Symmetries and conserved quantities of constrained mechanical systems

F. X. Mei · H. B. Wu · Y. F. Zhang

Abstract This is a concise review of the symmetries and conserved quantities of the constrained mechanical systems. It deals with three kinds of symmetries: the Noether symmetry, the Lie symmetry and the form invariance, and three kinds of conserved quantities: the Noether conserved quantity, the Hojman type conserved quantity and the new conserved quantity.

Keywords Symmetry · Conserved quantity · Constrained mechanical system

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1 Introduction

The most important task in dynamics is to find the solution of the differential equations of motion. If one can find all of the integrals of the equations, the solution of the equations can be given. An integral is a conserved quantity, therefore, people make efforts to find all of the conserved quantities of mechanical system.

Newtonian mechanics provides us three conservation laws, i.e. the conservation of momentum, the conservation of moment of momentum and the conservation of mechanical energy, by means of the analysis of forces. The three conservation laws have very clear physical meaning. Lagrangian mechanics provides us two conservation laws, i.e. the conservation of generalized momentum and the conservation of generalized energy, by means of the analysis of the form of Lagrangian. The conservation of generalized momentum

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may be a conservation of momentum, a conservation of moment of momentum, or neither. The physical meaning of the conservation law in Lagrangian mechanics is less clear than that in Newtonian mechanics, but the conserved quantities deduced by Lagrangian mechanics are more than that deduced by Newtonian mechanics. Since Noether published her well-known paper [1], the Noether symmetry method has become a modern method for seeking the conservation law of mechanical systems [2–18]. The physical meaning of the conservation law in the Noether symmetry method is less than that in Lagrangian mechanics, but the conserved quantities deduced by the Noether symmetry method are more than that deduced by Lagrangian mechanics.

The Noether symmetry is an invariance of the Hamilton action under the infinitesimal transformations of time and coordinates. Besides the Noether symmetry, there are two other important symmetries, i.e. the Lie symmetry and the form invariance. The Lie symmetry is a kind of invariance of the differential equations under the infinitesimal transformations of time and coordinates. The form invariance is a kind of invariance under which the transformed dynamical functions still satisfy the original differential equations of motion. A Noether symmetry can lead to a conserved quantity according to the Noether theory. A Lie symmetry or a form invariance can also lead to a conserved quantity under certain conditions. The conserved quantities deduced directly by the Noether symmetry, the Lie symmetry and the form invariance are called the Noether conserved quantity, the Hojman type conserved quantity and the new conserved quantity, respectively. The constrained mechanical systems in the review involves holonomic systems, nonholonomic systems and Birkhoffian systems.

The outline of this review is as follows: In Sect. 2, we study the Noether symmetry of constrained mechanical systems. In Sect. 3, we discuss the Lie symmetry of constrained

mechanical systems. In Sect. 4, we present the form invariance of constrained mechanical systems. Section 5 deals with the Noether conserved quantity deduced by the Noether symmetry. Section 6 provides the Hojman type conserved quantity deduced by the Lie symmetry. In Sect. 7, we work at a new conserved quantity deduced by the form invariance. Section 8 goes into the Non-Noether conserved quantity guided by the Noether symmetry. Section 9 is devoted to the Noether conserved quantity and the new conserved quantity deduced by the Lie symmetry. In Sect. 10, we specialize the Noether conserved quantity and the Hojman type conserved quantity deduced by the form invariance. Finally, we propose some topics for future research on the study of the symmetry in Sect. 11.

2 Noether symmetry

The Noether symmetry is an invariance of the Hamilton action under the infinitesimal transformations of time and the coordinates.

2.1 Noether symmetry of Lagrangian system

The equations of motion of Lagrangian system have the form

$$E_s(L) = 0 \quad (s = 1, \ldots, n), \tag{1}$$

where L is the Lagrangian and E_s the Euler operator

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}. \tag{2}$$

For the system (1), Arnold gave the Noether's theorem [19] as follows:

If the system (M, L) admits the one-parameter group of diffeomorphisms $h^s : M \to M, s \in R$, then the Lagrangian system of equations corresponding to L has a first integral $I : TM \to R$.

In local coordinates q on M, the integral I is written in the form

$$I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{\mathrm{d}h^s(q)}{\mathrm{d}s} \bigg|_{s=0}. \tag{3}$$

José and Saletan wrote " if a Lagrangian is invariant under a family of transformations, its dynamical system possesses a constant of the motion, and that constant can be found from a knowledge of the Lagrangian and the transformation" [20].

Therefore, there are two comprehensions for the Noether symmetry. One comprehension is an invariance of the Hamilton action; another is an invariance of the Lagrangian. We all think that the first one is more reasonable.

Introducing the infinitesimal transformations of time and the generalized coordinates as

$$t^* = t + \Delta t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s, \tag{4}$$

or

$$t^* = t + \varepsilon \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{5}$$

where ε is an infinitesimal parameter, and ξ_0, ξ_s the infinitesimal generators, the invariance of the Hamilton action leads to the Noether identity

$$L\dot{\xi}_0 + X^{(1)}(L) + \dot{G}_N = 0, \tag{6}$$

where $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is a gauge function and

$$X^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_s}. \tag{7}$$

If there exists a function G_N such that the identity (6) is satisfied, then the corresponding symmetry is called a Noether symmetry. We will see that the conserved quantities can be found by using the Noether symmetry.

2.2 Noether symmetry of general holonomic system

The equations of motion of general holonomic system can be written in the form

$$E_s(L) = Q_s \quad (s = 1, \ldots, n), \tag{8}$$

where $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are generalized forces. Under the infinitesimal transformations (5), the Noether symmetry of the system leads to the satisfaction of the following Noether identity

$$L\dot{\xi}_0 + X^{(1)}(L) + Q_s(\xi_s - \dot{q}_s\xi_0) + \dot{G}_N = 0. \tag{9}$$

Formula (9) is called the generalized Noether-Bessel-Hagen equation [8].

By means of the Noether symmetry, we can seek the conserved quantities of the general holonomic system.

2.3 Noether symmetry of nonholonomic system

The nonholonomic system is more complected than the holonomic system, e.g. see references [21–31].

Let the position of a mechanical system be determined by the n generalized coordinates q_s ($s = 1, \ldots, n$) and its motion be subject to the g ideal nonholonomic constraints of Chetaev's type

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \quad (\beta = 1, \ldots, g). \tag{10}$$

The differential equations of motion of the system can be written in the form [20,25,29–33]

$$E_s(L) = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{11}$$

where λ_β are the constraint multipliers. Before integrating the differential equations, from Eqs. (10) and (11) one can solve

the multipliers λ_β as functions of t, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. So, Eqs. (11) have the form

$$E_s(L) = Q_s + \Lambda_s, \tag{12}$$

where

$$\Lambda_s = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\frac{\partial f_\beta}{\partial \dot{q}_s} \tag{13}$$

express the nonholonomic constraint forces. Equations (12) are called the equations of holonomic system corresponding to the nonholonomic system (10), (11).

For the system (12), the Noether identity is

$$L\dot{\xi}_0 + X^{(1)}(L) + (Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \dot{G}_N = 0. \tag{14}$$

The restriction of nonholonomic constraints (10) on the generators has the form [13]

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0. \tag{15}$$

If the generators ξ_0, ξ_s and the gauge function G_N satisfy Eqs. (14) and (15), then the symmetry is a Noether one of the nonholonomic system. If they satisfy only Eq. (14), then the symmetry is a Noether symmetry of the corresponding holonomic system (12).

Considering Eq. (15), the identity (14) can be simplified as

$$L\dot{\xi}_0 + X^{(1)}(L) + Q_s(\xi_s - \dot{q}_s\xi_0) + \dot{G}_N = 0. \tag{16}$$

The complexity of the Noether symmetry of the nonholonomic system is to have the restriction of nonholonomic constraints on the generators.

The conserved quantities can be obtained by using the Noether symmetry.

2.4 Noether symmetry of Birkhoffian system

The Birkhoffian system is a kind of constrained mechanical systems. The Birkhoff equations have the form [34]

$$\Omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = 0 \quad (\mu, \nu = 1, \ldots, 2n), \tag{17}$$

where

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \tag{18}$$

and $B = B(t, \boldsymbol{a})$ is the Birkhoffian and $R_\mu = R_\mu(t, \boldsymbol{a})$ the Birkhoff's functions.

Introducing the infinitesimal transformations as

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{a}), \quad a^{\mu*} = a^\mu + \varepsilon\xi_\mu(t, \boldsymbol{a}) \tag{19}$$

for the system (17), the Noether identity has the form [14]

$$\left(\frac{\partial R_\mu}{\partial t}\dot{a}^\mu - \frac{\partial B}{\partial t}\right)\xi_0 + \left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu}\right)\xi_\mu - B\dot{\xi}_0$$
$$+ R_\mu\dot{\xi}_\mu + \dot{G}_N = 0, \tag{20}$$

where $G_N = G_N(t, \boldsymbol{a})$ is the gauge function.

The Noether symmetry of Birkhoffian system can lead to the conserved quantities.

The study on Birkhoffian mechanics can be seen in reference [35].

3 Lie symmetry

The Lie symmetry is an invariance of the differential equations of motion under the infinitesimal transformations. On the part of mathematics, the monographs are very important, e.g. see references [36–39]. There have been some important results on the study of the Lie symmetry of mechanical systems [40–48].

3.1 Lie symmetry of Lagrangian system

Suppose that the system (1) is nonsingular, i.e.

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0. \tag{21}$$

Then Eq. (1) can be written as

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{22}$$

Under the infinitesimal transformations (5), the determining equations of the Lie symmetry of Eq. (22) have the form

$$\ddot{q}_s - \dot{q}_s\ddot{\xi}_0 - 2\dot{\xi}_0\alpha_s = X^{(1)}(\alpha_s). \tag{23}$$

The Lie symmetry can lead to the conserved quantities under certain conditions. But it is not easy that one find the generators ξ_0, ξ_s from Eq. (23), because symmetry methods for differential equations, originally developed by Sophus Lie, are highly algorithmic [37].

For the Lagrangian system, all of the Noether symmetries are also Lie symmetries [49,50].

3.2 Lie symmetry of general holonomic system

Let Eq. (8) have the form

$$\ddot{q}_s = \beta_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{24}$$

The determining equations of the Lie symmetry of Eq. (24) have the form

$$\ddot{\xi}_s - \dot{q}_s\ddot{\xi}_0 - 2\dot{\xi}_0\beta_s = X^{(1)}(\beta_s). \tag{25}$$

If the infinitesimal generators ξ_0, ξ_s satisfy Eq. (25), then the Lie symmetry can lead to the conserved quantities under certain conditions.

For the general holonomic system, a Noether symmetry is a Lie symmetry under certain conditions [50].

3.3 Lie symmetry of nonholonomic system

From Eq. (12), we can solve all of the generalized accelerations as

$$\ddot{q}_s = \gamma_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{26}$$

Their determining equations of the Lie symmetry are

$$\ddot{\xi}_s - \dot{q}_s\ddot{\xi}_0 - 2\dot{\xi}_0\gamma_s = X^{(1)}(\gamma_s). \tag{27}$$

The restriction equations of nonholonomic constraints (10) have the form

$$X^{(1)}(f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})) = 0. \tag{28}$$

If the generators ξ_0, ξ_s satisfy the determining equations (27) and the restriction equations (28), then the symmetry is a Lie symmetry of the nonholonomic system. If the generators ξ_0, ξ_s satisfy only the determining equations (27), then the symmetry is a Lie symmetry of the corresponding holonomic system. The complexity of the Lie symmetry of the nonholonomic system is to have the restriction conditions (28) of the nonholonomic constraint to the infinitesimal generators.

3.4 Lie symmetry of Birkhoffian system

Suppose that

$$\det(\Omega_{\mu\nu}) \neq 0. \tag{29}$$

Then from Eq. (17), we can solve $\dot{a}^\mu$ as

$$\dot{a}^\mu - \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) = 0. \tag{30}$$

Under the infinitesimal transformations (19), the determining equations of the Lie symmetry of Eq. (30) have the form

$$\dot{\xi}_\mu - \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\dot{\xi}_0 = X^{(0)}\left\{\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\right\}, \tag{31}$$

where

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_\mu\frac{\partial}{\partial a^\mu}. \tag{32}$$

If the infinitesimal generators ξ_0, ξ_s satisfy the determining equations (31), then the symmetry is a Lie symmetry of the Birkhoffian system.

4 Form invariance

There are dynamical functions in the differential equations of motion of mechanical systems, e.g. the Lagrangian, the generalized forces, the generalized constraint forces in the holonomic and nonholonomic systems, the Birkhoffian, the

Birkhoff functions in the Birkhoffian system. If the transformed dynamical functions still satisfy the original differential equations of motion, then the symmetry is called a form invariance. There have been some important results on the study of the form invariance [51–61].

4.1 Form invariance of Lagrangian system

After the transformations (5), the Lagrangian $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ becomes $L(t^*, \boldsymbol{q}^*, \frac{d\boldsymbol{q}^*}{dt^*})$, we have

$$L\left(t^*, \boldsymbol{q}^*, \frac{d\boldsymbol{q}^*}{dt^*}\right) = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon X^{(1)}(L) + 0(\varepsilon^2). \tag{33}$$

Substituting (33) into Eq. (1) and neglecting ε^2 terms and the higher terms, we obtain

$$E_s\{X^{(1)}(L)\} = 0. \tag{34}$$

If the generators ξ_0, ξ_s satisfy the Eq. (34), then the symmetry is a form invariance of the Lagrangian system. Equation (34) are called the criterion equations of the form invariance of the Lagrangian system. The form invariance is different from the Noether symmetry and the Lie symmetry.

4.2 Form invariance of general holonomic system

After the transformations (5), the Lagrangian $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and the generalized forces $Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ become respectively $L(t^*, \boldsymbol{q}^*, \frac{d\boldsymbol{q}^*}{dt^*})$ and $Q_s(t^*, \boldsymbol{q}^*, \frac{d\boldsymbol{q}^*}{dt^*})$, and we have

$$L\left(t^*, \boldsymbol{q}^*, \frac{d\boldsymbol{q}^*}{dt^*}\right) = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon X^{(1)}(L) + 0(\varepsilon^2),$$
$$Q_s\left(t^*, \boldsymbol{q}^*, \frac{d\boldsymbol{q}^*}{dt^*}\right) = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon X^{(1)}(Q_s) + 0(\varepsilon^2). \tag{35}$$

Substituting (35) into Eq. (8) and neglecting ε^2 terms and the higher terms, we obtain

$$E_s\left\{X^{(1)}(L)\right\} = X^{(1)}(Q_s). \tag{36}$$

If the generators ξ_0, ξ_s satisfy the Eq. (36), then the symmetry is a form invariance of the general holonomic system. Equation (36) are called the criterion equations of the form invariance of the general holonomic system.

4.3 Form invariance of nonholonomic system

After the transformations (5), the Lagrangian $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, the generalized forces $Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and the generalized constraint forces $\Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ become respectively $L(t^*, \boldsymbol{q}^*, \frac{d\boldsymbol{q}^*}{dt^*})$, $Q_s(t^*, \boldsymbol{q}^*, \frac{d\boldsymbol{q}^*}{dt^*})$ and $\Lambda_s(t^*, \boldsymbol{q}^*, \frac{d\boldsymbol{q}^*}{dt^*})$, and we have

$$L\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon X^{(1)}(L) + 0(\varepsilon^2),$$

$$Q_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon X^{(1)}(Q_s) + 0(\varepsilon^2),$$

$$\Lambda_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon X^{(1)}(\Lambda_s) + 0(\varepsilon^2).$$

$$(37)$$

Substituting (37) into Eq. (12) and neglecting ε^2 terms and the higher terms, we obtain

$$E_s\left\{X^{(1)}(L)\right\} = X^{(1)}(Q_s) + X^{(1)}(\Lambda_s). \tag{38}$$

If the generators ξ_0, ξ_s satisfy the Eq. (38), then the symmetry is a form invariance of the corresponding holonomic system (12).

After the transformations (5), the constraint functions $f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ become $f_\beta^*(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*})$, and we have

$$f_\beta^* = f_\beta + \varepsilon X^{(1)}(f_\beta) + 0(\varepsilon^2). \tag{39}$$

Substituting (39) into the constraint equations (10) and neglecting ε^2 terms and the higher terms, we obtain

$$X^{(1)}(f_\beta) = 0. \tag{40}$$

If the generators ξ_0, ξ_s satisfy the criterion equations (38) and the restriction equations (40), then the symmetry is a form invariance of the nonholonomic system. The complexity of the form invariance of the nonholonomic system is to have the restriction (40) of the nonholonomic constraints on the generators.

4.4 Form invariance of Birkhoffian system

After the transformations (19), the Birkhoffian $B(t, \boldsymbol{a})$ and the Birkhoff functions $R_\mu(t, \boldsymbol{a})$ become respectively $B(t^*, \boldsymbol{a}^*)$ and $R_\mu(t^*, \boldsymbol{a}^*)$, and we have

$$B\left(t^*, \boldsymbol{a}^*\right) = B(t, \boldsymbol{a}) + \varepsilon X^{(0)}(B) + 0(\varepsilon^2),$$
$$R_\mu\left(t^*, \boldsymbol{a}^*\right) = R_\mu(t, \boldsymbol{a}) + \varepsilon X^{(0)}(R_\mu) + 0(\varepsilon^2). \tag{41}$$

Substituting (41) into Eq. (17) and neglecting ε^2 terms and the higher terms, we obtain

$$\left\{\frac{\partial}{\partial a^\mu}X^{(0)}(R_\nu) - \frac{\partial}{\partial a^\nu}X^{(0)}(R_\mu)\right\}\dot{a}^\nu - \frac{\partial}{\partial a^\mu}X^{(0)}(B)$$
$$- \frac{\partial}{\partial t}X^{(0)}(R_\mu) = 0. \tag{42}$$

If the generators ξ_0, ξ_μ satisfy the criterion equations (42), then the symmetry is a form invariance of the Birkhoffian system.

5 Noether conserved quantities deduced by Noether symmetry

In Sects. 5–10, we will present three kinds of conserved quantities deduced by using three kinds of symmetries for the constrained mechanical systems and give some practical examples to illustrate the application of the results.

The Noether symmetry can lead directly to the Noether conserved quantity by using the Noether theorem.

5.1 Lagrangian system

Proposition 5.1 *For the Lagrangian system* (1)*, if the infinitesimal generators* ξ_0, ξ_s *and the gauge function* G_N *satisfy the Noether identity* (6)*, then the Noether symmetry will lead to the Noether conserved quantity*

$$I_N = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s \xi_0) + G_N = const. \tag{43}$$

This is the Noether theorem of the Lagrangian system.

Example 5.1 The Lagrangian of a harmonic oscillator system with two-degree-of-freedom is [62]

$$L = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{1}{2}k\left(q_1^2 + q_2^2\right),$$

where m, k are constants.

The Noether identity (6) gives

$$\left\{\frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{1}{2}k\left(q_1^2 + q_2^2\right)\right\}\dot{\xi}_0 + m\dot{q}_1\left(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0\right)$$
$$+ m\dot{q}_2\left(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0\right) - kq_1\xi_1 - kq_2\xi_2 + \dot{G}_N = 0.$$

It has the following solutions:

$$\xi_0 = -1, \quad \xi_1 = \xi_2 = 0, \quad G_N = 0,$$
$$\xi_0 = 0, \quad \xi_1 = -q_2, \quad \xi_2 = q_1, \quad G_N = 0,$$
$$\xi_0 = -1, \quad \xi_1 = 0, \quad \xi_2 = -\dot{q}_2, \quad G_N = \frac{1}{2}m\dot{q}_2^2 - \frac{1}{2}kq_2^2,$$
$$\xi_0 = -1, \quad \xi_1 = -\dot{q}_1, \quad \xi_2 = 0, \quad G_N = \frac{1}{2}m\dot{q}_1^2 - \frac{1}{2}kq_1^2,$$
$$\xi_0 = 0, \quad \xi_1 = \dot{q}_1 + \frac{1}{2}\dot{q}_2, \quad \xi_2 = \dot{q}_2 + \frac{1}{2}\dot{q}_1,$$
$$G_N = -L + \frac{1}{2}kq_1q_2 - \frac{1}{2}m\dot{q}_1\dot{q}_2.$$

By using Proposition 5.1, we obtain the following Noether conserved quantities:

$$I_{N_1} = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}k(q_1^2 + q_2^2) = \text{const.},$$

$$I_{N_2} = m(q_1\dot{q}_2 - q_2\dot{q}_1) = \text{const.},$$

$$I_{N_3} = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}kq_1^2 = \text{const.},$$

$$I_{N_4} = \frac{1}{2}m\dot{q}_2^2 + \frac{1}{2}kq_2^2 = \text{const.},$$

$$I_{N_5} = \frac{1}{2}m\dot{q}_1\dot{q}_2 + \frac{1}{2}kq_1q_2 = \text{const.}$$

These five integrals are not independent, because [62]

$$I_{N_1} = I_{N_3} + I_{N_4},$$

$$I_{N_2} = 2\left\{\frac{m}{k}\left(I_{N_3}I_{N_4} - I_{N_5}^2\right)\right\}^{1/2}.$$

The integral I_{N_1} is the integral of energy, the integral I_{N_2} is the integral of momentum moment, the integrals I_{N_3} and I_{N_4} are the local integrals of energy. The Lagrangian mechanics only provides us the integral I_{N_1}.

Example 5.2 The Emden equation is

$$\ddot{q} + \frac{2}{t}\dot{q} + q^5 = 0.$$

This is a Lagrangian system whose Lagrangian has the form

$$L = t^2\left(\frac{1}{2}\dot{q}^2 - \frac{1}{6}q^6\right).$$

The Noether identity (6) gives

$$L\dot{\xi}_0 + t\left(\dot{q}^2 - \frac{1}{3}q^6\right)\xi_0 + t^2\dot{q}\left(\dot{\xi} - \dot{q}\dot{\xi}_0\right) - t^2q^5\xi + \dot{G}_N = 0.$$

It has the solution

$$\xi_0 = -2t, \quad \xi = q, \quad G_N = 0.$$

The Noether conserved quantity (43) gives

$$I_N = t^3\dot{q}^2 + t^2q\dot{q} + \frac{1}{3}t^3q^6 = \text{const.}$$

5.2 General holonomic system

Proposition 5.2 *For the general holonomic system* (8)*, if the infinitesimal generators* ξ_0, ξ_s *and the gauge function* G_N *satisfy the Noether identity* (9)*, then the Noether symmetry will lead to the Noether conserved quantity*

$$I_N = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_N = const. \tag{44}$$

This is the Noether theorem of the general holonomic system.

Example 5.3 The differential equation model of the armament race can be written in the form

$$\ddot{q} + (\alpha + \beta)\dot{q} + (\alpha\beta - kl)q = 0,$$

where α, β, k and l are constants. It is a particular case of the Lanchester model [63]. The equation can be considered as the equation of a holonomic system whose Lagrangian is

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}(\alpha\beta - kl)q^2,$$

and the generalized force is

$$Q = -(\alpha + \beta)\dot{q}.$$

The Noether identity (9) gives

$$\left\{\frac{1}{2}\dot{q}^2 - \frac{1}{2}(\alpha\beta - kl)q^2\right\}\dot{\xi}_0 + \dot{q}\left(\dot{\xi} - \dot{q}\dot{\xi}_0\right)$$
$$-(\alpha\beta - kl)q\xi - (\alpha + \beta)\dot{q}(\xi - \dot{q}\xi_0) + \dot{G}_N = 0.$$

which has the solution

$$\xi_0 = 0, \quad \xi = \exp At, \quad G_N = q(\alpha + \beta - A)\exp At,$$

where the constant A is the solution of the following algebraic equation

$$A^2 - (\alpha + \beta)A + \alpha\beta - kl = 0.$$

The Noether conserved quantity (44) gives

$$I_N = \{\dot{q} + (\alpha + \beta - A)\}\exp At = \text{const.}$$

Example 5.4 The differential equation is

$$\ddot{q} + knq^{n-1} = -\mu\left(a\dot{q} + b\dot{q}^2\right),$$

where n is a positive integer, a, b and k are constants and $\mu \ll 1$ [8].

The equation can be considered as the equation of a holonomic system whose Lagrangian is

$$L = \frac{1}{2}\dot{q}^2 - kq^n,$$

and the generalized force is

$$Q = -\mu\left(a\dot{q} + b\dot{q}^2\right).$$

There is no exact constant of motion in the example. In the following, we seek the approximate constant of motion of the system by using the Noether theorem. Let

$$\xi_0 = (\xi_0)_0 + \mu(\xi_0)_1 + \cdots,$$
$$\xi = (\xi)_0 + \mu(\xi)_1 + \cdots,$$
$$G_N = (G_N)_0 + \mu(G_N)_1 + \cdots.$$

Substituting them into the Noether identity (9), we obtain

$$-knq^{n-1}\{(\xi)_0 + \mu(\xi)_1 + \cdots\} + \dot{q}\left\{(\dot{\xi})_0 + \mu\left(\dot{\xi}\right)_1 + \cdots\right\}$$
$$+ \left(-\frac{1}{2}\dot{q}^2 - kq^n\right)\{(\dot{\xi}_0)_0 + \mu(\dot{\xi}_0)_1 + \cdots\}$$
$$-\mu\left(a\dot{q} + b\dot{q}^2\right)\{(\xi)_0 + \mu(\xi)_1 + \cdots - \dot{q}(\xi_0)_0$$
$$-\mu\dot{q}(\xi_0)_1 - \cdots\}$$
$$+ \left(\dot{G}_N\right)_0 + \mu\left(\dot{G}_N\right)_1 + \cdots = 0.$$

Separating the terms containing μ and the terms not containing μ, we obtain

$$-knq^{n-1}(\xi)_0+\dot{q}\left(\dot{\xi}\right)_0-\left(\frac{1}{2}\dot{q}^2+kq^n\right)(\dot{\xi}_0)_0+(\dot{G}_N)_0=0,$$

$$-knq^{n-1}(\xi)_1+\dot{q}\left(\dot{\xi}\right)_1-\left(\frac{1}{2}\dot{q}^2+kq^n\right)(\dot{\xi}_0)_1$$
$$-(a\dot{q}+b\dot{q}^2)[(\xi)_0-\dot{q}(\xi_0)_0]+(\dot{G}_N)_1=0.$$

which have the solution

$$(\xi_0)_0=1,\quad (\xi)_0=0,\quad (G_N)_0=0,$$
$$(\xi_0)_1=2bq+\frac{2ant}{2+n},\quad (\xi)_1=-\frac{2aq}{2+n},\quad (G_N)_1=2bk\frac{q^{n+1}}{1+n}.$$

The Noether conserved quantity (44) gives

$$I_N=\left(\frac{1}{2}\dot{q}^2-kq^n\right)\left\{1+2\mu\left(bq+\frac{ant}{2+n}\right)\right\}$$
$$+\dot{q}\left\{-\mu\frac{2aq}{2+n}-\dot{q}\left[1+2\mu\left(bq+\frac{ant}{2+n}\right)\right]\right\}$$
$$+2\mu bk\frac{q^{n+1}}{1+n},$$

where the μ^2 terms and the higher terms are neglected, therefore, I_N is not a constant of motion. But we have

$$\frac{\mathrm{d}I_N}{\mathrm{d}t}=\mu^2(\cdots)+\cdots\approx 0,$$

and I_N is a generalized adiabatic invariant [8].

5.3 Nonholonomic system

Proposition 5.3 *For the nonholonomic system* (10), (11), *if the infinitesimal generators* ξ_0, ξ_s *and the gauge function* G_N *satisfy Eqs.* (14) *and* (15), *then the Noether symmetry will lead to the Noether conserved quantity* (43) [32].

Example 5.5 The Lagrangian and the constraint equation of a nonholonomic system are respectively

$$L=\frac{1}{2}m\left(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2\right)-mgq_3,$$
$$f=\dot{q}_3-\left(\dot{q}_1^2+\dot{q}_2^2\right)^{1/2}=0.$$

This is the celebrated Appell–Hamel problem [22,25,26].

The Noether identity (16) and Eq. (15) give respectively

$$L\dot{\xi}_0+m\dot{q}_1\left(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0\right)+m\dot{q}_2\left(\dot{\xi}_2-\dot{q}_2\dot{\xi}_0\right)$$
$$+m\dot{q}_3\left(\dot{\xi}_3-\dot{q}_3\dot{\xi}_0\right)-mg\xi_3+\dot{G}_N=0,$$
$$\xi_3-\dot{q}_3\xi_0-\left(\dot{q}_1^2+\dot{q}_2^2\right)^{-1/2}$$
$$\times\{\dot{q}_1\left(\xi_1-\dot{q}_1\xi_0\right)+\dot{q}_2\left(\xi_2-\dot{q}_2\xi_0\right)\}=0.$$

They have the following solutions:

$$\xi_0=-1,\quad \xi_1=\xi_2=\xi_3=0,\quad G_N=0,$$
$$\xi_0=0,\quad \xi_1=-\frac{1}{\dot{q}_2},\quad \xi_2=\frac{\dot{q}_1}{\dot{q}_2^2},\quad \xi_3=0,\quad G_N=-m\frac{\dot{q}_1}{\dot{q}_2},$$
$$\xi_0=0,\quad \xi_1=0,\quad \xi_2=\dot{q}_3,\quad \xi_3=\dot{q}_2,$$
$$\quad G_N=mgq_2-m\dot{q}_2\dot{q}_3,$$
$$\xi_0=0,\quad \xi_1=\dot{q}_3,\quad \xi_2=0,\quad \xi_3=\dot{q}_1,$$
$$\quad G_N=mgq_1-m\dot{q}_1\dot{q}_3.$$

The corresponding Noether conserved quantities are respectively

$$I_{N_1}=\frac{1}{2}m\left(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2\right)+mgq_3=\text{const.},$$
$$I_{N_2}=-m\frac{\dot{q}_1}{\dot{q}_3}=\text{const.},$$
$$I_{N_3}=m\dot{q}_2\dot{q}_3+mgq_2=\text{const.},$$
$$I_{N_4}=m\dot{q}_1\dot{q}_3+mgq_1=\text{const.}$$

Example 5.6 The Lagrangian and the constraint equation of a nonholonomic system are respectively

$$L=\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right),$$
$$f=\dot{q}_1+bt\dot{q}_2-bq_2+t=0,$$

where b is a constant [25].

Equation (12) give

$$\ddot{q}_1=-\frac{1}{1+b^2t^2},\quad \ddot{q}_2=-\frac{bt}{1+b^2t^2}.$$

The Noether identity (14) becomes

$$\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right)\dot{\xi}_0+\dot{q}_1\left(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0\right)+\dot{q}_2\left(\dot{\xi}_2-\dot{q}_2\dot{\xi}_0\right)$$
$$-\frac{1}{1+b^2t^2}\left(\xi_1-\dot{q}_1\xi_0\right)-\frac{bt}{1+b^2t^2}\left(\xi_2-\dot{q}_2\xi_0\right)+\dot{G}_N=0.$$

Equation (15) gives

$$\xi_1-\dot{q}_1\xi_0+bt\left(\xi_2-\dot{q}_2\xi_0\right)=0.$$

The above two equations have the solution

$$\xi_0=0,\quad \xi_1=bt,\quad \xi_2=-1,\quad G_N=-bq_1.$$

Thus, the Noether conserved quantity (43) gives

$$I_N=\dot{q}_1bt-\dot{q}_2-bq_1=\text{const.}$$

5.4 Birkhoffian system

Proposition 5.4 [14,34] *For the Birkhoffian system* (17), *if the generators of infinitesimal transformations* ξ_0, ξ_μ *and the gauge function* G_N *satisfy the Noether identity* (20), *then the Noether symmetry will lead to the Noether conserved quantity*

$$I_N=R_\mu\xi_\mu-B\xi_0+G_N=const. \tag{45}$$

Example 5.7 [34] The Birkhoffian and the Birkhoff's functions of a system are respectively

$$B = \frac{1}{2}\left\{ \left(a^3\right)^2 + 2a^2a^3 - \left(a^4\right)^2 \right\},$$

$$R_1 = a^2 + a^3, \quad R_2 = 0, \quad R_3 = a^4, \quad R_4 = 0.$$

The Noether identity (20) gives

$$\left(\dot{a}^1 - a^3\right)\xi_2 + \left(\dot{a}^1 - a^2 - a^3\right)\xi_3 + \left(\dot{a}^3 + a^4\right)\xi_4 - B\dot{\xi}_0$$

$$+ \left(a^2 + a^3\right)\dot{\xi}_1 + a^4\dot{\xi}_3 + \dot{G}_N = 0.$$

It has the following solutions:

$$\xi_0 = 0, \quad \xi_1 = \cos t, \quad \xi_2 = \sin t, \quad \xi_3 = -\sin t, \quad \xi_4 = \cos t,$$
$$G_N = -a^3\cos t,$$
$$\xi_0 = 0, \quad \xi_1 = \sin t, \quad \xi_2 = -\cos t, \quad \xi_3 = \cos t, \quad \xi_4 = \sin t,$$
$$G_N = -a^3\sin t,$$
$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = \xi_3 = \xi_4 = 0, \quad G_N = 0,$$
$$\xi_0 = 0, \quad \xi_1 = -t, \quad \xi_2 = 0, \quad \xi_3 = -1, \quad \xi_4 = 0, \quad G_N = a^1.$$

The Noether conserved quantity (45) gives respectively

$$I_{N_1} = a^2\cos t - a^4\sin t = C_1,$$
$$I_{N_2} = a^2\sin t + a^4\cos t = C_2,$$
$$I_{N_3} = a^2 + a^3 = C_3,$$
$$I_{N_4} = a^1 - a^4 - \left(a^2 + a^3\right)t = C_4.$$

from which we obtain the solution as

$$a^1 = -C_1\sin t + C_2\cos t + C_3 t + C_4,$$
$$a^2 = C_1\cos t + C_2\sin t,$$
$$a^3 = -C_1\cos t - C_2\sin t + C_3,$$
$$a^4 = -C_1\sin t + C_2\cos t.$$

6 Hojman type conserved quantities deduced by Lie symmetry

Hojman gave a conserved quantity by using the Lie symmetry under the infinitesimal transformations where time is not changed [43]. Clearly, this result is applicable to the constrained mechanical systems.

6.1 Lagrangian system

Choosing the infinitesimal transformations as

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\tag{46}$$

the determining equations (23) of the Lie symmetry become

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial\alpha_s}{\partial q_k}\xi_k + \frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k,\tag{47}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial\dot{q}_s}.\tag{48}$$

Proposition 6.1 *For the Lagrangian system* (22), *if the infinitesimal generators ξ_s satisfy the determining equations* (47) *and there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that*

$$\frac{\partial\alpha_s}{\partial\dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0,\tag{49}$$

then the Lie symmetry will lead to the Hojman type conserved quantity

$$I_H = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial\dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = const.\tag{50}$$

The complexity of the above proposition is to find the generators ξ_s by Eq. (47) and the function μ by Eq. (49).

Example 6.1 The Lagrangian of a system is

$$L = \frac{1}{2}\dot{q}^2\exp(-\gamma t),$$

where γ is a constant.

Equation (22) give

$$\ddot{q} = \gamma\dot{q}.$$

Equation (47) give

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi = \gamma\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi.$$

Equation (49) gives

$$\gamma + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0.$$

from which one can obtain

$$\xi = (\gamma q - \dot{q})^2, \quad \mu = \exp(-\gamma t).$$

Substitution of them in (50) gives the Hojman type conserved quantity

$$I_H = 2\gamma(\gamma q - \dot{q}) = const.$$

6.2 General holonomic system

For the system (24), the determining equations of the Lie symmetry under the transformations (46) have the form

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial\beta_s}{\partial q_k}\xi_k + \frac{\partial\beta_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k.\tag{51}$$

Proposition 6.2 *For the general holonomic system* (24), *if the generators ξ_s of infinitesimal transformations satisfy the determining equations* (51) *of the Lie symmetry and there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that*

$$\frac{\partial\beta_s}{\partial\dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0,\tag{52}$$

then the Lie symmetry will lead to the Hojman type conserved quantity (50).

Example 6.2 The Lagrangian and the generalized forces of a system are respectively

$$L = \frac{1}{2} \left(\dot{q}_1^2 + \dot{q}_2^2 \right), \quad Q_1 = 0, \quad Q_2 = \dot{q}_1.$$

The differential equations of motion of the system have the form

$$\ddot{q}_1 = q_1, \quad \ddot{q}_2 = \dot{q}_1.$$

which are called the Whittaker equations.

Equations (51) and (52) give

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}t} \xi_1 = \xi_1, \quad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}t} \xi_2 = \frac{\mathrm{d}}{\mathrm{d}t} \xi_1, \quad 1 + \frac{\mathrm{d}}{\mathrm{d}t} \ln \mu = 0.$$

They have the following solutions

$$\xi_1 = 0, \quad \xi_2 = t, \quad \mu = q_1 - \dot{q}_2,$$
$$\xi_1 = \xi_2 = \exp t, \quad \mu = (q_1 + \dot{q}_1) \exp(-t),$$
$$\xi_1 = -\xi_2 = \exp(-t), \quad \mu = (q_1 - \dot{q}_1) \exp t.$$

The corresponding Hojman type conserved quantities are respectively

$$I_{H_1} = -(q_1 - \dot{q}_2)^{-1} = C_1,$$
$$I_{H_2} = 2(q_1 + \dot{q}_1)^{-1} \exp t = C_2,$$
$$I_{H_3} = 2(q_1 - \dot{q}_1)^{-1} \exp(-t) = C_3.$$

from which we obtain

$$q_1 = \frac{1}{C_2} \exp t + \frac{1}{C_3} \exp(-t),$$
$$q_2 = \frac{1}{C_2} \exp t - \frac{1}{C_3} \exp(-t) + \frac{1}{C_1} t + C_4.$$

6.3 Nonholonomic system

Under the infinitesimal transformations (46), the determining equations of the Lie symmetry of Eq. (26) have the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}t} \xi_s = \frac{\partial \gamma_s}{\partial q_k} \xi_k + \frac{\partial \gamma_s}{\partial \dot{q}_k} \frac{\mathrm{d}}{\mathrm{d}t} \xi_k, \tag{53}$$

Equation (28) become

$$\frac{\partial f_\beta}{\partial q_s} \xi_s + \frac{\partial f_\beta}{\partial \dot{q}_s} \frac{\mathrm{d}}{\mathrm{d}t} \xi_s = 0. \tag{54}$$

Proposition 6.3 *For the nonholonomic system* (10), (11), *if the generators ξ_s satisfy Eqs.* (53) *and* (54), *and there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that*

$$\frac{\partial \gamma_s}{\partial \dot{q}_s} + \frac{\mathrm{d}}{\mathrm{d}t} \ln \mu = 0, \tag{55}$$

then the Lie symmetry will lead to the Hojman type conserved quantity (50) [64].

Example 6.3 The Lagrangian and the constraint equation of a nonholonomic system are respectively

$$L = \frac{1}{2} \left(\dot{q}_1^2 + \dot{q}_2^2 \right), \quad f = \dot{q}_2 - t\dot{q}_1 = 0.$$

Equation (12) give

$$\ddot{q}_1 = -\frac{t}{1+t^2} \dot{q}_1, \quad \ddot{q}_2 = \frac{1}{1+t^2} \dot{q}_1.$$

Equations (53), (54) and (55) give

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}t} \xi_1 = -\frac{t}{1+t^2} \frac{\mathrm{d}}{\mathrm{d}t} \xi_1, \quad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}t} \xi_2 = \frac{1}{1+t^2} \frac{\mathrm{d}}{\mathrm{d}t} \xi_1,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \xi_2 - t \frac{\mathrm{d}}{\mathrm{d}t} \xi_1 = 0,$$
$$-\frac{t}{1+t^2} + \frac{\mathrm{d}}{\mathrm{d}t} \ln \mu = 0.$$

They have the following solutions

$$\xi_1 = 1, \quad \xi_2 = (q_2 - \dot{q}_1 - t\dot{q}_2)^2, \quad \mu = \left(1 + t^2\right)^{1/2},$$
$$\xi_1 = 0, \quad \xi_2 = \dot{q}_1 \left(1 + t^2\right)^{1/2} (q_2 - \dot{q}_1 - t\dot{q}_2), \quad \mu = \left(1 + t^2\right)^{1/2}.$$

The corresponding Hojman type conserved quantities are respectively

$$I_{H_1} = 2 (q_2 - \dot{q}_1 - t\dot{q}_2) = \mathrm{const.},$$
$$I_{H_2} = \dot{q}_1 \left(1 + t^2\right)^{1/2} = \mathrm{const.}$$

6.4 Birkhoffian system

Choose the infinitesimal transformations as

$$t^* = t, \quad a^{\mu*} \left(t^*\right) = a^\mu(t) + \varepsilon \xi_\mu(t, \boldsymbol{a}). \tag{56}$$

Under the transformations (56), the determining equations (31) of the Lie symmetry become

$$\frac{\mathrm{d}}{\mathrm{d}t} \xi_\mu = \frac{\partial}{\partial a^\rho} \left\{ \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} \right) \right\} \xi_\rho \quad (\mu, \nu, \rho = 1, \ldots, 2n), \tag{57}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} \right) \frac{\partial}{\partial a^\mu}. \tag{58}$$

Then we have

Proposition 6.4 *For the Birkhoffian system* (30), *if the infinitesimal generators ξ_μ satisfy Eq.* (57) *and there exists a function $\mu = \mu(t, \boldsymbol{a})$ such that*

$$\frac{\partial}{\partial a^\mu} \left\{ \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} \right) \right\} + \frac{\mathrm{d}}{\mathrm{d}t} \ln \mu = 0, \tag{59}$$

then the Lie symmetry will lead to the Hojman type conserved quantity

$$I_H = \frac{1}{\mu} \frac{\partial}{\partial a^\nu}(\mu \xi_\nu) = const. \qquad (60)$$

Example 6.4 The Birkhoffian and the Birkhoff's functions of a Birkhoffian system are respectively [34]

$$B = \frac{1}{2}\left\{\left(a^3\right)^2 + 2a^2 a^3 - \left(a^4\right)^2\right\},$$
$$R_1 = a^2 + a^3, \quad R_2 = 0, \quad R_3 = a^4, \quad R_4 = 0.$$

Equations (57) and (59) give respectively

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 = \xi_3, \quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 = \xi_4, \quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 = -\xi_4,$$
$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_4 = -\xi_2, \quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln \mu = 0.$$

They have the solutions

$$\xi_1 = \left\{a^1 - a^4 - \left(a^2 + a^3\right)t\right\}^2, \quad \xi_2 = \xi_3 = \xi_4 = 0, \quad \mu = 1,$$
$$\xi_1 = t, \quad \xi_2 = 0, \quad \xi_3 = 1, \quad \xi_4 = 0, \quad \mu = a^2 + a^3.$$

The conserved quantity (60) gives respectively

$$I_{H_1} = 2\left\{a^1 - a^4 - \left(a^2 + a^3\right)t\right\} = const.,$$
$$I_{H_2} = \left(a^2 + a^3\right)^{-1} = const.$$

The complexity of the study of the Hojman type conserved quantity is to find the generators ξ_s or ξ_μ by using the determining equations of the Lie symmetry and the suitable function μ.

7 New conserved quantities deduced by form invariance

For a holonomic system, a new conserved quantity can be deduced by the form invariance of the system [55]. We now present the new conserved quantity deduced by the form invariance for constrained mechanical systems, including the Lagrangian system, the general holonomic system, the nonholonomic system and the Birkhoffian system.

7.1 Lagrangian system

Proposition 7.1 *For the Lagrangian system* (1)*, if the generators* ξ_0, ξ_s *of the form invariance and the gauge function* $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy the structure equation*

$$\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \tilde{X}^{(1)}\left\{\tilde{X}^{(1)}(L)\right\} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_F = 0, \qquad (61)$$

where

$$\tilde{X}^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s}, \qquad (62)$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \alpha_s \frac{\partial}{\partial \dot{q}_s}, \qquad (63)$$

then the form invariance will lead to the new conserved quantity

$$I_F = \tilde{X}^{(1)}(L)\xi_0 + \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s \xi_0) + G_F = const. \qquad (64)$$

Example 7.1 The Lagrangian of a system is

$$L = t^2\left(\frac{1}{2}\dot{q}^2 - \frac{1}{6}q^6\right).$$

We have

$$X^{(1)}(L) = 2t\xi_0\left(\frac{1}{2}\dot{q}^2 - \frac{1}{6}q^6\right) + t^2\left\{\dot{q}\left(\dot{\xi} - \dot{q}\dot{\xi}_0\right) - q^5\xi\right\}.$$

Choosing the generators as

$$\xi_0 = t, \quad \xi = -\frac{1}{2}q,$$

and we have

$$\tilde{X}^{(1)}(L) = X^{(1)}(L) = -L,$$
$$E\left\{X^{(1)}(L)\right\} = -E(L) = 0,$$

therefore the above generators correspond to a form invariance of the system. Substituting them into the structure equation (61), we obtain

$$G_F = 0.$$

Then the conserved quantity (64) gives

$$I_F = \frac{1}{2}t^3\dot{q}^2 + \frac{1}{2}t^2 q\dot{q} + \frac{1}{6}t^3 q^6 = const.$$

It must be noted that the generators ξ_0, ξ are also the generators of a Noether symmetry, and the Noether conserved quantity deduced by them is

$$I_N = -I_F.$$

7.2 General holonomic system

Proposition 7.2 *For the general holonomic system* (8)*, if the generators* ξ_0, ξ_s *of the form invariance and the gauge function* $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy the structure equation*

$$\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \tilde{X}^{(1)}\left\{\tilde{X}^{(1)}(L)\right\} + \tilde{X}^{(1)}(Q_s)(\xi_s - \dot{q}_s \xi_0)$$
$$+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_F = 0, \qquad (65)$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \beta_s \frac{\partial}{\partial \dot{q}_s}, \tag{66}$$

then the form invariance will lead to the new conserved quantity (64).

Obviously, Proposition 7.1 is a particular case of Proposition 7.2.

Example 7.2 A system with two-degree-of-freedom is

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - q_2,$$

$$Q_1 = \frac{1}{1+t^2}\left(\dot{q}_2 - 2t\dot{q}_1 - t\right),$$

$$Q_2 = -\frac{1}{1+t^2}\left(t\dot{q}_2 + 2\dot{q}_1 - t^2\right).$$

We can obtain the following result

$$\xi_0 = \xi_1 = 0, \quad \xi_2 = t\dot{q}_1 + \dot{q}_2 + q_1 + t, \quad G_F = 0,$$

$$I_F = -\left(t\dot{q}_1 + \dot{q}_2 + q_1 + t\right) = \text{const}.$$

7.3 Nonholonomic system

Proposition 7.3 *For the nonholonomic system (10), (11), if the generators ξ_0, ξ_s of the form invariance and the gauge function $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfy the structure equation*

$$\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \tilde{X}^{(1)}\left\{\tilde{X}^{(1)}(L)\right\} + \tilde{X}^{(1)}(Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0)$$

$$+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_F = 0, \tag{67}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \gamma_s \frac{\partial}{\partial \dot{q}_s}, \tag{68}$$

then the form invariance will lead to the conserved quantity (64).

Example 7.3 Novoselov problem is [25]

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right), \quad f = \dot{q}_1 + bt\dot{q}_2 - bq_2 + t = 0.$$

By Proposition 7.3, we can obtain the result

$$\xi_0 = 0, \quad \xi_1 = \dot{q}_2 - bt\dot{q}_1 + bq_1 + bt, \quad \xi_2 = 1, \quad G_F = -b^2 t,$$

$$I_F = b(\dot{q}_2 - bt\dot{q}_1 + bq_1) = \text{const}.$$

7.4 Birkhoffian system

Proposition 7.4 *For the Birkhoffian system (17), if the generators ξ_0, ξ_μ of the form invariance and the gauge function $G_F = G_F(t, \boldsymbol{a})$ satisfy the structure equation*

$$X^{(0)}\left\{X^{(0)}(R_\mu)\right\}\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) - X^{(0)}\left\{X^{(0)}(B)\right\}$$

$$- X^{(0)}(B)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + X^{(0)}(R_\mu)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_F = 0, \tag{69}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\frac{\partial}{\partial a^\mu}, \tag{70}$$

then the form invariance will lead to the conserved quantity

$$I_F = X^{(0)}(R_\mu)\xi_\mu - X^{(0)}(B)\xi_0 + G_F = \text{const}. \tag{71}$$

Example 7.4 A Birkhoffian system of four order is

$$B = \left(a^2 \sin t + a^4 \cos t\right)^2,$$

$$R_1 = 0, \quad R_2 = a^1 - a^4 - \left(a^2 + a^3\right)t$$

$$\qquad + \left(a^2 \sin t + a^4 \cos t\right)\cos t,$$

$$R_3 = a^1 - a^4 - \left(a^2 + a^3\right)t,$$

$$R_4 = -\left(a^2 \sin t + a^4 \cos t\right)\sin t.$$

We can obtain the result

$$\xi_0 = 0, \quad \xi_1 = (a^2 + a^3)t + 1, \quad \xi_3 = a^2 + a^3,$$

$$\xi_2 = \xi_4 = 0, \quad G_F = 0, \quad I_F = a^2 + a^3 = \text{const}.$$

The complexity of the study on the new conserved quantity for the constrained mechanical systems is to find the generators ξ_0, ξ_s (or ξ_μ) of the form invariance and the suitable gauge function G_F.

8 Non-Noether conserved quantities deduced by Noether symmetry

In Sect. 5, we have presented the Noether conserved quantity deduced by the Noether symmetry for the constrained mechanical systems. In fact, we can also find the non-Noether conserved quantity, say the Hojman type conserved quantity and the new conserved quantity, by using the Noether symmetry for the constrained mechanical systems.

8.1 Lagrangian system

Under the infinitesimal transformations (46), the Noether identity (6) of the Lagrangian system (1) becomes

$$\frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \dot{G}_N = 0. \tag{72}$$

Proposition 8.1 *For the Lagrangian system (1), if the generators ξ_s of the Noether symmetry satisfy Eq. (47) and there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying Eq. (49), then the Noether symmetry will lead to the Hojman type conserved quantity (50).*

Example 8.1 The Lagrangian of a system is

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - q_2.$$

Identity (72), Eqs. (47) and (49) give respectively

$$\dot{q}_1 \dot{\xi}_1 + \dot{q}_2 \dot{\xi}_2 - \xi_2 + \dot{G}_N = 0,$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_1 = 0, \quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_2 = 0,$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0.$$

which have the solutions

$$\xi_1 = 1, \quad \xi_2 = 0, \quad G_N = 0, \quad \mu = q_1 - \dot{q}_1 t,$$

$$\xi_1 = 0, \quad \xi_2 = 1, \quad G_N = t, \quad \mu = q_2 - \dot{q}_2 t - \frac{1}{2}t^2.$$

The conserved quantity (50) gives respectively

$$I_{H_1} = (q_1 - \dot{q}_1 t)^{-1} = \text{const.},$$

$$I_{H_2} = \left(q_2 - \dot{q}_2 t - \frac{1}{2}t^2 \right)^{-1} = \text{const.},$$

and the Noether conserved quantity are respectively

$$I_{N_1} = \dot{q}_1 = \text{const.},$$

$$I_{N_2} = \dot{q}_2 + t = \text{const.}$$

Proposition 8.2 *For the Lagrangian system* (1), *if the generators* ξ_0, ξ_s *of the Noether symmetry satisfy Eq.* (34) *and there exists a gauge function* $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying Eq.* (61), *then the Noether symmetry will lead to the new conserved quantity* (64).

8.2 General holonomic system

Under the infinitesimal transformations (46), the Noether identity (9) of the general holonomic system becomes

$$\frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s + Q_s \xi_s + \dot{G}_N = 0. \tag{73}$$

Proposition 8.3 *For the general holonomic system* (8), *if the generators* ξ_s *of the Noether symmetry satisfy Eq.* (51) *and there exists a function* $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying Eq.* (52), *then the Noether symmetry will lead to the Hojman type conserved quantity* (50).

Proposition 8.4 *For the general holonomic system* (8), *if the generators* ξ_0, ξ_s *of the Noether symmetry satisfy Eq.* (36) *and there exists a gauge function* $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying Eq.* (65), *then the Noether symmetry will lead to the new conserved quantity* (64).

Example 8.2 The Lagrangian and the generalized forces of a holonomic system are respectively

$$L = \frac{1}{2} \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 \right) - q_3,$$

$$Q_1 = -\dot{q}_2^2, \quad Q_2 = \dot{q}_1^2, \quad Q_3 = 0.$$

The Noether identity (9) gives

$$L\dot{\xi}_0 + \dot{q}_1 \left(\dot{\xi}_1 - \dot{q}_1 \dot{\xi}_0 \right) + \dot{q}_2 \left(\dot{\xi}_2 - \dot{q}_2 \dot{\xi}_0 \right) + \dot{q}_3 \left(\dot{\xi}_3 - \dot{q}_3 \dot{\xi}_0 \right) - \xi_3$$
$$- \dot{q}_2^2 \left(\xi_1 - \dot{q}_1 \xi_0 \right) + \dot{q}_1^2 \left(\xi_2 - \dot{q}_2 \xi_0 \right) + \dot{G}_N = 0.$$

It has the solution

$$\xi_0 = \xi_1 = \xi_2 = 0, \quad \xi_3 = -\dot{q}_3 - t, \quad G_N = \frac{1}{2}\dot{q}_3^2 - \frac{1}{2}t^2.$$

We have

$$\tilde{X}^{(1)}(L) = \dot{q}_3 + t,$$

$$E_s \left\{ \tilde{X}^{(1)}(L) \right\} = \tilde{X}^{(1)}(Q_s) = 0 \quad (s = 1, 2, 3),$$

$$G_F = 0,$$

and the conserved quantity (64) gives

$$I_F = -\dot{q}_3 - t = \text{const.}$$

It must be noted that the Noether conserved quantity deduced by the Noether symmetry is

$$I_N = -\frac{1}{2}\dot{q}_3^2 - \dot{q}_3 t - \frac{1}{2}t^2 = \text{const.}$$

Therefore, the conserved quantity I_F is a non-Noether one.

8.3 Nonholonomic system

Under the infinitesimal transformations (46), the Noether identity (14) of the nonholonomic system becomes

$$\frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s + (Q_s + \Lambda_s)\xi_s + \dot{G}_N = 0, \tag{74}$$

and Eq. (15) become

$$\frac{\partial f_\beta}{\partial \dot{q}_s} \xi_s = 0. \tag{75}$$

Proposition 8.5 *For the nonholonomic system* (10), (11), *if the generators* ξ_s *of the Noether symmetry satisfy Eqs.* (53), (54) *and there exists a function* $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying Eq.* (55), *then the Noether symmetry will lead to the Hojman type conserved quantity* (50).

Proposition 8.6 *For the nonholonomic system* (10), (11), *if the generators* ξ_0, ξ_s *of the Noether symmetry satisfy Eqs.* (38), (40) *and there exists a gauge function* $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying Eq.* (67), *then the Noether symmetry will lead to the new conserved quantity* (64).

Example 8.3 Rolling sphere problem.

The Lagrangian and the constraint equations of the problem are respectively

$$L = \frac{1}{2}m \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} \frac{2}{5} ma^2 \left(\dot{\psi}^2 + \dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\psi}\dot{\varphi}\cos\theta \right),$$

$$f_1 = \dot{x} + a \left(\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \right) = 0,$$

$$f_2 = \dot{y} + a \left(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \right) = 0,$$

where a is the radius of the sphere, m the mass, x and y the coordinates of the center of the sphere, ψ, θ and φ the Euler angles.

Let

$$q_1 = \psi, \quad q_2 = \theta, \quad q_3 = \varphi, \quad q_4 = x, \quad q_5 = y.$$

Then Eq. (11) give

$$\frac{2}{5}ma^2 \left(\dot{q}_1 + \dot{q}_3 \cos q_2\right)^{\cdot} = 0,$$

$$\frac{2}{5}ma^2 \ddot{q}_2 + \frac{2}{5}ma^2 \dot{q}_1 \dot{q}_3 \sin q_2 = -\lambda_1 a \sin q_1 + \lambda_2 a \cos q_1,$$

$$\frac{2}{5}ma^2 \left(\dot{q}_3 + \dot{q}_1 \cos q_2\right)^{\cdot} = \lambda_1 a \sin q_2 \cos q_1$$

$$+\lambda_2 a \sin q_2 \sin q_1,$$

$$m\ddot{q}_4 = \lambda_1,$$

$$m\ddot{q}_5 = \lambda_2.$$

By using the constraint equations, we obtain [32]

$$\lambda_1 = \lambda_2 = 0,$$

and then

$$Q_s + \Lambda_s = 0 \quad (s = 1, 2, 3, 4, 5).$$

All of the accelerations $\ddot{q}_s$ can be solved as

$$\ddot{q}_1 = \frac{1}{\sin q_2}\dot{q}_2\dot{q}_3 - \frac{\cos q_2}{\sin q_2}\dot{q}_1\dot{q}_2,$$

$$\ddot{q}_2 = -\dot{q}_1\dot{q}_3 \sin q_2,$$

$$\ddot{q}_3 = \frac{1}{\sin q_2}\dot{q}_1\dot{q}_2 - \frac{\cos q_2}{\sin q_2}\dot{q}_2\dot{q}_3,$$

$$\ddot{q}_4 = 0,$$

$$\ddot{q}_5 = 0.$$

Equation (53) give

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 = \frac{1}{\sin q_2}\left(\dot{q}_3\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 + \dot{q}_2\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3\right) - \frac{\cos q_2}{\sin^2 q_2}\dot{q}_2\dot{q}_3\xi_2$$

$$- \frac{\cos q_2}{\sin q_2}\left(\dot{q}_1\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 + \dot{q}_2\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1\right) + \frac{1}{\sin^2 q_2}\dot{q}_1\dot{q}_2\xi_2,$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 = -\dot{q}_1\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 \sin q_2 - \dot{q}_3\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 \sin q_2 - \dot{q}_1\dot{q}_3\xi_2 \cos q_2,$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 = \frac{1}{\sin q_2}\left(\dot{q}_1\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 + \dot{q}_2\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1\right) - \frac{\cos q_2}{\sin^2 q_2}\dot{q}_1\dot{q}_2\xi_2$$

$$- \frac{\cos q_2}{\sin q_2}\left(\dot{q}_2\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 + \dot{q}_3\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2\right) + \frac{1}{\sin^2 q_2}\dot{q}_2\dot{q}_3\xi_2,$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_4 = 0,$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_5 = 0.$$

Equation (54) show

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_4 + a\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 \sin q_2 \cos q_1 - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 \sin q_1\right]$$

$$+ a\left[\dot{q}_3\xi_2 \cos q_2 \cos q_1 - \dot{q}_3\xi_1 \sin q_2 \sin q_1 - \dot{q}_2\xi_1 \cos q_1\right] = 0,$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_5 + a\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 \sin q_2 \sin q_1 + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 \cos q_1\right]$$

$$+ a\left[\dot{q}_3\xi_2 \cos q_2 \sin q_1 + \dot{q}_3\xi_1 \sin q_2 \cos q_1 - \dot{q}_2\xi_1 \sin q_1\right] = 0.$$

Identity (74) becomes

$$-\frac{2}{5}ma^2\dot{q}_1\dot{q}_3\xi_2 \sin q_2 + \frac{2}{5}ma^2\dot{q}_1\dot{\xi}_1 + \frac{2}{5}ma^2\dot{q}_2\dot{\xi}_2 + \frac{2}{5}ma^2\dot{q}_3\dot{\xi}_3$$

$$+ \frac{2}{5}ma^2\left(\dot{q}_1\dot{\xi}_3 + \dot{q}_3\dot{\xi}_1\right)\cos q_2 + m\dot{q}_4\dot{\xi}_4 + m\dot{q}_5\dot{\xi}_5 + \dot{G}_N = 0.$$

Eq. (55) gives

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln \mu = 0.$$

The above 9 equations have the solutions

$$\xi_4 = 1, \quad \xi_1 = \xi_2 = \xi_3 = \xi_5 = 0, \quad \mu = q_4 - \dot{q}_4 t,$$

$$\xi_5 = 1, \quad \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0, \quad \mu = q_5 - \dot{q}_5 t.$$

The Hojman type conserved quantity are respectively

$$I_{H_1} = (q_4 - \dot{q}_4 t)^{-1} = \text{const.},$$

$$I_{H_2} = (q_5 - \dot{q}_5 t)^{-1} = \text{const.}$$

8.4 Birkhoffian system

Under the infinitesimal transformations (56), the Noether identity (20) becomes

$$\left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu}\right)\xi_\mu + R_\mu\dot{\xi}_\mu + \dot{G}_N = 0. \qquad (76)$$

Proposition 8.7 *For the Birkhoffian system (17), if the generators ξ_μ of the Noether symmetry satisfy Eq. (57) and there exists a function $\mu = \mu(t, \boldsymbol{a})$ satisfying Eq. (59), then the Noether symmetry will lead to the Hojman type conserved quantity (60).*

Proposition 8.8 *For the Birkhoffian system (17), if the generators ξ_0, ξ_μ of the Noether symmetry satisfy Eq. (42) and the generators and the gauge function $G_F = G_F(t, \boldsymbol{a})$ satisfy Eq. (69), then the Noether symmetry will lead to the new conserved quantity (71).*

Example 8.4 A four order Birkhoffian system is

$$B = a^2 + \frac{1}{2}\left\{\left(a^3\right)^2 + \left(a^4\right)^2\right\},$$

$$R_1 = a^3, \quad R_2 = a^4, \quad R_3 = R_4 = 0.$$

The Noether identity (20) gives

$$\left(\dot{a}^1 - a^3\right)\xi_3 + \left(\dot{a}^2 - a^4\right)\xi_4 - \xi_2$$
$$- \left\{a^2 + \frac{1}{2}\left[\left(a^3\right)^2 + \left(a^4\right)^2\right]\right\}\dot{\xi}_0 + a^3\dot{\xi}_1$$
$$+ a^4\dot{\xi}_2 + \dot{G}_N = 0.$$

Equation (42) give

$$\left(\frac{\partial\xi_4}{\partial a^1} - \frac{\partial\xi_3}{\partial a^2}\right)\dot{a}^2 - \frac{\partial\xi_3}{\partial a^3}\dot{a}^3 - \frac{\partial\xi_3}{\partial a^4}\dot{a}^4$$
$$- \left(\frac{\partial\xi_2}{\partial a^1} + a^3\frac{\partial\xi_3}{\partial a^1} + a^4\frac{\partial\xi_4}{\partial a^1}\right) = 0,$$
$$\left(\frac{\partial\xi_3}{\partial a^2} - \frac{\partial\xi_4}{\partial a^1}\right)\dot{a}^1 - \frac{\partial\xi_4}{\partial a^3}\dot{a}^3$$
$$- \frac{\partial\xi_4}{\partial a^4}\dot{a}^4 - \left(\frac{\partial\xi_2}{\partial a^2} + a^3\frac{\partial\xi_3}{\partial a^2} + a^4\frac{\partial\xi_4}{\partial a^2}\right) = 0,$$
$$\frac{\partial\xi_3}{\partial a^3}\dot{a}^1 + \frac{\partial\xi_4}{\partial a^3}\dot{a}^2 - \left(\frac{\partial\xi_2}{\partial a^3} + \xi_3 + a^3\frac{\partial\xi_3}{\partial a^3} + a^4\frac{\partial\xi_4}{\partial a^3}\right) = 0,$$
$$\frac{\partial\xi_3}{\partial a^4}\dot{a}^1 + \frac{\partial\xi_4}{\partial a^4}\dot{a}^2 - \left(\frac{\partial\xi_2}{\partial a^4} + \xi_4 + a^3\frac{\partial\xi_3}{\partial a^4} + \frac{\partial\xi_4}{\partial a^4}\right) = 0.$$

Eq. (69) becomes

$$\left(\frac{\partial\xi_3}{\partial a^\mu}\xi_\mu + \frac{\partial\xi_3}{\partial t}\xi_0\right)a^3 + \left(\frac{\partial\xi_4}{\partial a^\mu}\xi_\mu + \frac{\partial\xi_4}{\partial t}\xi_0\right)a^4$$
$$- X^{(0)}\left(\xi_2 + a^3\xi_3 + a^4\xi_4\right)$$
$$- \left(\xi_2 + a^3\xi_3 + a^4\xi_4\right)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \xi_3\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 + \xi_4\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2$$
$$+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_F = 0.$$

These 6 equations have the following solution

$$\xi_0 = \xi_1 = \xi_3 = 0, \quad \xi_2 = a^4, \quad \xi_4 = -1, \quad G_F = -t.$$

The conserved quantity (71) gives

$$I_F = -a^4 - t = \text{const.}$$

This is a new conserved quantity deduced by the Noether symmetry. It must be noted that the Noether conserved quantity deduced by the Noether symmetry is

$$I_N = \frac{1}{2}\left(a^4\right)^2 + a^2 = \text{const.},$$

therefore, the conserved quantity I_F is a non-Noether one.

9 Conserved quantities deduced by Lie symmetry

In Sect. 6, we have presented the Hojman type conserved quantities deduced by the Lie symmetry for the constrained mechanical systems. In fact, the Noether conserved quantity

and the new conserved quantity can also be deduced by using the Lie symmetry for the systems.

9.1 Lagrangian system

Proposition 9.1 *For the Lagrangian system* (1), *if the generators* ξ_0, ξ_s *of the Lie symmetry and the gauge function* $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy the Noether identity* (6), *then the Lie symmetry will lead to the Noether conserved quantity* (43).

Proposition 9.2 *For the Lagrangian system* (1), *if the generators* ξ_0, ξ_s *of the Lie symmetry satisfy Eq.* (34) *and the generators and the gauge function* $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy Eq.* (61), *then the Lie symmetry will lead to the new conserved quantity* (64).

Example 9.1 The Lagrangian of a system is

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \left(q_1^2 + q_2^2 + q_1^4 + q_2^4 + 6q_1^2q_2^2\right).$$

Equation (23) give

$$\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 + 2\dot{\xi}_0\left(2q_1 + 4q_1^3 + 12q_1q_2^2\right) = -2\xi_1$$
$$- 12q_1^2\xi_1 - 12\xi_1q_2^2 - 24q_1q_2\xi_2,$$
$$\ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 + 2\dot{\xi}_0\left(2q_2 + 4q_2^3 + 12q_2q_1^2\right) = -2\xi_2$$
$$- 12q_2^2\xi_2 - 12\xi_2q_1^2 - 24q_1q_2\xi_1.$$

The Noether identity (6) gives

$$L\dot{\xi}_0 + \dot{q}_1\left(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0\right) + \dot{q}_2\left(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0\right) - 2q_1\xi_1 - 2q_2\xi_2$$
$$- 4q_1^3\xi_1 - 4q_2^3\xi_2 - 12q_1q_2^2\xi_1 - 12q_1^2q_2\xi_2 + \dot{G}_N = 0.$$

These three equations have the solution

$$\xi_0 = 0, \quad \xi_1 = \dot{q}_2, \quad \xi_2 = \dot{q}_1,$$
$$G_N = -\dot{q}_1\dot{q}_2 + 2q_1q_2 + 4q_1q_2\left(q_1^2 + q_2^2\right).$$

The Noether conserved quantity (43) gives

$$I_N = \dot{q}_1\dot{q}_2 + 2q_1q_2 + 4q_1q_2\left(q_1^2 + q_2^2\right) = \text{const.}$$

9.2 General holonomic system

For the general holonomic system, the Lie symmetry can lead to the Noether conserved quantity and the new conserved quantity. We have

Proposition 9.3 *For the general holonomic system* (8), *if the generators* ξ_0, ξ_s *of the Lie symmetry and the gauge function* $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy the Noether identity* (9), *then the Lie symmetry will lead to the Noether conserved quantity* (43).

Proposition 9.4 *For the general holonomic system* (8), *if the generators* ξ_0, ξ_s *of the Lie symmetry satisfy Eq.* (36) *and the generators and the gauge function* $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy Eq.* (65), *then the Lie symmetry will lead to the new conserved quantity* (64).

Example 9.2 The Lagrangian and the generalized forces of a system with three-degree-of-freedom are respectively

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) - q_3,$$
$$Q_1 = -\dot{q}_2^3, \quad Q_2 = \dot{q}_1^3, \quad Q_3 = 0.$$

Equation (25) give

$$\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 - 2\dot{\xi}_0\left(-\dot{q}_2^3\right) = -3\dot{q}_2^2\left(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0\right),$$
$$\ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 - 2\dot{\xi}_0\left(-\dot{q}_1^3\right) = 3\dot{q}_1^2\left(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0\right),$$
$$\ddot{\xi}_3 - \dot{q}_3\ddot{\xi}_0 - 2\dot{\xi}_0(-1) = 0.$$

They have the solution

$$\xi_0 = \xi_1 = \xi_2 = 0, \quad \xi_3 = \dot{q}_3 + t,$$

and then we have

$$\tilde{X}^{(1)}(L) = -\dot{q}_3 - t, \quad E_s\left\{\tilde{X}^{(1)}(L)\right\} = 0,$$
$$\tilde{X}^{(1)}(Q_s) = 0 \quad (s = 1, 2, 3).$$

Equation (65) gives

$$G_F = 0.$$

Thus, the conserved quantity (64) gives

$$I_F = -\dot{q}_3 - t = \text{const.}$$

The conserved quantity is a non-Noether one, because the Noether conserved quantity deduced by the above generators is

$$I_N = \frac{1}{2}\dot{q}_3^2 + t\dot{q}_3 + \frac{1}{2}t^2 = \text{const.}$$

9.3 Nonholonomic system

For the nonholonomic system, the Lie symmetry can lead to the Noether conserved quantity and the new conserved quantity. We have

Proposition 9.5 *For the nonholonomic system* (10), (11), *if the generators* ξ_0, ξ_s *of the Lie symmetry and the gauge function* $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy the Noether identity* (14) *and Eq.* (15), *then the Lie symmetry will lead to the Noether conserved quantity* (43).

Proposition 9.6 *For the nonholonomic system* (10), (11), *if the generators* ξ_0, ξ_s *of the Lie symmetry satisfy Eqs.* (38) *and* (40), *and the generators and the gauge function* $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy Eq.* (67), *then the Lie symmetry will lead to the new conserved quantity* (64).

Example 9.3 Chaplygin sled problem.

The Lagrangian and the nonholonomic constraint equation are respectively [32]

$$L = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}J\dot{q}_3^2 - \frac{1}{2}kq_3^2,$$
$$f = \dot{q}_2 - \dot{q}_1\tan q_3 = 0.$$

Equation (27) give

$$\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 - 2\dot{\xi}_0(-\dot{q}_1\dot{q}_3\tan q_3) = -\frac{\dot{q}_1\dot{q}_3}{\cos^2 q_3}\xi_3$$
$$+ (\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)(-\dot{q}_3\tan q_3) + (\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0)(-\dot{q}_1\tan q_3),$$
$$\ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 - 2\dot{\xi}_0(\dot{q}_1\dot{q}_3) = \dot{q}_3(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_1(\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0),$$
$$\ddot{\xi}_3 - \dot{q}_3\ddot{\xi}_0 - 2\dot{\xi}_0\left(-\frac{k}{J}q_3\right) = -\frac{k}{J}\xi_3.$$

Equation (28) shows

$$\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0 - (\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)\tan q_3 - \frac{\dot{q}_1}{\cos^2 q_3}\xi_3 = 0.$$

Identity (14) becomes

$$L\dot{\xi}_0 + m\dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + m\dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) + J\dot{q}_3(\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0)$$
$$- kq_3\xi_3 - m\dot{q}_1\dot{q}_3(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)\tan q_3 + m\dot{q}_1\dot{q}_3(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)$$
$$+ \dot{G}_N = 0.$$

Equation (15) provide

$$\xi_2 - \dot{q}_2\xi_0 - (\xi_1 - \dot{q}_1\xi_0)\tan q_3 = 0.$$

These six equations have the solution

$$\xi_0 = -1, \quad \xi_1 = \xi_2 = \xi_3 = 0, \quad G_N = 0,$$

and the conserved quantity (43) gives

$$I_N = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}J\dot{q}_3^2 + \frac{1}{2}kq_3^2 = \text{const.},$$

which is the integral of energy of the system.

9.4 Birkhoffian system

For the Birkhoffian system, the Lie symmetry can lead to the Noether conserved quantity and the new conserved quantity. We have

Proposition 9.7 *For the Birkhoffian system* (17), *if the generators* ξ_0, ξ_μ *of the Lie symmetry and the gauge function* $G_N = G_N(t, \boldsymbol{a})$ *satisfy the Noether identity* (20), *then the Lie symmetry will lead to the Noether conserved quantity* (45).

Proposition 9.8 *For the Birkhoffian system* (17), *if the generators* ξ_0, ξ_μ *of the Lie symmetry satisfy Eq.* (42), *and the generators and the gauge function* $G_F = G_F(t, \boldsymbol{a})$ *satisfy Eq.* (69), *then the Lie symmetry will lead to the new conserved quantity* (71).

Example 9.4 The Birkhoffian and the Birkhoff's functions of a Birkhoffian system of four order are respectively

$$B = \left(a^2 \sin t + a^4 \cos t\right)^2,$$

$$R_1 = 0, \quad R_2 = a^1 - a^4 - \left(a^2 + a^3\right) t$$
$$+ \left(a^2 \sin t + a^4 \cos t\right) \cos t,$$

$$R_3 = a^1 - a^4 - \left(a^2 + a^3\right) t,$$

$$R_4 = - \left(a^2 \sin t + a^4 \cos t\right) \sin t.$$

Equation (31) give

$$\dot{\xi}_1 - a^3 \dot{\xi}_0 = \xi_3, \quad \dot{\xi}_2 - a^4 \dot{\xi}_0 = \xi_4,$$
$$\dot{\xi}_3 + a^4 \dot{\xi}_0 = -\xi_4, \quad \dot{\xi}_4 + a^2 \dot{\xi}_0 = -\xi_2.$$

which have the solution

$$\xi_0 = \xi_2 = \xi_4 = 0, \quad \xi_1 = \left(a^2 + a^3\right) t + 1, \quad \xi_3 = a^2 + a^3,$$

and then we have

$$X^{(0)}(B) = X^{(0)}(R_1) = X^{(0)}(R_4) = 0,$$
$$X^{(0)}(R_2) = X^{(0)}(R_3) = 1,$$
$$G_F = 0.$$

The conserved quantity (71) gives

$$I_F = a^2 + a^3 = \text{const.}$$

10 Conserved quantities deduced by form invariance

In Sect. 7, we have presented the new conserved quantity deduced by the form invariance for the constrained mechanical systems. In fact, the Noether conserved quantity and the Hojman type conserved quantity can also be deduced by using the form invariance for the systems.

10.1 Lagrangian system

Proposition 10.1 *For the Lagrangian system* (1), *if the generators* ξ_0, ξ_s *of the form invariance and the gauge function* $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy the Noether identity* (6), *then the form invariance will lead to the Noether conserved quantity* (43).

Proposition 10.2 *For the Lagrangian system* (1), *if the generators* ξ_s *of the form invariance satisfy Eq.* (47), *and there exists a function* $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying Eq.* (49), *then the form invariance will lead to the Hojman type conserved quantity* (50).

Example 10.1 The Lagrangian of a system is

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{b}\dot{q}_1 \arctan bt + \frac{1}{2b}\dot{q}_2 \ln\left(1 + b^2 t^2\right),$$

where b is a constant.

Taking the generators as

$$\xi_0 = \xi_2 = 0,$$
$$\xi_1 = \frac{1}{2}\left\{q_1 + \int \frac{1}{b}\arctan bt\,dt - t\left[\dot{q}_1 + \frac{1}{b}\arctan bt\right]\right\}^2,$$

and then

$$X^{(1)}(L) = 0,$$

therefore the generators are form invariance ones. In this case, Eq. (47) are verified. Equation (49) gives

$$\frac{\bar{d}}{dt}\ln\mu = 0,$$

and we have

$$\mu = 1.$$

The Hojman type conserved quantity (50) gives

$$I_H = q_1 + \int \frac{1}{b}\arctan bt\,dt - t\left[\dot{q}_1 + \frac{1}{b}\arctan bt\right]$$
$$= \text{const.}$$

10.2 General holonomic system

Proposition 10.3 *For the general holonomic system* (8), *if the generators* ξ_0, ξ_s *of the form invariance and the gauge function* $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy the Noether identity* (9), *then the form invariance will lead to the Noether conserved quantity* (43).

Proposition 10.4 *For the general holonomic system* (8), *if the generators* ξ_s *of the form invariance satisfy Eq.* (51) *and there exists a function* $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying Eq.* (52), *then the form invariance will lead to the Hojman type conserved quantity* (50).

Example 10.2 The Lagrangian and the generalized forces of a system with two-degree-of-freedom are respectively

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - f(t, q_2),$$
$$Q_1 = -\dot{q}_2, \quad Q_2 = Q_2(t, q_2, \dot{q}_2).$$

Taking the generators as

$$\xi_0 = \xi_2 = 0, \quad \xi_1 = 1,$$

and we have

$$X^{(1)}(L) = X^{(1)}(Q_1) = X^{(1)}(Q_2) = 0.$$

Identity (9) gives

$$G_N = q_2,$$

and the conserved quantity (43) gives

$$I_N = \dot{q}_1 + q_2 = \text{const.},$$

which is a Noether conserved quantity deduced by the form invariance.

10.3 Nonholonomic system

Proposition 10.5 *For the nonholonomic system* (10), (11), *if the generators* ξ_0, ξ_s *of the form invariance and the gauge function* $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfy the Noether identity* (14) *and Eq.* (15), *then the form invariance will lead to the Noether conserved quantity* (43).

Proposition 10.6 *For the nonholonomic system* (10), (11), *if the generators* ξ_s *of the form invariance satisfy Eq.* (53) *and* (54), *and there exists a function* $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying Eq.* (55), *then the form invariance will lead to the Hojman type conserved quantity* (50) [65].

Example 10.3 Novoselov problem is

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right), \quad f = \dot{q}_1 + bt\dot{q}_2 - bq_2 + t = 0.$$

Taking the generators as

$$\xi_0 = 0, \quad \xi_1 = (\dot{q}_2 - bt\dot{q}_1 + bq_1)^2, \quad \xi_2 = 0,$$

we can verify that the generators are form invariant and Lie symmetric. Then Eq. (55) gives

$$\mu = 1,$$

and the conserved quantity (50) gives

$$I_H = 2b\left(\dot{q}_2 - bt\dot{q}_1 + bq_1\right) = \text{const.}$$

10.4 Birkhoffian system

Proposition 10.7 *For the Birkhoffian system* (17), *if the generators* ξ_0, ξ_μ *of the form invariance and the gauge function* $G_N = G_N(t, \boldsymbol{a})$ *satisfy the Noether identity* (20), *then the form invariance will lead to the Noether conserved quantity* (45).

Proposition 10.8 *For the Birkhoffian system* (17), *if the generators* ξ_μ *of the form invariance satisfy Eq.* (57), *and there exists a function* $\mu = \mu(t, \boldsymbol{a})$ *satisfying Eq.* (59), *then the form invariance will lead to the Hojman type conserved quantity* (60).

Example 10.4 The Birkhoffian and the Birkhoff's functions of a Birkhoffian system are respectively

$$B = \frac{1}{2}\frac{1}{m + f(t)}\left\{k_2\left(a^2\right)^2 - k_1\left(a^1\right)^2\right\},$$
$$R_1 = a^2, \quad R_2 = 0,$$

where m, k_1 and k_2 are constants.

Taking the generators as

$$\xi_0 = 0, \quad \xi_1 = k_2 a^2, \quad \xi_2 = k_1 a^1,$$

we can verify that they are form invariance generators. Identity (20) gives

$$G_N = -\frac{1}{2}\left\{k_1\left(a^1\right)^2 + k_2\left(a^2\right)^2\right\},$$

and the conserved quantity (45) gives

$$I_N = \frac{1}{2}\left\{k_2\left(a^2\right)^2 - k_1\left(a^1\right)^2\right\} = \text{const.}$$

The methods for finding the conserved quantities in the review are called the symmetry methods. The symmetry methods are important modern integration methods in dynamics. By using the methods, one can find not only the conserved quantities deduced by Newtonian mechanics and Lagrangian mechanics, but also the conserved quantities which can not be deduced by them. Of course, when using the methods the mathematical difficulties can crop up. The complexity in the Noether symmetry method is to seek the generators and the gauge function by the Noether identity. The complexity in the Lie symmetry method is to find the generators in the determining equations and the function μ, and the difficulty in the form invariance method is to find the generators and the gauge function in the structure equation. The Noether symmetry method is more convenient than the two others, because a Noether symmetry corresponds, in general, to a Noether conserved quantity.

We have presented three kinds of symmetries, i.e. the Noether symmetry, the Lie symmetry and the form invariance, three kinds of conserved quantities, i.e. the Noether conserved quantity, the Hojman type conserved quantity and the new conserved quantity, for the Lagrangian system, the general holonomic system, the nonholonomic system and the Birkhoffian system. Every symmetry can lead to three kinds of conserved quantities. One can study the conserved quantity of other constrained mechanical systems, say, the Hamiltonian system, the holonomic system with redundant coordinates, the variable mass system, the dynamical system in the event space, the generalized Hamiltonian system, the nonholonomic system with the nonholonomic constraints of non-Chetaev's type, the vakonomic system, etc.

There is a symmetry involving equivalent Lagrangians of the Lagrangian system and a conserved quantity can be found by using the symmetry [66]. This idea can be generalized to other constrained mechanical systems. There is a symmetry under which a non-Noether conserved quantity can be found [57,67,68]. The mathematical tools used in the review are simpler analytical ones. There are differential geometry methods in the study of the symmetry, e.g. see references [15,19,69–81].

11 Open problems

Below, we summarize some problems for future research in the area of the symmetry and the conserved quantity of constrained mechanical systems:

1. Find new symmetry methods;
2. Solve the general differential equations by using the symmetry methods;
3. Study the stability of motion by using the symmetry methods;
4. Study the symmetry and the conserved quantity of the vakonomic system.

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