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Distributed control for cooperative hyperbolic systems involving Schrödinger operator

A. H. Qamlo

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Abstract In this paper, some hyperbolic systems involving Schrödinger operator defined on R^n have been established. The existence and uniqueness for the state of these systems have been proved, Then the necessary and sufficient conditions of optimality for such systems have been obtained by a set of equations and inequalities.

Keywords Cooperative hyperbolic systems · Optimal control · Schrödinger operator · Distributed control problem

Mathematics Subject Classification 49 J 20

1 Introduction

The necessary and sufficient conditions of optimality for systems governed by elliptic, parabolic, and hyperbolic operators have been studied by Lions in [7,8]. The considered systems in these problems are in the scalar case (system of one equation).

The discussion is extended to 2 × 2systems for example in [1,5,9,10,12] and to $n \times n$ systems in [11].

Optimal control problem for systems involving Schrödinger operators has been studied for the following elliptic system of distributed type [10]:

$$\begin{cases} (-\Delta + q)\, y_1 = a y_1 + b y_2 + f_1 & \text{in } R^n, \\ (-\Delta + q)\, y_2 = c y_1 + d y_2 + f_2 & \text{in } R^n, \\ y_1, y_2 \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{1}$$

and for parabolic system of boundary type in [1].

A. H. Qamlo (✉)
Mathematics Department, Faculty of Applied sciences, Umm Al-Qura University, P.O. Box 1348, Makkah 21955, Saudi Arabia
e-mail: drahqamlo@yahoo.com

Systems with different potentials and positive weight function is studied in [12] and with variable coefficients is studied in [9].

The existence of optimal control for systems like (1) has been proved with $q(x) = 0$ in [6], and for semi linear cooperative systems in [5].

Time-optimal control problem for cooperative hyperbolic systems involving the Laplace operator is studied in [2].

Here, we consider the following 2 × 2 cooperative hyperbolic systems involving Schrödinger operator:

$$\begin{cases} \frac{\partial^2 y_1(x)}{\partial t^2} + (-\Delta + q)\, y_1 = a y_1 + b y_2 + f_1(x,t) & \text{in } Q, \\ \frac{\partial^2 y_2(x)}{\partial t^2} + (-\Delta + q)\, y_2 = c y_1 + d y_2 + f_2(x,t) & \text{in } Q, \\ y_1, y_2 \to 0 \quad \text{as} \quad |x| \to \infty, \\ y_1|_\Sigma = y_2|_\Sigma = 0, \\ y_1(x,0) = y_{1,0}(x), \quad y_2(x,0) = y_{2,0}(x) & \text{in } R^n, \\ \frac{\partial y_1(x,0)}{\partial t} = y_{1,1}(x), \quad \frac{\partial y_2(x,0)}{\partial t} = y_{2,1}(x) & \text{in } R^n. \end{cases} \tag{2}$$

with

$$y_1, y_2 \in L^2\left(0, T; V_q\left(R^n\right)\right), \quad \frac{\partial y_1}{\partial t}, \frac{\partial y_2}{\partial t} \in L^2\left(0, T; V_q'\left(R^n\right)\right).$$

where

$$a, b, c \text{ and } d \text{ are given numbers such that } b, c > 0, \tag{3}$$

(in this case, we say that the system (2) is cooperative)

$$q(\mathrm{x}) \text{ is a positive function and tending to } \infty \text{ at infinity}, \tag{4}$$

and $Q = R^n \times]0, T[$ with boundary $\Sigma = \Gamma \times]0, T[$.

We first prove the existence and uniqueness of the state for these systems, then we introduce the optimal control of distributed type for these systems.

2 Some concepts and results

In this paper, we shall consider some results introduced in [3], [10] concerning the eigenvalue problem

$$\begin{cases} (-\Delta + q)\,\phi = \lambda(q)\phi \quad \text{in} \quad R^n \\ \phi\,(x) \to 0 \quad \text{as} \quad |x| \to \infty, \quad \phi \succ 0 \end{cases} \tag{5}$$

The associated variational space is $V_q\,(R^n)$, the completion of $D\,(R^n)$, with respect to the norm:

$$\|y\|_q = \left(\int_{R^n} \left[|\nabla y|^2 + q\,|y|^2 \right] dx \right)^{1/2} \tag{6}$$

Since the imbedding of $V_q\,(R^n)$ in to $L^2\,(R^n)$ is compact, then the operator $(-\Delta + q)$ considered as an operator in $L^2\,(R^n)$ is positive self-adjoint with compact inverse. Hence its spectrum consists of an infinite sequence of positive eigenvalues, tending to infinity; moreover the smallest one which is called the principal eigenvalue denoted by $\lambda(q)$ is simple and is associated with an eigenfunction which does not change sign in R^n. It is characterized by:

$$\lambda\,(q) \int_{R^n} |y|^2\, dx \le \int_{R^n} \left[|\nabla y|^2 + q\,|y|^2 \right] dx \quad \forall y \in V_q\left(R^n\right). \tag{7}$$

We have the following embedding :

$$V_q\left(R^n\right) \times V_q\left(R^n\right) \subseteq L^2\left(R^n\right) \times L^2\left(R^n\right) \subseteq V_q'\left(R^n\right) \times V_q'\left(R^n\right)$$

which is continuous and compact .

Let us introduce the space $L^2(0,T;V_q(R^n))$of measurable function $t \to f(t)$ which is defined on open interval (0,T), since the variable $t \in (0,T)$ and $T < \infty$ denotes the time .

On (0,T) with Lebesgue measure dt we have the norm:

$$\|f(t)\|_{L^2(0,T;V_q(R^n))} = \left(\int_{(0,T)} \|f(t)\|^2_{V_q(R^n)}\, dt \right)^{1/2} \prec \infty$$

and the scalar product

$$(f(t), g(t))_{L^2(0,T;V_q(R^n))} = \int_{(0,T)} (f(t), g(t))_{V_q(R^n)}\, dt,$$

the space$L^2\left(0,T;V_q(R^n)\right)$with the scalar product and the norm above is a Hilbert space .

Analogously, we can define the spaces $L^2(0,T;L^2(R^n)) = L^2(Q)$, with the scalar product

$$(f(t), g(t))_{L^2(Q)} = \int_{(0,T)} (f(t), g(t))_{L^2(R^n)}\, dt = \int_Q f(t).g(t)\, dx\, dt$$

then we have the following embedding

$$\left(L^2(0,T;V_q(R^n))\right)^2 \subseteq \left(L^2(Q)\right)^2 \subseteq \left(L^2(0,T;V_q'(R^n))\right)^2$$

3 Existence and uniqueness of solution

We introduce the bilinear form

$$\pi\,(t;y,\psi) = (A(t)\,y,\psi)_{(L^2(R^n))^2},$$

$$y = \{y_1, y_2\}, \quad \psi = \{\psi_1, \psi_2\} \in \left(V_q\left(R^n\right)\right)^2,$$

$$A\,(t)\,y \in \left(V_q'\left(R^n\right)\right)^2$$

where

$$A\,(t)\,y(x) = \{(-\Delta + q)\,y_1 - ay_1 - by_2, \; (-\Delta + q)\,y_2 - cy_1 - dy_2\}$$

then

$$\begin{aligned} \pi\,(t;y,\psi) &= \frac{1}{b} \int_{R^n} [\nabla y_1 \nabla \psi_1 + q y_1 \psi_1]\, dx \\ &+ \frac{1}{c} \int_{R^n} [\nabla y_2 \nabla \psi_2 + q y_2 \psi_2]\, dx \\ &- \int_{R^n} y_1 \psi_2\, dx - \frac{d}{c} \int_{R^n} y_2 \psi_2\, dx \\ &- \frac{a}{b} \int_{R^n} y_1 \psi_1\, dx - \int_{R^n} y_2 \psi_1\, dx. \end{aligned} \tag{8}$$

For all $y, \psi \in \left(V_q\,(R^n)\right)^2$, the function $t \to \pi\,(t;y,\psi)$ is measurable on (0,T).

By using the necessary and sufficient conditions for having the maximum principle and existence of positive solutions for cooperative system (1) which have been obtained by Fleckinger [4] and take the form

$$\begin{cases} a \prec \lambda\,(q)\;,\; d \prec \lambda\,(q)\;, \\ (\lambda\,(q) - a)\,(\lambda\,(q) - d) \succ bc \;, \end{cases} \tag{9}$$

the coerciveness condition of the bilinear form (8) in $\left(V_q\,(R^n)\right)^2$has been proved by Serag [10], that means

$$\pi\,(t;y,y) \ge C\left(\|y_1\|^2_{q,m} + \|y_2\|^2_{q,m}\right), \quad C \succ 0 \tag{10}$$

Theorem 1 *Under the hypotheses (3) and (10), if* $f_1, f_2 \in L^2(0,T;V_q'(R^n))$, $y_{1,0}(x), y_{2,0}(x) \in V_q(R^n)$ *and* $y_{1,1}(x), y_{2,1}(x) \in V_q'(R^n)$, *then there exists a unique solution* $y = \{y_1, y_2\} \in \left(L^2\left(0,T;V_q\left(R^n\right)\right)\right)^2$ *for system (2).*

Proof Let $\psi \to L(\psi)$ be a continuous linear form defined on $\left(L^2(Q)\right)^2$ by

$$L(\psi) = \frac{1}{b}\int\limits_Q f_1(x,t)\,\psi_1(x)dx\,dt + \frac{1}{c}\int\limits_Q f_2(x,t)\,\psi_2(x)dx\,dt + \frac{1}{b}\int\limits_{R^n} y_{1,1}(x)\,\psi_1(x,0)\,dx + \frac{1}{c}\int\limits_{R^n} y_{2,1}(x)\,\psi_2(x,0)\,dx$$

$$\forall\, \psi = \{\psi_1, \psi_2\} \in \left(L^2\left(0,T;V_q\left(R^n\right)\right)\right)^2, \tag{11}$$

then by Lax–Milgram lemma, there exists a unique element $y = \{y_1, y_2\} \in \left(L^2\left(0,T;V_q\left(R^n\right)\right)\right)^2$ such that

$$\frac{1}{b}\left(\frac{\partial^2 y_1}{\partial t^2}, \psi_1\right) + \frac{1}{c}\left(\frac{\partial^2 y_2}{\partial t^2}, \psi_2\right) + \pi\,(t; y, \psi) = L(\psi)$$

$$\forall \psi = \{\psi_1, \psi_2\} \in \left(L^2\left(0,T;V_q\left(R^n\right)\right)\right)^2, \tag{12}$$

Now, let us multiply both sides of first equation of system (2) by $\frac{1}{b}\psi_1(x)$, and the second equation by $\frac{1}{c}\psi_2(x)$ then integration over Q, we have:

$$\frac{1}{b}\int\limits_Q \left[\frac{\partial^2 y_1(x)}{\partial t^2} + (-\Delta + q)\,y_1 - ay_1 - by_2\right]\psi_1 dx\,dt = \frac{1}{b}\int\limits_Q f_1(x,t)\,\psi_1 dx\,dt \quad ,$$

$$\frac{1}{c}\int\limits_Q \left[\frac{\partial^2 y_2(x)}{\partial t^2} + (-\Delta + q)\,y_2 - cy_1 - dy_2\right]\psi_2 dx\,dt = \frac{1}{c}\int\limits_Q f_2(x,t)\,\psi_2 dxdt \quad .$$

By applying Green's formula:

$$\frac{1}{b}\int\limits_Q \frac{\partial^2 y_1(x)}{\partial t^2}\psi_1(x)\,dx + \frac{1}{b}\int\limits_Q \nabla y_1 \nabla \psi_1 dxdt - \frac{1}{b}\int\limits_\Sigma \psi_1 \frac{\partial y_1}{\partial \nu_A}d\Sigma - \frac{1}{b}\int\limits_{R^n}\psi_1(x,0)\,\frac{\partial y_1(x,0)}{\partial t}dx + \int\limits_Q \left(\frac{q}{b}y_1 - \frac{a}{b}y_1 - y_2\right)\psi_1 dxdt = \frac{1}{b}\int\limits_Q f_1(x,t)\,\psi_1 dxdt\ ,$$

$$\frac{1}{c}\int_{R^n}\frac{\partial^2 y_2(x)}{\partial t^2}\psi_2(x)\,dx + \frac{1}{c}\int\limits_Q \nabla y_2 \nabla \psi_2 dxdt - \frac{1}{c}\int\limits_\Sigma \psi_2 \frac{\partial y_2}{\partial \nu_A}d\Sigma - \frac{1}{c}\int\limits_{R^n}\psi_2(x,0)\,\frac{\partial y_2(x,0)}{\partial t}dx + \int\limits_Q \left(\frac{q}{c}y_2 - y_1 - \frac{d}{c}y_2\right)\psi_2 dxdt = \frac{1}{c}\int\limits_Q f_2(x,t)\,\psi_2 dxdt.$$

By sum the two equations, then comparing the summation with (8), (11) and (12) we get:

$$-\frac{1}{b}\int\limits_\Sigma \psi_1 \frac{\partial y_1}{\partial \nu_A}d\Sigma - \frac{1}{c}\int\limits_\Sigma \psi_2 \frac{\partial y_2}{\partial \nu_A}d\Sigma - \frac{1}{b}\int\limits_{R^n}\psi_1(x,0)\,\frac{\partial y_1(x,0)}{\partial t}dx - \frac{1}{c}\int\limits_{R^n}\psi_2(x,0)\,\frac{\partial y_2(x,0)}{\partial t}dx = \frac{1}{b}\int\limits_{R^n} y_{1,1}(x)\,\psi_1(x,0)\,dx + \frac{1}{c}\int\limits_{R^n} y_{2,1}(x)\,\psi_2(x,0)\,dx,$$

then we deduce that:

$$y_1|_\Sigma = y_2|_\Sigma = 0$$
$$\frac{\partial y_1(x,0)}{\partial t} = y_{1,1}(x), \quad \frac{\partial y_2(x,0)}{\partial t} = y_{2,1}(x) \quad \text{in} \quad R^n.$$

which completes the proof. □

4 Formulation of the control problem

The space $L^2(Q) \times L^2(Q)$ is the space of controls. For a control $u = \{u_1, u_2\} \in (L_2(Q))^2$, the state $y(u) = \{y_1(u), y_2(u)\} \in \left(L^2\left(0,T;V_q\left(R^n\right)\right)\right)^2$ of the system (2) is given by the solution of

$$\begin{cases} \frac{\partial^2 y_1(u)}{\partial t^2} + (-\Delta + q)\,y_1(u) = ay_1(u) + by_2(u) + f_1 + u_1 & \text{in } Q, \\ \frac{\partial^2 y_2(u)}{\partial t^2} + (-\Delta + q)\,y_2(u) = cy_1(u) + dy_2(u) + f_2 + u_2 & \text{in } Q, \\ y_1, y_2 \to 0 \quad \text{as} \quad |x| \to \infty, \\ y_1(u)|_\Sigma = y_2(u)|_\Sigma = 0, \\ y_1(x,0,u) = y_{1,0}(x), \quad y_2(x,0,u) = y_{2,0}(x) & \text{in} \quad R^n, \\ \frac{\partial y_1(x,0,u)}{\partial t} = y_{1,1}(x), \quad \frac{\partial y_2(x,0,u)}{\partial t} = y_{2,1}(x) & \text{in} \quad R^n. \end{cases} \tag{13}$$

with

$$y_1(u), y_2(u) \in L^2\left(0,T;V_q\left(R^n\right)\right)\ ,$$
$$\frac{\partial y_1(u)}{\partial t}, \frac{\partial y_2(u)}{\partial t} \in L^2\left(0,T;V_q'\left(R^n\right)\right)$$

The observation equation is given by $z(u) = \{z_1(u), z_2(u)\} = y(u) = \{y_1(u), y_2(u)\}$.

For a given $z_d = \{z_{d1}, z_{d2}\} \in \left(L^2(Q)\right)^2$, the cost function is given by

$$J(v) = \|y_1(v) - z_{d1}\|^2_{L^2(Q)} + \|y_2(v) - z_{d2}\|^2_{L^2(Q)} + (Nv, v)_{(L^2(Q))^2} \tag{14}$$

where $N \in L\left(\left(L^2(Q)\right)^2, \left(L^2(Q)\right)^2\right)$ is a Hermitian positive definite operator:

$$(Nu, u)_{(L^2(Q))^2} \geq \gamma \ \|u\|^2_{(L^2(Q))^2}, \quad \gamma \succ 0. \tag{15}$$

The control problem then is to find $u = \{u_1, u_2\} \in U_{ad}$ such that $J(u) \leq J(v)$,
where U_{ad} is a closed convex subset of $\left(L^2(Q)\right)^2$.

Since the cost function (14) can be written as (see [7]):

$$J(v) = a(v, v) - 2L(v) + \|y(0) - z_d\|^2_{(L^2(Q))^2},$$

where $a(v, v)$ is a continuous coercive bilinear form and $L(v)$ is a continuous linear form on $\left(L^2(Q)\right)^2$. Then using the general theory of Lions [7], there exists a unique optimal control $u \in U_{ad}$ such that $J(u) = \inf J(v)$ for all $v \in U_{ad}$. Moreover, we have the following theorem which gives the necessary and sufficient conditions of optimality :

Theorem 2 *Assume that (10) and (15) hold. If the cost function is given by (14), the optimal control $u = \{u_1, u_2\} \in (L_2(Q))^2$ is then characterized by the following equations and inequalities:*

$$\begin{cases} \frac{\partial^2 p_1(u)}{\partial t^2} + (-\Delta + q)\, p_1(u) - ap_1(u) - cp_2(u) = y_1(u) - z_{d1} & \text{in } Q, \\ \frac{\partial^2 p_2(u)}{\partial t^2} + (-\Delta + q)\, p_2(u) - bp_1(u) - dp_2(u) = y_2(u) - z_{d2} & \text{in } Q, \\ p_1, p_2 \to 0 \quad as \quad |x| \to \infty, \\ p_1(u)|_\Sigma = p_2(u)|_\Sigma = 0, \\ p_1(x, T, u) = p_2(x, T, u) = 0 & \text{in } R^n, \\ \frac{\partial p_1(x,T,u)}{\partial t} = \frac{\partial p_2(x,T,u)}{\partial t} = 0 & \text{in } R^n. \end{cases} \tag{16}$$

with

$$p_1(u), p_2(u) \in L^2\left(0, T; V_q\left(R^n\right)\right),$$
$$\frac{\partial p_1(u)}{\partial t}, \frac{\partial p_2(u)}{\partial t} \in L^2\left(0, T; V_q'\left(R^n\right)\right)$$
$$(p_1(u), v_1 - u_1)_{L^2(Q)} + (p_2(u), v_2 - u_2)_{L^2(Q)} + (Nu, v - u)_{(L^2(Q))^2} \geq 0 \quad \forall\, v = \{v_1, v_2\} \in U_{ad} \tag{17}$$

together with (13), where $p(u) = \{p_1(u), p_2(u)\}$ is the adjoint state .

Proof The optimal control $u = \{u_1, u_2\} \in (L_2(Q))^2$ is characterized by (see[7])

$$J\prime(u)\ (v - u) \geq 0 \quad \forall\, v \in U_{ad},$$

which is equivalent to:

$$(y(u) - z_d, y(v) - y(u))_{(L^2(Q))^2} + (Nu, v - u)_{(L^2(Q))^2} \geq 0$$

i.e.

$$(y_1(u) - z_{d1}, y_1(v) - y_1(u))_{L^2(Q)} + (y_2(u) - z_{d2}, y_2(v) - y_2(u))_{L^2(Q)} + (Nu, v - u)_{(L^2(Q))^2} \geq 0$$

this inequality can be written as

$$\int_0^T (y_1(u) - z_{d1}, y_1(v) - y_1(u))_{L^2(R^n)}\, dt + \int_0^T (y_2(u) - z_{d2}, y_2(v) - y_2(u))_{L^2(R^n)}\, dt + (Nu, v - u)_{(L^2(Q))^2} \geq 0 \tag{18}$$

Now, since

$$(p, By)_{(L^2(Q))^2} = \int_0^T \left(p_1(u), \frac{\partial^2 y_1(u)}{\partial t^2} + (-\Delta + q)\, y_1(u) - ay_1(u) - by_2(u)\right)_{L^2(R^n)} dt + \int_0^T \left(p_2(u), \frac{\partial^2 y_2(u)}{\partial t^2} + (-\Delta + q)\, y_2(u) - cy_1(u) - dy_2(u)\right)_{L^2(R^n)} dt$$

where

$$By(u) = B\{y_1(u), y_2(u)\} = \left\{\frac{\partial^2 y_1(u)}{\partial t^2} + (-\Delta + q)\, y_1(u) - ay_1(u) - by_2(u),\ \frac{\partial^2 y_2(u)}{\partial t^2} + (-\Delta + q)\, y_2(u) - cy_1(u) - dy_2(u)\right\}$$

by using Green formula and (13), we get

$$(p, By)_{(L^2(Q))^2} = \int_0^T \left(\frac{\partial^2 p_1(u)}{\partial t^2} + (-\Delta + q)\, p_1(u)\right.$$

$$-ap_1(u) - cp_2(u), y_1(u)\Big)_{L^2(R^n)} dt$$

$$+ \int_0^T \left(\frac{\partial^2 p_2(u)}{\partial t^2} + (-\Delta + q)\, p_2(u) - bp_1(u) \right.$$

$$\left. -dp_2(u), y_2(u) \right)_{L^2(R^n)} dt$$

$$= \left(B^* p, y\right)_{(L^2(Q))^2}$$

then

$$B^* p(u) = B^* \{p_1(u), p_2(u)\}$$
$$= \left\{ \frac{\partial^2 p_1(u)}{\partial t^2} + (-\Delta + q)\, p_1(u) - ap_1(u) - cp_2(u)\, , \right.$$
$$\left. \frac{\partial^2 p_2(u)}{\partial t^2} + (-\Delta + q)\, p_2(u) - bp_1(u) - dp_2(u) \right\}$$

and

$$A^* p(u) = A^* \{p_1(u), p_2(u)\}$$
$$= \{(-\Delta + q)\, p_1(u) - ap_1(u) - cp_2(u)\, ,$$
$$(-\Delta + q)\, p_2(u) - bp_1(u) - dp_2(u)\}$$

since the adjoint equation takes the form:

$$\frac{\partial^2 p(u)}{\partial t^2} + A^* p(u) = y(u) - z_d$$

and from Theorem1, we get a unique solution $p(u) \in \left(L^2\left(0, T; V_q(R^n)\right)\right)^2$which satisfies

$$p_1(u), p_2(u) \in L^2\left(0, T; V_q\left(R^n\right)\right)\, ,$$
$$\frac{\partial p_1(u)}{\partial t}, \frac{\partial p_2(u)}{\partial t} \in L^2\left(0, T; V_q'\left(R^n\right)\right).$$

This proves system (16).

Now, we transform (18) by using (16) as follows:

$$\int_0^T \left(\frac{\partial^2 p_1(u)}{\partial t^2} + (-\Delta + q)\, p_1(u) - ap_1(u) \right.$$

$$\left. -cp_2(u), y_1(v) - y_1(u) \right)_{L^2(R^n)} dt$$

$$+ \int_0^T \left(\frac{\partial^2 p_2(u)}{\partial t^2} + (-\Delta + q)\, p_2(u) - bp_1(u) \right.$$

$$\left. -dp_2(u), y_2(v) - y_2(u) \right)_{L^2(R^n)} dt$$

$$+ (Nu, v - u)_{(L^2(Q))^2} \geq 0.$$

Using Green formula, we obtain

$$\int_0^T \left(p_1(u), \left(\frac{\partial^2}{\partial t^2} + (-\Delta + q) \right) y_1(v) - y_1(u) \right)_{L^2(R^n)} dt$$

$$+ \int_0^T -a\, (p_1(u), y_1(v) - y_1(u))_{L^2(R^n)}\, dt$$

$$+ \int_0^T -c\, (p_2(u), y_1(v) - y_1(u))_{L^2(R^n)}\, dt$$

$$+ \int_0^T \left(p_2(u), \left(\frac{\partial^2}{\partial t^2} + (-\Delta + q) \right) y_2(v) - y_2(u) \right)_{L^2(R^n)} dt$$

$$+ \int_0^T -b\, (p_1(u), y_2(v) - y_2(u))_{L^2(R^n)}\, dt$$

$$+ \int_0^T -d\, (p_2(u), y_2(v) - y_2(u))_{L^2(R^n)}\, dt$$

$$+ (Nu, v - u)_{(L^2(Q))^2} \geq 0.$$

Using (13), we have

$$\int_0^T (p_1(u)\, , v_1 - u_1)_{L^2(R^n)}\, dt$$

$$+ \int_0^T (p_2(u)\, , v_2 - u_2)_{L^2(R^n)}\, dt$$

$$+ (Nu\, , v - u)_{(L^2(Q))^2} \geq 0$$

which is equivalent to

$$(p_1(u)\, , v_1 - u_1)_{L^2(Q)} + (p_2(u)\, , v_2 - u_2)_{L^2(Q)}$$
$$+ (Nu\, , v - u)_{(L^2(Q))^2} \geq 0$$

Thus the proof is complete. □

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