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A plea for a mathematical approach in psychology

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Abstract No conception can reach a scientific status unless it is supported by a mathematical theory. Clearly, this universal rule is as inescapable in psychology as in any other field. In psychology, a theory already exists, namely Vygotsky's Activity Theory, which has a potential axiomatic structure. Unfortunately, until now, this structure has never been made explicit, so that it is necessary to make it apparent. In fact, moving from the units of analysis of Vygotskian theory, it is possible to give psychology the consistency of a rigorous formal system. Of course, to do that doesn't mean avoiding the necessity, complementary and inescapable, of submitting the system thus obtained to empirical control, and to adjust it step by step according to factual evidence. However, a formal system also helps empirical and experimental verification, in that it makes concepts and their relationship clearer, more precise and more apparent.

Keywords Mathematics · Axiomatic structure · Psychology · Formal control

Science is formed by combining empirical observation with hypotheses. The observations must be as numerous and varied as possible—as well as, obviously, as accurate as possible—in the attempt to rule out the existence of facts that belie or alter the conclusions that are drawn from them: in essence, such facts invalidate the idea of their universal nature and validity. The hypotheses must be as comprehensive and overarching as possible in order to produce a universally valid, systematic framework within the context of the phenomena being investigated.

The degree to which these two requirements are satisfied guarantees the formation of cohesive systems of knowledge tending to exhaustiveness in the areas of interest, and coherent with all other available pertinent knowledge. The generalisation of the body of observations and hypotheses taken as a whole is the means for assuring the maximum capacity of these systems to embrace all pertinent phenomena. When such systems are validated by the empirical evidence to a degree considered satisfactory enough to render them undeniable and substantially true, the set of the related hypotheses are considered to be theories.

In their turn, theories are considered practically to be expressions of truth (at an ingenuous level, of pure and simple truth), in the sense that they can be used as descriptions and explanations of the actual, unquestionable (or the best available) phenomena that they relate to, until new evidence or conceptions intervene to disprove them, or lead to their rectification.

Systems that are adequately cohesive and exhaustive within the limits indicated can be formalised in such a way that the set of empirical data and ideas that constitute them as knowledge is given a mathematical structure.[1] The formalisation allows deductive reasoning and logical verification; the empirical basis provides the system's content of reality and the possibility of verifying it concretely with ulterior observations and, in the most fundamental and systematic manner, with experiments.

The experiments are distinguished from simple experiences since these are artificially provoked on the basis of

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[1] This formalisation does not necessarily have to be numerical, since it need only guarantee the logical rigour of the conceptual developments and formal means of verification.

hypotheses that tend to be explicit—the more valid as they are more clearly formulated and systematic—which also guide their interpretations. Thus, both the premises as well as the predictions of the experiments are based solely and equally on hypotheses and on previous observations; likewise, the verifications of the related procedures and the results obtained must be complementary in terms of formal logic and empirical content.

As pertinent empirical observations are gradually carried out, and the formal aspects are deepened and developed, these can come to be confirmed, integrated or questioned until they are finally rejected. However, it is always difficult, if not almost impossible, to establish whether the refutations of a theory are real or apparent, definitive or provisional until the arrival of further theoretical or empirical knowledge, effectively due to empirical evidence or to interpretations of these that might themselves later turn out to be erroneous.

Thus we see that in science, thought and knowledge of the empirical contents dealt with are necessarily interdependent and co-evolve, and that the scientific procedure involves interdependence and circular (or better, spiral) co-evolution of inductive and deductive thought.[2]

Precisely for this reason, a fundamental part of science consists in explicating as clearly and rigorously as possible the hypothetical-theoretical components, that is, the concepts utilised. Any related omission and unverified residue of ideas constitutes a source of weakness and risk of error. The procedure that is indispensable for assuring the maximum control of these components is their formalisation, that is to say, their explication by means of formal logical formulations.

Now let us come to pedagogy, or to be more exact, for the considerations that will follow, to psychopedagogy.

Pedagogy and Psychology are the disciplines that, in principle (and that is, notwithstanding from the limits that prejudice the conceptual rigour of the disciplines), are, or should be, furnished by the formal theories regarding the nature, respectively, of education and psychic structures.

With regard to the concrete factual plane, using upper case initials to indicate the disciplines and lower case initials for their respective spheres of reality and their informal concepts, Pedagogy is used to indicate generically the set of pedagogical practices, and Psychology to indicate in an equally generic fashion psychic structures and processes.

On the existential plane, Pedagogy precedes Psychology because it is in fact the conditions of growth of individuals and educative practices that determine the formation of psychological qualities. Instead, on the abstract theoretical plane, the relationship is inverted, and it is Psychology that precedes Pedagogy,[3] given the fact that we appeal to theories, or if these are lacking, to concepts that are more or less generic and unreflective regarding structures and psychological processes (or on 'character' and the behaviour of people) to decide the possibilities and needs of educative interventions.

In essence, while the conditions of the growth of individuals and the ways of educating them form personalities, the capacity to reflect on such conditions and terms, and to guide them with theories, is provided solely by systematic rational thought, and even more, by the discipline of Psychology. On a practical level, first of all we find ourselves in a state of existential conditions of fact, and we act. Instead, on the level of awareness and reflection, we are in the plane of ideas and theories about their most evolved forms, and we begin from them to choose and guide operative actions and practices.

Traditionally, and as Psychology gradually came to constitute a discipline independent of Philosophy, a similar theoretical precedence attributed to it over educational practices and Pedagogy sprung from two concealed presuppositions—concealed because taken for granted—which thus remained to a large extent unconscious and inadequately reflected upon. One is that human nature is essentially predetermined in a way that is static, universal and eternal. The other, closely related, is that it is determined biologically, with culture having no, or only a secondary, impact on it.

Although human nature has been often thought of as essentially spiritual, in fact the biologistic components have always, or almost always, prevailed, especially at the level of common sense, according to which—refractory as it is to the theories—bodies, their growth and practical behaviours have a more immediately perceptible and more incisive impact on psychic processes, which remain more hidden, and of cultural influences, which are less palpable. Thus, spirituality has been, for the most part and more analytically, only theorised while naïve biologistic naturalism has constituted the most immediate and dominant expression of nativism and of the preformism of the belief in the predisposition of 'character' (a substitutive generic idea of concepts, more closely related to theorisation), 'personality' and 'human nature'.

The awareness, of relatively recent acquisition and far from universally widespread, that human or psychological

[2] We could just as well say 'deductive and inductive', since we cannot identify a theoretical or factual beginning in either of them.

[3] This occurs according to the well-known law of the inversion of the course of becoming aware of practical skills and knowledge with respect to that of their operative formation, by which, using the example cited by Piaget in stating these laws, while topological structures are the first to be formed in the sphere of 'spontaneous' or 'natural' mathematical thought, topology was the last branch of mathematics to be recognised and formalised.

formation is carried out by the internalisation of functional systems, of which I will say more below, renders comprehension of the dual relationships between Psychology and Pedagogy more penetrating, leading us to revise and specify them in concrete scientific terms rather than terms that are abstract philosophical or prejudicial common sense.

The solution that has progressively come to be imposed over the course of the twentieth century, though often in a manner that is partial or unclear, was first of all to conceive of that relationship in the strict terms of a unitary discipline thought of as Psychopedagogy, broadened in the last decades to the idea (although, it appears to me, above all to the elocution) of 'science of education' (or of formation, or similar terms) in place of that of Pedagogy.[4]

Here I will limit myself to a consideration of the relationship between Pedagogy (including there a realistic vision of its applications), and Psychology, by reason of how much one is necessary to the other for a co-evolutionary scientific formulation of the practices and theories of education. All of the remaining disciplines that work to delineate the related overall framework are set aside in order to simplify the discussion, but they are not excluded; indeed, they are kept well in mind in the way it is carried out.

Within the restricted prospective assumed, the reflections carried out to this point should indicate that rendering Psychology as scientific as possible involves an interest that is pedagogical as well. To argue this statement, I will begin with the consideration that science's need to give formal structure to its own constructions of knowledge finds its best solution in axiomatic systems. In fact, only axiomatic systems make it possible to establish with maximum clarity and brevity the foundations of such constructions and follow their developments with the required rigour, guiding them and verifying them in a way that is complementary.

Now, Psychology, as a discipline of the nature of psychological faculties and processes, should constitute the point of departure for a significant formalisation, also in pedagogical terms. Thus, there is also a pedagogical interest in creating an axiomatic system in Psychology.

Unfortunately, as much as every theory constitutes an axiomatic or is a prelude to it in proportion to the degree of cohesion and completeness given to it by the character of the system, no current psychological theory has an axiomatic form. However, the 'Activity Theory' created by Lev Vygotsky and developed by his collaborators and followers suggests precisely the possibility of putting it in such a form. In particular, the developments made to it by Alexei Leontiev give a strong, precise impression that that theory is underlain by an unexpressed axiomatic structure that needs only to be rendered explicit.

Vygotsky, with his attempt to establish the units of analysis that would serve as the basis for constructing the psychological investigation,[5] had already seen human activity as the first of those units, had carried out a series of reflections, and produced specifically psychological concepts that already provided references that were sufficiently precise and coordinated to permit a glimpse of the outline of an axiomatic, although one that was still incomplete and not explicitly formulated. The most significant concepts in this regard were the 'functional system', and of 'transplanting' or 'grafting' of functional systems into individual minds as means or instruments of construction and development of the capacity to think. We can grasp the particular importance of such concepts by taking into account that Vygotsky's thinking is entirely centred, in Marxian terms, precisely on the analysis of how the quality of the instruments used in the activities both practical and intellectual determines the quality of these activities as well as their results.

In his turn, Leontiev had analysed in greater depth the concept of 'activity', identifying within it salient moments or components in 'actions' and 'operations'. These two concepts, while recalling the meanings of the corresponding terms in normal usage, generalise and specify them into two respective psychological categories: actions as phases of the performance of the activities, and operations as coordinated actions in unitary systems endowed with their own functional automatism.

Any system of this kind constitutes a 'functional system'. This, with its own internal organisation, guides and restrains in its own characteristic manner its application—that is, that of the sequence of operations that compose it—in the execution of any activity. Correlatively, functional systems are characterised by their typical structures and ways of functioning. Among functional systems, those that become conventional assume particular relevance because that tends to render them available to all, so that anyone who makes adequate use of them becomes capable of obtaining the results produced by their combined operations (anyone who learns certain cultural notions or contents becomes capable, in proportion to the degree that they are learned and of other remaining capacities, of the same operations, that is, he acquires capacities that tend to be identical for everyone).

[4] It is in such a sense that the related courses in university departments, degree programs and secondary schools have been renamed.

[5] He did this following Marx's many psychological intuitions, which are so fundamental and integral, not to mention revolutionary, that they can be composed into a genuine theory, in spite of their philosophical nature [1, 3]. In fact, out of them arose Activity Theory, which was also constructed on experimental bases.

As can be seen, functional systems play a central role in determining psychological capacity; the relative concept plays a corresponding role in explaining cognitive development and in indicating possibilities and ways of promoting it. In consequence, to the extent to which it is possible to compare the outcomes of the application of the various functional systems in the same kind of activity (more precisely, in activities with identical aims), it is also possible to define the possibilities and best ways to carry it out. That means that it is also possible to understand which of these, when internalised—that is, converted into personal ways of acting—best furnish the possibilities and ways for intellectual growth.

Essentially two point thus emerge. The first is that in Vygotsky's day the concept of 'internalisation', through which practical actions are converted into mental operations, had already become part of official Psychology: specifically, it had been theorised by Piaget, who was held to be the most eminent psychologist and almost emblematic of psychology itself until the end of the twentieth century.[6] The second is that the concept of functional system leads to conceiving culture as a set of all conventional functional systems and the historical development of culture as the progressive evolution of functional systems made, in principle, available for all individuals in any given period.

In order for a functional system to be internalised, it is obviously necessary to learn it, that is to say, to acquire it as one's own way of organising practical and mental behaviours. Generally speaking, the acquisition takes place through both informal means, with imitation induced by participation in practice and in social discourse, and in formal ways through the reception of explicit, purposeful teaching. In both ways takes place what Vygotsky referred to metaphorically as the transplanting or grafting of a functional system. In the way in which he used these terms, learning functional systems simply consists in restructuring one's mental processes in conformity with the sequences of operations of the systems.

In general, mental development occurs systematically and (basically) only through the effect of the internalisation of the most evolved functional systems of those possessed at the moment. Even the cognitive advances that individuals can achieve on their own, as results of their independent mental operations, are only possible using learned functional systems.

This means that mental development is the product of the internalisation of culture, in consequence of participation of social life and formal learning. However, it also means that it is then the characteristics of societies and cultures that determine the psychological characteristics of individuals.[7]

There are also other consequences than those discussed up to now. The historical evolution of conventional functional systems, as more generally the evolution of any given functional system, consists in the progressive differentiations of the operations that compose it into others that are more analytical and refined, and of coordination of these with the consequent levels of increasing specialisation thus reached. Given that mental development is due to their internalisation, this too consists in corresponding progressive differentiations of the representative structures and their higher coordination. The process is also valid for culture (by definition, since the culture is the sum of the conventional functional systems).

Now, however mind is conceived, it is certain that the mental faculties operate on the symbolic plane that is the same as that of the culture. In psychology 'schematic' mental structures (those that are most primitive, beginning with the levels that are closest to biological reactions) are distinguished from 'conceptual' mental structures,[8] and mental development is held to be characterised as progress from the former to the latter.

In this view, psychology is reconnected to biology, and specifically to the evolutionism already conceived by Darwin as a unitary process of progressive differentiations of the forms gradually assumed by organs and organisms, with correlated coordination with the consequent levels of increasing superiority, of the organs in each organism and of organisms among competitive or symbiotic species and between these and their environments. As in biology speciation and the increasing specialisations are thus explained, so too in psychology the evolution of the mental structures of individuals is explained in correspondence to the evolution of the functional structures of societies and of cultures. Both of the latter, which are also reciprocally

[6] Still today, if we were to ask anyone capable of naming a psychologist to do so, he or she would almost certainly name Piaget before anyone else.

[7] This brief statement would need to be analysed within the complex relationships of the myriad of social situations that influence each individual, from the dyad of the relationship of a child with a parent or between two siblings or two friends, to family groups and micro-societies of all kinds: school classes, associations, groups of neighbours or of friends and so forth.

[8] Traditionally, and improperly, it is customary to call any mental category 'concepts', while instead a rigorous conception requires the application of that term solely to mental categories that belong to conscious rational systems and thus 'logical' or rational in the strict meaning of the term, distinguishing them from schemas that are essentially unconscious and analogous. In reality, even these last possess a logic of their own. The latter term indicates both (in the most generic manner) any type of coherence or connection of the elements of a system and, in a more restricted and specific key, the systematic nature that corresponds to formal logic: to the actual state of the concepts and the relative linguistic usages, which engenders a confusion of terminology that needs to be overcome.

interconnected, in one way are in fact produced by individual mental structures, while in another way, registering the mental products that are most evolved and best endure the trials of application, they provide the functional systems that are necessary to the mental development of individuals.

The connection between psychology and biology that is thus established configures the evolutionistic continuity of one to the other that, in principle, for the first time in history, dispels the mystery of nature and the formation of psychological faculties. The traditional concepts of these phenomena was metaphysical and identified with the very vague idea of 'spirit', without succeeding in formulating any scientifically tenable hypothesis as to how they are constituted and develop. In contrast, making psychology derive from biology allows us to identify the processes of psychological formation and evolution, to study them experimentally as well, and to document them with empirical evidence.[9]

Conceiving practical and mental processes in terms of functional systems opens the perspective of being able to arrive at establishing the different degree of evolution comparatively along a continuous line of unconditioned, innate reflections (that is, from the 'primary thinking' of dynamic psychology) to thought proper (the 'secondary thinking' mediated by cultural forms: in fact, by functional systems) to its highest rational, systematic and abstract expressions.

One consequence of the theorisation with as much regard to psychology as well as to the history of cultural and of cultural comparability is that, in order to evaluate the level of the intellectual development of cultures and the mental development of individuals by means of analytical, scientific methodologies, it is necessary to be able to identify their respective structures and the cognitive processes both proper to and permitted by them. The outcome is guaranteed to the highest level where the relative procedures and results can be formalised.

Regarding the sphere of pedagogy, with reference to education of all kinds, no educative and didactic program can be decided scientifically if it cannot be articulated in the relative functional systems. This alone makes it possible to identify the mental faculties that provide the acquisition of all skills and knowledge.

To my knowledge, in the attempt to create a formal system of psychopedagogy, the units of analysis of Activity Theory present themselves as the first axioms to be adopted, those which provide the most promising point of departure traceable to psychology. However, there are also other axioms to be integrated which reflect areas of knowledge and concepts from both psychology and other pertinent disciplines, as well as simple ideas not yet clarified at the level of concepts but still indispensable for discourse and normally used in that sense, in the lack of adequate knowledge, in various scientific areas. For example, given the current state of knowledge, I don't know how far it is possible to consider a concept the physical idea of 'energy', whose sense of 'potential work' (in our case the state of being capable of responding to stimuli by determining specific actions or processes) leaves the nature of it completely vague. However, that idea is indispensable for reasoning as to processes (or changes of state), just as another equally vague idea is indispensable—that of 'quantum of energy'—which alone makes it possible to establish sufficiently precise correspondences between the determining conditions of a process and its effects.

Instead, both measuring and qualifying energy (in terms of motion, heat, electromagnetic potential) are so much a part of practices both physical and physiological that this suggests, if not the possibility of arriving to doing so in psychology as well, that it is at least legitimate to avail ourselves of the abstract idea of quantum of energy to mediate between two different psychic states that are reciprocally connected.

There are then concepts in the proper sense both psychological and of the various educational sciences, as well as historical, social and cultural sciences in general, that can be used to delineate a psychological axiomatic that makes it possible to elaborate the procedures of deductive discourse. There are, for example, Piaget's 'assimilation' and 'accommodation' of psychic structures, indispensable for describing how these transform, and the so-called 'memory' (intended as equal to the overall mind as functional disposition, not as a part of this in a conception that hypostatises both). Also, and again taken as examples, are the concepts proper of 'Markov processes' and 'non-Markov processes', by means of which it is possible to distinguish genres of knowledge that make it possible to reconstruct the genesis from those that do not.

If it were possible to identify the minimum indispensable number of concepts and notions of any conceptual level compatible with the state of the respective sectors of knowledge implied, which could be assumed as axioms for defining the basis of an axiomatic system, this would be the first step in the direction towards the formalisation of psychopedagogy. As always, both such axioms and the entire systems would need to be gradually verified and possibly, but inevitably, rectified, modified,

[9] Even though the undertaking involves problems that are still open, specifically regarding how the symbolic emerges from the biologic, recognising the validity of the principle provides the line of investigation for explaining human nature. This is what Marx did in his day, but on the level of philosophy and not on that of the discipline of psychology. It is on this level instead that Activity Theory, in an experimental framework, is situated.

substituted, integrated and completed, both when defects of logic are discovered, and in function of the results of the empirical verification and the acquisition of new knowledge.

Any formal, axiomatic system provides the conceptual framework for the domain of application, but it can never be exhaustive, much less serve as a substitute for the empirical part of the theory. This is provided exclusively by the empirical investigations carried out through observations and experiments and can never be considered complete. This would only be the case when there is a cessation of all new experiences procured, even by the deepening of those already known, and only when the situation reached is not already in itself susceptible to new developments, the premise for the occurrence of additional situations.

The fact remains that a formal system furnishes the guide for the procedures to be performed in experiments both applicative and of research, and is susceptible to corrections and rectifications in consequence of their results. The dialectic that is thus established between formal and empirical components of the theory gradually modifies both co-evolutionarily with each further development on either plane. Further, a formal system constructed beginning with the unit of analysis such as the one indicated is itself already specialised, by reason of the psychological contents of the axioms.

Among such units of analysis, one that is particularly pertinent and relevant regards the concept of 'autopoiesis' [4]. This allows us to frame fundamental questions such as: Why does cardiac tissue pulse independently from its embryological constitution? Why do cells follow evolutionary paths that specialise them differently in function of their localisation in areas that are reciprocally contiguous and reciprocally influential? Why are the localisations of those of the central nervous system initially so fluid that they are subject to migrations that mark the specialisations assumed?

These are questions that, even when unanswered, are easily and inevitably identified as aimed at the great turning points between organic and inorganic and from the biological to the psychological plane, indicative respectively of the two points of passage from one level to the other. Long before their processes can be explained, what is immediately clear is the absolute inconsistency of the metaphysical justifications sought through recourse to the expressions accepted for centuries as rational explanations of the genre of 'soul, spirit, force or vital thrust', and of intellectual 'soul' or 'spirit'. We can immediately see that extremely vague and substantially empty terms are only manifestations of metaphorical utterings to indicate inferences to contents that are totally unknown.

The modern evolutionistic view does not yet free us from a large part of that ignorance, but at least configures a more reliable and more promising approach to the related problems.

Putting acquired knowledge into a scientifically treatable form would make a determining contribution to their full utilisation in operative practices and in the investigation of problems that have yet to be solved. Analytically detailing the components of any given notion and the operations (the functional system) of any given practical or mental activity (considering that, beside all else, any practical activity is accompanied by a mental activity and both implies and produces it) makes it possible to establish in an evolutionary key the levels of differentiation and specialised organisation. Being able to do it comparatively for two different states of evolutionary levels of differentiation of the same type of knowledge (content, process, skill) would permit an equally analytical understanding of the effects that differentiation and specialised organisation can exert on an individual, situated in the lower level, introduce him to the higher level by involving him in an appropriate activity of either practice or study, and indicate how this should work if done in the proper manner.

To set this enterprise in motion, it would be above all necessary to define, in a form that would make it possible to derive developments that are basically automatic and concepts that can in any case be rigorously deduced and verified, such as 'structure', 'system', 'state of a system', 'function and functioning', 'modality of variation of a system', its 'development' (or 'memory') as the conservation of acquired states and functions, 'autopoiesis', 'recursivity', 'interaction', 'internalisation', 'psychism', 'representation', 'abstraction' and 'communication'. Many of these are already well defined or easily defined with the knowledge available. What is necessary, now, for these as for still others to be formulated, is to render their meanings rigorously unambiguous so that they can be inserted into a deductive system.

I have personally carried out such an attempt with operative functions of verification and an initial example, albeit limited and rudimentary because I lacked some of the many skills required [2]. Taking the initiative to a profitable degree of development would require the intervention of mathematicians who are expert in formalisation. In any case, the operation that I performed confirmed for me (I hope this was not a mere illusion) the plausibility of the hints of a potential axiomatic system that I had appeared to have glimpsed in Activity Theory.

If I am not mistaken, that first step could both suggest the idea as well as pave the way for a line of research, one that in my opinion is most promising and worthwhile.

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Graziano Cavallini previously a senior lecturer in pedagogy in the Department of Physics of the Università Statale of Milan, has led several research projects about the teaching and learning of physics and science, in collaboration with physicists, biologists, mathematicians, geologists, and a variety of teachers from primary schools to university level. In his work, he has mainly dealt with Vygotsky's Activity Theory and history and epistemology of science, connecting them in a unified approach to understand the nature of misconceptions, and to find out the conditions and process of their origin and ways of overcoming them. In such an inquiry, he actively joined research groups working internationally in the field, trying to promote awareness of the interconnection between psychological theories of activity and of schemes, and how useful they are when applied.