\delta,\delta-1}(T, \rho(0))\right]} \sum_{\xi=1}^{T} e_{\vartheta,\delta,\delta-1}(T, \rho(\xi)) h(\xi). \tag{3.8}$$

Substituting the expression for C from (3.8) in (3.7), we obtain (3.3). The proof is complete. □

Theorem 3.2 *The boundary value problem* (3.2) *has a unique solution*

$$y(x) = \sum_{\xi=1}^{T} \mathcal{G}_{C}(x, \xi) h(\xi), \quad x \in \mathbb{N}_0^T, \tag{3.9}$$

where

$$\mathcal{G}_{C}(x, \xi) = \begin{cases} \mathcal{G}_{C_1}(x, \xi), & x \in \mathbb{N}_0^{\rho(\xi)}, \\ \mathcal{G}_{C_2}(x, \xi), & x \in \mathbb{N}_\xi^T. \end{cases} \tag{3.10}$$

Here

$$\mathcal{G}_{C_1}(x, \xi) = \frac{e_{\vartheta,\delta,0}(x, 0)}{\left[1 - e_{\vartheta,\delta,0}(T, 0)\right]} e_{\vartheta,\delta,\delta-1}(T, \rho(\xi)), \tag{3.11}$$

and

$$\mathcal{G}_{C_2}(x, \xi) = \mathcal{G}_{C_1}(x, \xi) + e_{\vartheta,\delta,\delta-1}(x, \rho(\xi)). \tag{3.12}$$

Proof From Theorem 2.5, the nonhomogeneous difference equation in (3.2) has a general solution

$$y(x) = C e_{\vartheta,\delta,0}(x, 0) + \sum_{\xi=1}^{x} e_{\vartheta,\delta,\delta-1}(x, \rho(\xi)) h(\xi), \quad x \in \mathbb{N}_0. \tag{3.13}$$

Using the boundary condition $y(0) = y(T)$ in (3.13) and rearranging the terms, we get

$$C = \frac{1}{\left[1 - e_{\vartheta,\delta,0}(T, 0)\right]} \sum_{\xi=1}^{T} e_{\vartheta,\delta,\delta-1}(T, \rho(\xi)) h(\xi). \tag{3.14}$$

Substituting the expression for C from (3.14) in (3.13), we obtain (3.9). The proof is complete. □

Remark 1 Note that

$$\sum_{\xi=1}^{x} e_{\vartheta,\delta,\delta-1}(x,\rho(\xi)) = \sum_{\xi=1}^{x}\left[\sum_{j=0}^{\infty}\vartheta^{j} H_{\delta j+\delta-1}(x,\rho(\xi))\right] = \sum_{j=0}^{\infty}\vartheta^{j}\left[\sum_{\xi=1}^{x} H_{\delta j+\delta-1}(x,\rho(\xi))\right] = \sum_{j=0}^{\infty}\vartheta^{j} H_{\delta j+\delta}(x,0) = e_{\vartheta,\delta,\delta}(x,0).$$

Lemma 3.3 *$\mathcal{G}_{RL}(x,\xi)$ has the following properties:*

(1) $\mathcal{G}_{RL}(x,\xi) > 0$, $(x,\xi) \in \mathbb{N}_0^T \times \mathbb{N}_1^T$;
(2) $\sum_{\xi=1}^{T}\mathcal{G}_{RL}(x,\xi) = \frac{e_{\vartheta,\delta,\delta-1}(x,\rho(0))}{\left[\frac{1}{1-\vartheta} - e_{\vartheta,\delta,\delta-1}(T,\rho(0))\right]} e_{\vartheta,\delta,\delta}(T,0) + e_{\vartheta,\delta,\delta}(x,0)$, $x \in \mathbb{N}_0^T$.

Proof The proof of 3.3 follows from Lemma 2.6. To prove 3.3, for $x \in \mathbb{N}_0^T$, consider

$$\begin{aligned}\sum_{\xi=1}^{T}\mathcal{G}_{RL}(x,\xi) &= \sum_{\xi=1}^{x}\mathcal{G}_{RL_2}(x,\xi) + \sum_{\xi=x+1}^{T}\mathcal{G}_{RL_1}(x,\xi)\\ &= \sum_{\xi=1}^{T}\mathcal{G}_{RL_1}(x,\xi) + \sum_{\xi=1}^{x} e_{\vartheta,\delta,\delta-1}(x,\rho(\xi))\\ &= \frac{e_{\vartheta,\delta,\delta-1}(x,\rho(0))}{\left[\frac{1}{1-\vartheta} - e_{\vartheta,\delta,\delta-1}(T,\rho(0))\right]}\\ &\quad \sum_{\xi=1}^{T} e_{\vartheta,\delta,\delta-1}(T,\rho(\xi)) + \sum_{\xi=1}^{x} e_{\vartheta,\delta,\delta-1}(x,\rho(\xi))\\ &= \frac{e_{\vartheta,\delta,\delta-1}(x,\rho(0))}{\left[\frac{1}{1-\vartheta} - e_{\vartheta,\delta,\delta-1}(T,\rho(0))\right]}\\ &\quad e_{\vartheta,\delta,\delta}(T,0) + e_{\vartheta,\delta,\delta}(x,0) \quad \text{(By Remark 1).}\end{aligned}$$

The proof is complete. □

Lemma 3.4 *$\mathcal{G}_{C}(x,\xi)$ has the following properties:*

(1) $\mathcal{G}_{C}(x,\xi) > 0$, $(x,\xi) \in \mathbb{N}_0^T \times \mathbb{N}_1^T$;
(2) $\sum_{\xi=1}^{T}\mathcal{G}_{C}(x,\xi) = \frac{e_{\vartheta,\delta,0}(x,0)}{[1-e_{\vartheta,\delta,0}(T,0)]} e_{\vartheta,\delta,\delta}(T,0) + e_{\vartheta,\delta,\delta}(x,0)$, $x \in \mathbb{N}_0^T$.

Proof The proof is similar to that of Lemma 3.3. □

Remark 2 Denote by

$$\begin{aligned}\vartheta_1 &= \max_{x\in\mathbb{N}_0^T}\sum_{\xi=1}^{T}\mathcal{G}_{RL}(x,\xi)\\ &= \max_{x\in\mathbb{N}_0^T}\left[\frac{e_{\vartheta,\delta,\delta-1}(x,\rho(0))}{\left[\frac{1}{1-\vartheta} - e_{\vartheta,\delta,\delta-1}(T,\rho(0))\right]} e_{\vartheta,\delta,\delta}(T,0) + e_{\vartheta,\delta,\delta}(x,0)\right],\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\vartheta_2 &= \max_{x\in\mathbb{N}_0^T}\sum_{\xi=1}^{T}\mathcal{G}_{C}(x,\xi)\\ &= \max_{x\in\mathbb{N}_0^T}\left[\frac{e_{\vartheta,\delta,0}(x,0)}{[1-e_{\vartheta,\delta,0}(T,0)]} e_{\vartheta,\delta,\delta}(T,0) + e_{\vartheta,\delta,\delta}(x,0)\right].\end{aligned} \tag{3.16}$$

4 Existence of solutions

The existence of solutions to (1.1) and (1.2) is established by the sufficient conditions set forth in this section. Theorems 3.1 and 3.2 imply the equivalence between

(i) the solutions of (1.1) and the solutions of the summation equation

$$y(x) = \sum_{\xi=1}^{T}\mathcal{G}_{RL}(x,\xi) f(\xi, y(\rho(\xi))), \quad x \in \mathbb{N}_0^T;$$

(ii) the solutions of (1.2) and the solutions of the summation equation

$$y(x) = \sum_{\xi=1}^{T}\mathcal{G}_{C}(x,\xi) f(\xi, y(\rho(\xi))), \quad x \in \mathbb{N}_0^T,$$

respectively. Let $\mathcal{B}$ be the set of all real-valued functions defined on $\mathbb{N}_0^T$. Define the operators $S_1, S_2 : \mathcal{B} \to \mathcal{B}$ by

$$(S_1 y)(x) = \sum_{\xi=1}^{T}\mathcal{G}_{RL}(x,\xi) f(\xi, y(\rho(\xi))), \quad x \in \mathbb{N}_0^T,$$

$$(S_2 y)(x) = \sum_{\xi=1}^{T}\mathcal{G}_{C}(x,\xi) f(\xi, y(\rho(\xi))), \quad x \in \mathbb{N}_0^T.$$

Clearly, y is a fixed point of S_1 (or S_2) if and only if y is a solution of (1.1) [or (1.2)]. Observe that $\mathcal{B}$ is equivalent to $\mathbb{R}^{T+1}$. We know that $\mathcal{B}$ is a Banach space equipped with the maximum norm defined by

$$\|y\| = \max_{x\in\mathbb{N}_0^T}|y(x)|.$$

Let

$$K = \left\{y \in \mathcal{B} : y(0) = y(T) \text{ and } \|y\| \le r \text{ for all } x \in \mathbb{N}_0^T,\ r > 0\right\}.$$

Clearly, K is a nonempty bounded closed convex subset of the finite-dimensional normed space $\mathcal{B}$. First, we apply the Brouwer fixed-point theorem [26] to discuss the existence of solutions to (1.1) and (1.2).

Theorem 4.1 *Assume that*

(A1) $|f(x, y)| \leq M$ *for all* $(x, y) \in \mathbb{N}_0^T \times K$.

Choose

$$r \geq M\vartheta_1.$$

Then, (1.1) *has a solution.*

Proof To show that $S_1 : K \to K$, take $y \in K$, $x \in \mathbb{N}_0^T$ and consider

$$\begin{aligned}
|(S_1 y)(x)| &= \left| \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) f(\xi, y(\rho(\xi))) \right| \\
&\leq \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \, |f(\xi, y(\rho(\xi)))| \\
&\leq M \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \\
&\leq M\vartheta_1 \quad \text{(By Remark 2)} \\
&\leq r,
\end{aligned}$$

implying that $|(S_1 y)(x)| \leq r$ for all $x \in \mathbb{N}_0^T$. Also, $(S_1 y)(0) = (S_1 y)(T)$. As a result, $S_1 : K \to K$. The continuity of S_1 follows from the continuity of f. Thus, by Brouwer fixed-point theorem, (1.1) has a solution y in K. The proof is complete. □

Theorem 4.2 *Assume that (A1) holds. Choose*

$$r \geq M\vartheta_2.$$

Then, (1.2) *has a solution.*

Proof The proof is similar to that of Theorem 4.1. □

Next, we apply the Leray–Schauder nonlinear alternative [26] to discuss the existence of solutions to (1.1) and (1.2).

Theorem 4.3 *Assume that*

(C 1) *There exist* $\hat{\phi} : \mathbb{N}_1^T \to [0, \infty)$ *and a nondecreasing function* $\hat{\psi} : [0, \infty) \to [0, \infty)$ *such that*

$$|f(x, s)| \leq \hat{\phi}(x)\hat{\psi}(|s|), \quad (x, s) \in \mathbb{N}_1^T \times \mathbb{R}.$$

(C 2) *There exists* $M_1 > 0$ *such that*

$$\frac{M_1}{\vartheta_1 \overline{\Omega} \hat{\psi}(M_1)} > 1,$$

where

$$\overline{\Omega} = \max_{x \in \mathbb{N}_1^T} \hat{\phi}(x).$$

Then, the boundary value problem (1.1) *has a solution defined on* $\mathbb{N}_0^T$.

Proof We first show that S_1 maps bounded sets into bounded sets. By (C 1), for $x \in \mathbb{N}_0^T$ and $y \in K$,

$$\begin{aligned}
\left|(S_1 y)(x)\right| &\leq \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \, |f(\xi, y(\rho(\xi)))| \\
&\leq \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi)\hat{\phi}(\xi)\hat{\psi}(|y(\rho(\xi))|) \\
&\leq \hat{\psi}(\|y\|) \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi)\hat{\phi}(\xi) \\
&\leq \vartheta_1 \overline{\Omega} \hat{\psi}(r),
\end{aligned}$$

implying that

$$\left|(S_1 y)(x)\right| \leq \vartheta_1 \overline{\Omega} \hat{\psi}(r).$$

Thus, S_1 maps K into a bounded set. Since $\mathbb{N}_0^T$ is a discrete set, it follows immediately that S_1 maps K into an equicontinuous set. Therefore, by the Arzela–Ascoli theorem, S_1 is completely continuous. Next, we suppose $y \in \mathcal{B}$ and that for some $0 < \lambda < 1$, $y = \lambda S_1 y$. Then, for $x \in \mathbb{N}_0^T$, and again by (C 1),

$$\begin{aligned}
|y(x)| &= \left|\lambda (S_1 y)(x)\right| \\
&\leq \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \, |f(\xi, y(\rho(\xi)))| \\
&\leq \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi)\hat{\phi}(\xi)\hat{\psi}(|y(\rho(\xi))|) \\
&\leq \hat{\psi}(\|y\|) \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi)\hat{\phi}(\xi) \\
&\leq \vartheta_1 \overline{\Omega} \hat{\psi}(\|y\|),
\end{aligned}$$

implying that

$$\frac{\|y\|}{\vartheta_1 \overline{\Omega} \hat{\psi}(\|y\|)} \leq 1.$$

It follows from (C 2) that $\|y\| \neq M_1$. If we set

$$U = \left\{ y \in \mathcal{B} : \|y\| < M_1 \right\},$$

then the operator $S_1 : \bar{U} \to \mathcal{B}$ is completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda S_1 y$ for some $0 < \lambda < 1$. It follows from Leray–Schauder nonlinear alternative that S_1 has a fixed point $y_0 \in \bar{U}$, which is a desired solution of (1.1). □

Theorem 4.4 *Assume that (C 1) and*

(C 3) *There exists $M_2 > 0$ such that*

$$\frac{M_2}{\vartheta_2 \overline{\Omega} \hat{\psi}\,(M_2)} > 1,$$

where

$$\overline{\Omega} = \max_{x \in \mathbb{N}_1^T} \hat{\phi}(x).$$

Then, the boundary value problem (1.2) has a solution defined on $\mathbb{N}_0^T$.

Proof The proof is similar to the proof of Theorem 4.3, so we omit it. □

Now, we apply the Banach fixed-point theorem [26] to discuss the existence and uniqueness of solutions to (1.1) and (1.2).

Theorem 4.5 *The conditions*

(A2) *f is Lipschitz w.r.t. the second variable with $\overline{\kappa}$ as the Lipschitz constant on $\mathbb{N}_0^T \times K$;*

(A3) *Let*

$$\max_{x \in \mathbb{N}_0^T} |f(x, 0)| = P,$$

and

$$\max_{(x,y) \in \mathbb{N}_0^T \times K} |f(x, y)| = Q.$$

(A4) $\overline{\kappa}\vartheta_1 < 1,$

with

$$r \geq \frac{P\vartheta_1}{1 - \overline{\kappa}\vartheta_1},$$

or

$$r \geq Q\vartheta_1,$$

yield a unique solution for (1.1).

Proof Clearly, $S_1 : K \to \mathcal{B}$. To show that S_1 is a contraction mapping, take $y, w \in K$, $x \in \mathbb{N}_0^T$ and consider

$$\begin{aligned}
&|(S_1 y)(x) - (S_1 w)(x)| \\
&= \left| \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \left[f(\xi, y(\rho(\xi))) - f(\xi, w(\rho(\xi))) \right] \right| \\
&\leq \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \left| f(\xi, y(\rho(\xi))) - f(\xi, w(\rho(\xi))) \right| \\
&\leq \overline{\kappa}\, \|y - w\| \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \\
&\leq \overline{\kappa}\vartheta_1 \|y - w\|,
\end{aligned}$$

implying that

$$\|S_1 y - S_1 w\| \leq \overline{\kappa}\vartheta_1 \|y - w\|.$$

Since $\overline{\kappa}\vartheta_1 < 1$, S_1 is a contraction mapping. Now, we show that $S_1 : K \to K$. Let $y \in K$, $x \in \mathbb{N}_0^T$ and consider

$$\begin{aligned}
|(S_1 y)(x)| &= \left| \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) f(\xi, y(\rho(\xi))) \right| \\
&\leq \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \left| f(\xi, y(\rho(\xi))) - f(\xi, 0) \right| \\
&\quad + \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \left| f(\xi, 0) \right| \\
&\leq \overline{\kappa} \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \left| y(\rho(\xi)) \right| + P \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \\
&\leq (\overline{\kappa} r + P)\vartheta_1 \\
&\leq r,
\end{aligned}$$

implying that $S_1 : K \to K$. Also, consider

$$\begin{aligned}
|(S_1 y)(x)| &= \left| \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) f(\xi, y(\rho(\xi))) \right| \\
&\leq \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \left| f(\xi, y(\rho(\xi))) \right| \\
&\leq Q \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x, \xi) \\
&\leq Q\vartheta_1 \\
&\leq r,
\end{aligned}$$

implying that $S_1 : K \to K$. So, there exists a unique solution for (1.1) by Banach fixed-point theorem. □

Theorem 4.6 *The conditions (A2), (A3) and*

(A5) $\overline{\kappa}\vartheta_2 < 1$,

with

$$r \geq \frac{P\vartheta_2}{1-\overline{\kappa}\vartheta_2},$$

or

$$r \geq Q\vartheta_2,$$

yield a unique solution for (1.2).

Proof The proof is similar to the proof of Theorem 4.5, so we omit it. □

Theorem 4.7 *The conditions*

(A6) *f is Lipschitz w.r.t. the second variable with L as the Lipschitz constant on $\mathbb{N}_0^T \times \mathcal{B}$;*
(A7) $L\vartheta_1 < 1$,

yield a unique solution for (1.1).

Proof Clearly, $S_1 : \mathcal{B} \to \mathcal{B}$. To show that S_1 is a contraction mapping, take $y, w \in \mathcal{B}$, $x \in \mathbb{N}_0^T$ and consider

$$\begin{aligned} &|(S_1 y)(x) - (S_1 w)(x)| \\ &\quad = \left|\sum_{\xi=1}^{T} \mathcal{G}_{RL}(x,\xi)\left[f(\xi, y(\rho(\xi))) - f(\xi, w(\rho(\xi)))\right]\right| \\ &\quad \leq \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x,\xi)\left|f(\xi, y(\rho(\xi))) - f(\xi, w(\rho(\xi)))\right| \\ &\quad \leq L\,\|y-w\| \sum_{\xi=1}^{T} \mathcal{G}_{RL}(x,\xi) \\ &\quad \leq L\,\|y-w\|\,\vartheta_1, \end{aligned}$$

implying that

$$\|S_1 y - S_1 w\| \leq L\vartheta_1\,\|y-w\|.$$

Since $L\vartheta_1 < 1$, S_1 is a contraction mapping. Then, there exists a unique solution for (1.1), by Banach fixed-point theorem. □

Theorem 4.8 *The conditions (A6) and*

(A8) $L\vartheta_2 < 1$,

yield a unique solution for (1.2).

Proof The proof is similar to the proof of Theorem 4.7, so we omit it. □

Table 1 Evaluation of ϑ_1

x	$e_{\vartheta,\delta,\delta-1}(x,\rho(0))$	$e_{\vartheta,\delta,\delta}(x,0)$	$\sum_{\xi=1}^{10}\mathcal{G}_{RL}(x,\xi)$
0	0.909091	0	3.07902
1	0.413223	0.909091	2.30865
2	0.291134	1.32231	2.30836
3	0.230944	1.61345	2.39564
4	0.19382	1.84439	2.50084
5	0.168157	2.03821	2.60775
6	0.149141	2.20637	2.7115
7	0.134371	2.35551	2.81062
8	0.122504	2.48988	2.9048
9	0.112722	2.61239	2.99417
10	0.104493	2.72511	3.07902

5 Examples

Example 1 Consider (1.1) with $\delta = 0.5$, $\vartheta = -0.1$, $T = 10$ and

$$f(x,z) = (0.25)\left(x + \tan^{-1} z\right).$$

Clearly, f is Lipschitz w.r.t. the second variable with $L = 0.25$ as the Lipschitz constant on $\mathbb{N}_0^{10} \times \mathcal{B}$. Table 1 shows the calculations for the evaluation of $\sum_{\xi=1}^{10}\mathcal{G}_{RL}(x,\xi)$ using Mathematica:

From Table 1, we have

$$\vartheta_1 = \max_{x\in\mathbb{N}_0^{10}} \sum_{\xi=1}^{10}\mathcal{G}_{RL}(x,\xi) = 3.07902.$$

Then, $L\vartheta_1 < 1$. All assumptions of Theorem 4.7 hold. As a result, there exists a unique solution for (1.1).

Example 2 Consider (1.2) with $\delta = 0.5$, $\vartheta = -0.1$, $T = 10$ and $f(x,z) = \frac{1}{11}\left(x + \tan^{-1} z\right)$. Clearly, f is Lipschitz w.r.t. the second variable with $L = \frac{1}{11}$ as the Lipschitz constant on $\mathbb{N}_0^{10} \times \mathcal{B}$. Table 2 shows the calculations for the evaluation of $\sum_{\xi=1}^{10}\mathcal{G}_C(x,\xi)$ using Mathematica:

From Table 2, we have

$$\vartheta_2 = \max_{x\in\mathbb{N}_0^{10}} \sum_{\xi=1}^{10}\mathcal{G}_C(x,\xi) = 10.4132.$$

Table 2 Evaluation of ϑ_2

x	$e_{\vartheta,\delta,0}(x,0)$	$e_{\vartheta,\delta,\delta}(x,0)$	$\sum_{\xi=1}^{10}\mathcal{G}_C(x,\xi)$
0	1	0	10
1	0.909091	0.909091	10
2	0.909091	1.32231	10.4132
3	0.838655	1.61345	10
4	0.815561	1.84439	10
5	0.796179	2.03821	10
6	0.779363	2.20637	10
7	0.764449	2.35551	10
8	0.751012	2.48988	10
9	0.738761	2.61239	10
10	0.727489	2.72511	10

Then, $L\vartheta_2 < 1$. All assumptions of Theorem 4.8 hold. Thus, there exists a unique solution for (1.2).

Example 3 Consider the boundary value problem

$$\begin{cases}\left(\nabla^{0.5}_{\rho(0)}y\right)(x) = -\frac{1}{10}y(x) + xy^2(\rho(x)), & x \in \mathbb{N}_1^{10},\\ y(0) = y(10).\end{cases} \tag{5.1}$$

Here $T = 10$, $\delta = 0.5$, $\vartheta = -\frac{1}{10}$ and $f(x,\xi) = x\xi^2$. Clearly,

$$|f(x,\xi)| \le \hat{\phi}(x)\hat{\psi}(|\xi|), \quad (x,\xi) \in \mathbb{N}_1^{10} \times \mathbb{R},$$

where

$$\hat{\phi}(x) = x, \quad x \in \mathbb{N}_1^{10},$$

and

$$\hat{\psi}(|\xi|) = |\xi|^2 = \xi^2, \quad \xi \in \mathbb{R}.$$

Also, $\hat{\phi} : \mathbb{N}_1^{10} \to [0,\infty)$ and $\hat{\psi} : [0,\infty) \to [0,\infty)$ is a nondecreasing function. Thus, the assumption (C 1) of Theorem 4.3 holds. Further, we have

$$\overline{\Omega} = \max_{x\in\mathbb{N}_1^{10}} \hat{\phi}(x) = 10.$$

Using Mathematica, we found that $\vartheta_1 = 3.07902$. There exists $0 < M_1 < \frac{1}{31}$ such that

$$\frac{M_1}{(3.07902)(10)M_1^2} > 1,$$

implying that the assumption (C 2) of Theorem 4.3 holds. Therefore, by Theorem 4.3, the boundary value problem (1.1) has a solution defined on $\mathbb{N}_0^{10}$.

Example 4 Consider the boundary value problem

$$\begin{cases}\left(\nabla^{0.5}_{0*}y\right)(x) = -\frac{1}{10}y(x) + xy^2(\rho(x)), & x \in \mathbb{N}_1^{10},\\ y(0) = y(10).\end{cases} \tag{5.2}$$

Here $T = 10$, $\delta = 0.5$, $\vartheta = -\frac{1}{10}$ and $f(x,\xi) = x\xi^2$. Clearly,

$$|f(x,\xi)| \le \hat{\phi}(x)\hat{\psi}(|\xi|), \quad (x,\xi) \in \mathbb{N}_1^{10} \times \mathbb{R},$$

where

$$\hat{\phi}(x) = x, \quad x \in \mathbb{N}_1^{10},$$

and

$$\hat{\psi}(|\xi|) = |\xi|^2 = \xi^2, \quad \xi \in \mathbb{R}.$$

Also, $\hat{\phi} : \mathbb{N}_1^{10} \to [0,\infty)$ and $\hat{\psi} : [0,\infty) \to [0,\infty)$ is a nondecreasing function. Thus, the assumption (C 1) of Theorem 4.4 holds. Further, we have

$$\overline{\Omega} = \max_{x\in\mathbb{N}_1^{10}} \hat{\phi}(x) = 10.$$

Using Mathematica, we found that $\vartheta_2 = 10.4132$. There exists $0 < M_2 < \frac{1}{101}$ such that

$$\frac{M_2}{(10.4132)(10)M_2^2} > 1,$$

implying that the assumption (C 3) of Theorem 4.4 holds. Therefore, by Theorem 4.4, the boundary value problem (1.2) has a solution defined on $\mathbb{N}_0^{10}$.

Conclusion and future scope

This article considered two simple nabla fractional relaxation equations with related periodic boundary conditions. We provided sufficient conditions for the existence of solutions to the problems under consideration through relevant fixed-point theorems with adequate restrictions. We also offered a few examples to further illustrate the applicability of our findings. To our knowledge, such work has yet to be reported in the case of fractional differences.

The current work can also be extended to obtain sufficient conditions for multiple positive solutions of the considered boundary value problems due to the corresponding Green functions' positivity.

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