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Archimedes and double contradiction proof

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Abstract This article examines Archimedes' proofs in his quadrature of various plane and solid figures which use double contradiction proof (usually known as exhaustion method), and emphasizes the diversity of Archimedes' approach. Though it is widely believed that Archimedes established a standard method of quadrature in his mature work *on Conoids and Spheroids*, an analysis of the final part of the *method* (his last work) reveals that he failed to see a common quantitive property among the solids he was treating, which would have greatly simplified his arguments. In short, Archimedes did not have a systematic 'method' for quadrature, and each new figure constituted a new challenge for him.

Keywords Archimedes · Eudoxus · Quadrature · Exhaustion method

1 Double contradiction proof

The most important result of Archimedes in geometry is the determination of magnitudes (areas and volumes) of plane and solid figures comprised by curved lines and surfaces such as parabolic segments, spheres, paraboloids. In order to achieve these results, Archimedes resorted to a particular mode of reasoning that is known as the 'method of exhaustion'. In effect, both the concept of 'method' and that of 'exhaustion' are dated many centuries after Archimedes (essentially, to the seventeenth century): for this reason I prefer to refer to the Archimedean model of reasoning as the 'double contradiction proof' (if you prefer Latin, double *reductio ad absurdum*).

Schematically, the model can be described in the following terms. Let P be the figure whose magnitude we wish to determine (for example, a sphere), and X be a 'better known' figure (for example, a cylinder) to which P is equal (in Archimedes and in Greek geometry the concepts of 'area' and 'volume' are lacking: the measurement always occurs by direct comparison of two magnitudes). Two series of figures are constructed, I and C, respectively inscribed in and circumscribed about P such that they satisfy two conditions:

1. $I < X < C$;
2. The difference $C - I$ can be made infinitely small: given a magnitude E, there can be an inscribed figure I and a circumscribed figure C such that $C - I < E$.

In this case it is easy to prove that P is equal to X. In fact, if P is less than X, let $E = X - P$; by condition 2, it is possible to take C and I such that $C - I < E$. Then we would have $X - I < C - I < E = X - P$, that is, $P < I$, which is impossible because I is inscribed in P. From the assumption $P > X$, a similar reasoning leads to the existence of a C that would satisfy $P > C > X$, which contradicts the fact that C is circumscribed about P.

This is the essence of the reasoning for 'double contradiction proof', which is often thought of as something very complicated. We must, however, underline from the very beginning that what we have described is a general scheme that can be derived—not without straining—from the various proofs undertaken by Archimedes. We will examine various examples, beginning with the one that comes closest to the abstract model.

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2 Examples of double reduction

Now let us see a few examples of reasoning by double reduction in Archimedes, as well in Euclid's *Elements*.

2.1 Paraboloid or right-angled conoid

In his work *on Conoids and Spheroids* (one of his last writings, as we shall see), Archimedes proves that a parabolic segment (a right-angled conoid to use the terminology of Archimedes' day) is two-thirds of the inscribed cone, that is, half of the cylinder that circumscribes the paraboloid (Fig. 1).

Let *ABC* be a parabola, and *BD* its axis. Let P be the paraboloid generated by the revolution of ABC about axis BD. The inscribed solid I and circumscribed solid C, so that they differ by less than an assigned magnitude, are constructed in the following way: Let the axis BD be divided in half, then the halves bisected. Continuing this process of division, and at the points of division let there be planes parallel to the base AC. The cylinder that circumscribes the paraboloid is divided into small cylinders all equal to QC, and the division is continued until there is a cylinder that is less than E. Now are constructed the inscribed solid I, consisting of small cylinders $I_1, \ldots, I_{n-1}$, and the circumscribed solid C, which consists of small cylinders $C_1, \ldots, C_n$. The difference $C - I$ is obviously equal to small cylinder QC which lies at the bases of the segment, and thus is less than the given magnitude.

Now, from the properties of the parabola, it results that the small cylinders that constitute the inscribed and circumscribed solid ($I_1, I_2, \ldots, I_{n-1}$; $C_1, C_2, \ldots, C_n$) form an arithmetic progression whose smallest term is equal to the difference between two adjacent terms. That is, setting $a = I_1 = C_1$, we have $I_2 = C_2 = 2a$, $I_3 = C_3 = 3a$, etc. Summing the terms of this progression proves that the inscribed solid I (that is, $\sum_{k=1}^{n-1} I_k$) is less than half of the complete cylinder consisting of n small cylinders all equal

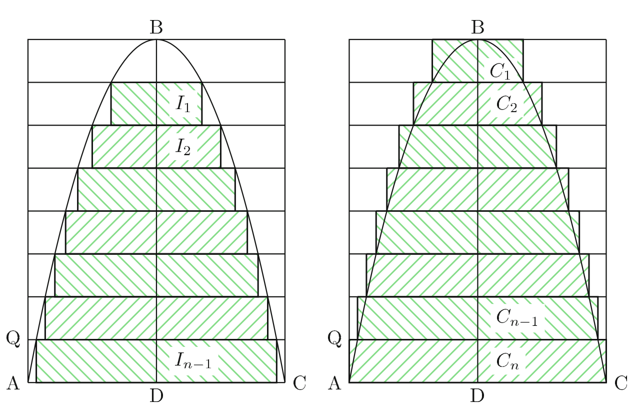


Fig. 1 Inscribed and circumscribed solids to a paraboloid

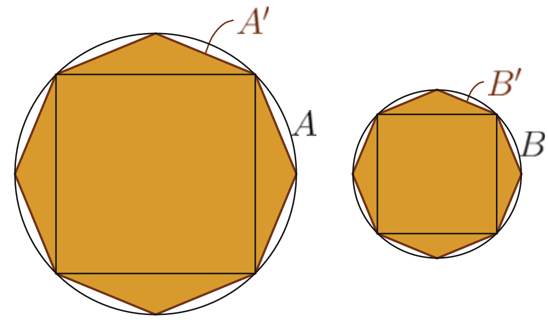


Fig. 2 Similar polygons inscribed in two circles A and B

to QC. Similarly, the circumscribed solid C (that is, $\sum_{k=1}^{n} C_k$), is larger than half of the complete cylinder.[1]

The rest is a simple application of the reasoning by double contradiction proof, and makes it possible to prove that the paraboloid is half of the cylinder that circumscribes it.

2.2 Relationship between two circles (Eudoxus)

Reasoning by double contradiction proof was not an invention of Archimedes. Book XII of Euclid's *Elements* contains several theorems attributed to Eudoxus (a mathematician a few years older than Aristotle) which are proved using precisely this kind of reasoning. To be more exact, Book XII proves that circles are to one another as the squares on their diameters (XII.2); that pyramids that have triangular bases and are of the same height are to one another other as their bases (XII.5); that a cone is one-third of a cylinder having the same base and height (XII.10); and that cones and cylinders of the same height are to one another as their bases (XII.11).

Here let us summarise XII.2.

In the preceding proposition (XII.1) it was proved that two similar regular polygons are to one another as the squares on the diameters of the circles they are inscribed in. Let A and B be two circles, and $q(A)$ and $q(B)$ the squares on their diameters. If the proportion $q(A){:}q(B) = A{:}B$ were not valid, there would be a magnitude X such that $q(A){:}a(B) = A{:}X$ with X either greater or smaller than B.

Let us suppose (the first absurdity) that X is smaller than B. We inscribe a square in circle B: it turns out to be larger than half of B. Now we construct the inscribed octagon by dividing in half the arcs of circles lying between the vertexes of the square (Fig. 2). Thus is removed more than half of what remains of the circle, after having taken away the square.

[1] Algebraically: $\overbrace{a + 2a + \cdots + (n-1)a}^{(n-1)\text{terms}} < \frac{1}{2}\overbrace{(na + na + \cdots + na)}^{n\text{terms}}$
$< \overbrace{a + 2a + \cdots + na}^{n\text{terms}}$.

Continuing in this way to divide the arcs of circles in half and conjoining the lines, the sum of the segments of circle B that remain outside of the inscribed polygon become infinitely small, and thus we can construct a polygon B' such that $B - B'$ is less than the difference $B - X$.[2] Thus, B' is greater than X. Let A' be the polygon similar to B', inscribed in circle A. Then, by the preceding proposition XII.1, we have $A : X = q(A){:}q(B) = A'{:}B'$. *Permuting* the inner terms of the proportion $A{:}X = A'{:}B'$, we also have $A{:}A' = X{:}B$. Since circle A is larger than polygon A inscribed in it, X must also be greater than B', but this is impossible because we constructed B' larger than X.

Another hypothesis, that X is greater than B (the second absurdity), easily leads to a contradiction. *Inverting*, we have $X{:}A = q(B){:}q(A)$ and $X > B$. Thus there exists an area Y less than circle A such that $B{:}Y = q(B){:}q(A)$ and the reasoning reduces to that of the first case, in which X was assumed to be less than B.

In this proposition is clearly recognisable the nucleus of the reasoning by double contradiction proof used later by Archimedes. However, several important elements are lacking with respect to the procedure used in *on Conoids and Spheroids*. In particular, there is no sum of a progression because here we have the comparison of two circles, figures of the same kind, and it is only necessary to compare two similar polygons. It is thus not necessary to attempt to obtain the partial sum of a series to evaluate the magnitude of the inscribed figure. Another difference worth noting is that the circumscribed figure is not used, because the second hypothesis reduces to the first, by exchanging the two circles in question.

2.3 The parabolic segment

Let us go back to Archimedes. In his *Quadrature of the Parabola*, we find an application of the double contradiction proof that is less coherent with both the abstract model and the procedure given in *on Conoids and Spheroids*. In the parabolic segment *ABC* let there be inscribed triangle ABC having the same base and height as the segment ('same height' means that point B is the 'vertex' of the parabolic segment, that is, the point farthest away from the base AC on the parabolic curve between A and C; in other words, the tangent to the parabola at point B is parallel to the base AC). Archimedes proves that the parabolic segment ABC is four-thirds of the triangle ABC (Fig. 3).

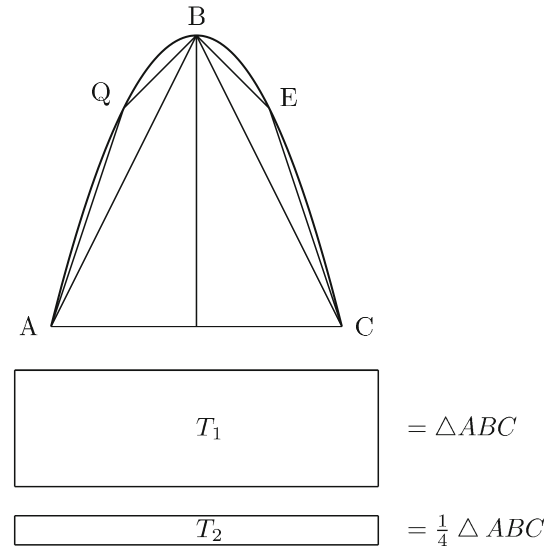


Fig. 3 Inscribe successively the triangles into a segment of a parabola

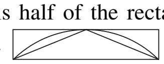
Let us say that area T_1 is equal to triangle ABC. The parabolic segment ABC turns out to be composed of triangle ABC and by the two residual segments of parabola, AQB and BEC. Let Q and E be the vertexes of these segments respectively, and construct triangles AQB and BEC. It is proved that the two triangles AQB and BEC constructed within the segments, taken together, are equal to one-fourth of T_1. Let us consider surface T_2 equal to these two triangles: we have that $T_2 = T_1/4$. In the four residual segments between AQ, QB, BE and EC are constructed four triangles in the same manner. It is proved that these four triangles taken together are one-fourth of T_2; let us say they are equal to surface T_3. In this way we can continue to construct triangles in the residual segments. The inscribed figure I, constructed in this way, is the sum of the geometric series of ratio 1/4:

$$I = T_1 + \frac{1}{4}T_1 + \frac{1}{4^2}T_1 + \cdots + \frac{1}{4^n}T_1 + \cdots$$

We immediately conclude that the whole parabolic segment is equal to:

$$\sum_{n=0}^{\infty} T_n = \frac{1}{1 - \frac{1}{4}}T_1 = \frac{4}{3}T_1$$

Archimedes, who had at his disposal neither the concept of limits nor the sum of an infinite series (which derives from it), resorts to double contradiction proof. Let P be the parabolic segment, and let $K = \frac{4}{3}T_1$. Assume:

[2] At the moment the octagon is constructed, the four triangles added to the square are more than half of the segments of circles that are outside the square, because the triangle is half of the rectangle that circumscribes the segment of the circle.  Thus the segments of the circle outside the octagon are less than half of the segment. The same is true in each doubling of the sides of the inscribed polygon and the segments of the circle become smaller than any given magnitude.

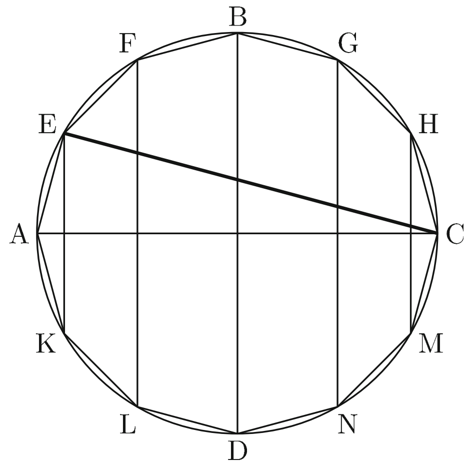


Fig. 4 Sphere and the solid inscribed in it, consisting of cones and truncated cones

that $P - I$ can be smaller than any given area;[3]

1. $K > I$;
2. $K - I$ can be smaller than any given area.

So, if $P > K$, from (1) can be taken an inscribed figure I such that $P > I > K$, which is contrary to (2). If instead $P < K$, I could be taken such that $P < I < K$, which contradicts the fact that I is a figure inscribed in P. Thus, $P = K$.

3 Variety and novelty in Archimedean reasoning

In the proof just seen Archimedes does not construct the circumscribed figure. In fact, the reasonings that he adopts to arrive at the double reduction are variable and not always strictly conform with the paradigmatic form that we set out at the beginning. As we have already observed, the model is based on *on Conoids and Spheroids*, which is a mature work of Archimedes. In other works—the *Quadrature of the Parabola* is just one example among many—the route followed diverges from that model.

It is not in fact indispensable to construct the circumscribed figure C. To prove that the figure P in question is equal to X, it is sufficient that one can construct the inscribed figure I such that the differences $P - I$ and $X - I$ are smaller than any given magnitude (i.e., plane or solid figure). Further, in place of using the fact that the difference $C - I$ can be made smaller than a given magnitude, in certain circumstances it is more convenient to take advantage of the possibility of taking C and I such that they have a smaller relation than that of two magnitudes a, b ($a > b$), that is $a{:}b > C{:}I$. That occurs in the determination of the surface of the sphere, in which C and I are similar figures and it is easier to consider their relationship.

The flexibility with which Archimedes modifies various parts of a presumed general scheme of proof that is always applicable in a uniform manner and his adherence to the particularities of the figures in question suggests to us that he did not have at his disposal a method that was automatically applicable to all the figures he treated.

We will return to this point shortly. What we want to emphasize here is the most remarkable and important novelty that Archimedes introduces into the reasoning by double contradiction proof with respect to the form used by its inventor, Eudoxus. That novelty consists, in my opinion, in the use of the sum of (finite) series to evaluate inscribed and circumscribed solids. If one looks only at *Quadrature of the Parabola* and *on Conoids and Spheroids*, one has the impression that Archimedes' reasoning can be divided cleanly into two distinct parts: the construction of the inscribed figures (and possibly also of the circumscribed figures) and the calculation of the sum of the figures that constitute the inscribed and circumscribed figures. This second part leads us inevitably to recall integral calculus: for us trained in modern mathematics, it is rather difficult to not see the sum of the Riemannian integral in the Archimedean procedure for calculating the sum of the little cylinders I_k/C_k that constitute the inscribed/circumscribed solid whose volume is sought. In reality, Archimedes did not have such a clear and uniform plan of attack. After having used the sum of the geometric series for the parabolic segment, treating the sphere in *On the Sphere and Cylinder* (a work that is later than *Quadrature of the Parabola*) he finds himself faced with the problem of obtaining the sum of a solid inscribed in the sphere composed of cones and truncated cones: in Fig. 4, the circle and the inscribed polygon are rotated about the axis AC, thus generating the sphere and the inscribed solid.

Archimedes reduces the problem of volume of this solid to that of its surface, and then proves that this surface is equal to a circle. The square constructed on the radius of that circle is equal to a rectangle that has as its first side the side AE of the polygon, while its second side is the sum of all the perpendicular chords, that is EK + FL + BD + GN + HM! Then, taking advantage of the similarity of the triangles, he shows that this rectangle is equal to the rectangle comprised by AC and CE.

With this extremely ingenious reasoning—one which never ceases to amaze his readers even after 2,300 years—Archimedes gets around the difficulty of calculating the sum of the parts of the inscribed solid. For him, the essence of the determination of the magnitude of curved figures by means of the double reduction consisted in finding a suitable inscribed figure (and possibly a circumscribed figure as well). It does not

[3] Each triangle constructed in a segment of parabola is greater than one-half of the segment. The reasoning made for a segment of circle (note 2) is also valid for a segment of parabola.

appear that he had at heart that which for us is the most important and useful thing: separating the calculation of the sum of the series from the other parts of the proof; that is, separating the quantitative or algebraic reasoning from the geometric. This undertaking that constitutes the nucleus of integral calculus, which would be arrived at many centuries later, inspired in part by the works of Archimedes.

4 Archimedes 'in difficulty'

It was not easy to use the sum of a series to determine the magnitude of inscribed and circumscribed solids. This point must not be neglected.

In the presentation in Sect. 2.1 above of the reasoning with which Archimedes proves that the paraboloid is half of the cylinder that circumscribes it, we explained the proof as if Archimedes had directly taken the sum of the cylinders that form an arithmetic progression. The reader's impression will have been that this calculation is no more difficult that adding $a + 2a + \cdots + na$.

In reality it was not that simple. The property of the parabola that Archimedes had at his disposal was not an equality but a proportion, that is, a relationship between four magnitudes. In consequence, the magnitude (the volume) of each little cylinder that makes up the solid inscribed in and circumscribed about the parabola appears only as a term of a proportion.

Let's look more closely at the problem. The whole cylinder T is divided by means of parallel plane into n little cylinders $T_1, T_2, \ldots, T_n$ which are all equal to each other. Then, from the property of the parabola by which the squares of the ordinates are proportional to the abscissas, we have:

$$I_1 : T_1 = P_1Q_1 : AD$$
$$I_2 : T_2 = P_2Q_2 : AD$$
$$I_3 : T_3 = P_3Q_3 : AD$$

……………………………

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in which P_1Q_1, P_2Q_2, P_3Q_3, etc. are in arithmetic progression, as can be seen in Fig. 5.

To obtain the sum $I_1 + I_2 + \cdots + I_{n-1}$, it is necessary to add the terms of the proportions. But when there are several proportions, it is not always possible to find the sum of the corresponding terms (for example, it is true that 1:2 = 3:6 and 3:2 = 6:4 but, if we take the sum of corresponding terms, the result is 4:4 = 9:10, which is obviously false). Archimedes thus had to specify the condition that made it possible to find the sum of corresponding terms of several proportions (this is the first proposition of *on Conoids and Spheroids*).

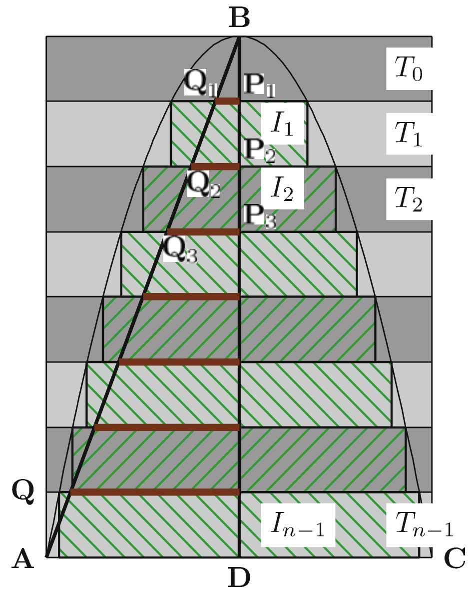

Fig. 5 Consider the sum of the cylinders I_1 to $I_{\{n-1\}}$ which make up the inscribed solid

This is not a banal problem related to mere differences in language between Archimedes and us. One proof of this is the fact that for Archimedes the cases of the ellipsoid and the hyperboloid turn out to be much more complicated. In the case of the hyperboloid (Fig. 6), the properties of the hyperbola provide the following proportion between the little cylinders I_k and T_k $(1 \le k \le n-1)$:[4]

$$I_k : T_k = BP_k(a + BP_k) : BD = a + BD$$

where a is the segment of the axis comprised between the vertices of the two branches of the hyperbola (in modern terms), and Archimedes called its half "the line adjacent to the axis" (that is, the segment between the vertex B and the centre of the hyperbola). Summing the proportions from $k = 1$ to $k = n-1$, results in the relation of the sum:

$$\sum_{k=1}^{n-1} \left(kab + (kb)^2\right)$$

to the sum $(n-1)\left(nab + (nb)^2\right)$ where $b = BP_1 = BD/n$.

For Archimedes, who had at his disposal neither algebraic expressions nor symbols in superscript, dealing with these magnitudes much have been very challenging, and in fact the related proof turns out to be quite complicated. It appears to be precisely in reference to this 'difficulty' that the Syracusan mathematician writes in the preface to *on Conoids and Spheroids*:

> In this book I have set forth and send you the proofs of the remaining theorems not included in what I sent

[4] T_k is an nth part of the complete cylinder that circumscribes the hyperboloid. $T_1 = T_2 = \ldots T_n$.

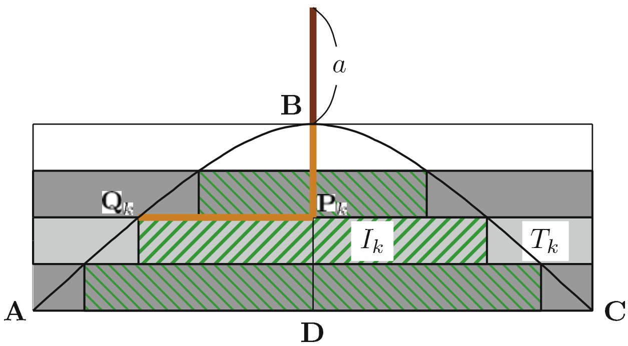

Fig. 6 Inscribed solid to a hyperboloid

> you before, and also of some others discovered later which, though I had often tried to investigate them previously, I had failed to arrive at because I found their discovery attended with some difficulty [1].

5 Archimedes as a precursor of integral calculus?

We have seen the difficulty that Archimedes had to face in determining the volume of solids. One might think (and until recently it was indeed thought) that Archimedes had overcome this difficulty with *on Conoids and Spheroids*, and established a very general method for determining area and volume. However, this interpretation depends heavily on the fact that in *on Conoids and Spheroids* the reasonings about the three solids—paraboloid, ellipsoid and hyperboloid—are practically the same. It is then tempting to interpret that in *on Conoids and Spheroids* Archimedes had arrived at a general method for determining volume, overcoming the difficulty of summing the solids that constitute the inscribed and circumscribed solids. The 'only' things that Archimedes lacked were the algebraic expressions and a concept of limits.

However, one must not be hasty in attributing the label of precursor of integral calculus to him. At the very least, it is first necessary to analyse what he did after *on Conoids and Spheroids*.

Let us begin with the chronology of Archimedes' works. Fortunately, all of his works known today that deal with the determination of area and volume are accompanied by a preface in the form of a letter. Five of these are addressed to a certain Dositheus, and the order in which they were written is determined by the preface. Only the *Method* is dedicated to Eratosthenes, the famous mathematician who directed the Library of Alexandria, and this work is almost certainly dated after the five dedicated to Dositheus. Of these five, the first—not necessarily the absolute first because there are works without a dedication (and thus of uncertain date) and youthful works in all likelihood lost—is the *Quadrature of the Parabola*, in which Archimedes speaks of the death of a friend of his, the mathematician Conon, which had occurred shortly before (we know that Conon was alive in 246 B.C.). The dedicatory letter of *on Spirals*, the fourth of the five works sent to Dositheus, again speaks of Conon, but gives it to be understood that Conon had by then been dead quite some time. Subsequent to *Spiral Lines*, in which Archimedes promised to send within a short time theorems relative to the paraboloid, is *on Conoids and Spheroids*, the last work dedicated to Dositheus, in which the Syracusan mathematician speaks of his 'difficulty' (it is reasonable to suppose a certain delay in the completion of this book).

Therefore, it was already rather late, probably at least after 230 B.C., that Archimedes arrived at completing *on Conoids and Spheroids*. In the *Method*, written even later, Archimedes expresses the hope that future mathematicians will develop his results as well as find new ones: for this reason he explains to Eratosthenes the approach he used. In short, *Method* is his swan song.

To establish the extent to which Archimedes can be considered a precursor of integral calculus, it is thus necessary to see if in the *Method* he proposes a systematic approach to the determination of the volumes of solids. It is here that he describes the approach he used to discover the results proven in the works sent to Dositheus. Therefore, even if the *Method* was sent to Eratosthenes later than the works sent to Dositheus, its content precedes that of the earlier works.

Archimedes' approach has two characteristics that render it invalid as a proving method: (1) the use of an ideal balance, that is, the introduction of a principle of mechanics into geometry; (2) the decomposition of the solid into plane sections without height (in the case of a plane area, into linear sections without thickness) and their recomposition into a solid (area). In other words, to use the language introduced by Bonaventura Cavalieri (1598–1647) nineteen centuries later: the use of *indivisibles*.

It must also be added that the approach using an ideal balance provides only the result (for example, the sphere is two-thirds of the cylinder that circumscribes it) without giving any hint of the geometric proof. The *Method* gave Archimedes a kind of magic carpet that allowed him to see where the top of the mountain he was attempting to scale was, but the trails to follow down here on earth in order to reach it had to be found in a completely different way.

It has to be noted in any case that the figure is question is cut by planes or lines that are parallel (and are also often perpendicular to the balance). Up to this point, the reasoning is similar to the geometric proofs of *on Conoids and Spheroids*. It could then seem reasonable to think that to Archimedes the approach of cutting the figure with parallel planes (or lines) was a familiar one, and when he devoted himself to the study of conoids and spheroids, this approach was given pride of place in the investigation of volume and area. However, recent studies of the final part of the *Method* suggest the opposite. The work contained, after the exposition of the approach by means of the balance, two other results: the last ones that he discovered. The two theorems regard the volume of the intersection of two cylinders and that of a solid that is called today a 'hoof'. The deteriorated state of the codex—the famous palimpsest of Constantinople, discovered in 1906, then lost once again and rediscovered at the end of the twentieth century—has allowed us to retrieve only part of the folia in which these results are dealt with. In a recent study, undertaken together with Pier Daniele Napolitani [2], I reached the following conclusions.

Almost immediately after the discovery of the *Method* in 1906 it was also discovered that the 'hoof' can be obtained by cutting the intersection of cylinders into eight parts. Studying the reasoning followed by Archimedes conserved in his surviving pages and estimating the number of pages lost, we concluded that Archimedes began by grappling with the problem of the intersections of cylinders. To obtain the volume, he divided this solid in eight parts, thus finding the 'hoof'. This solid was not simple to deal with, but with ingenious and laborious reasoning he was able to get to the bottom of it, determining its volume and, consequently, also that of the intersection of cylinders. In the context of our present discussion, the most important thing to observe is that, if Archimedes had attempted the approach of cutting the intersection of cylinders with parallel planes following an opportune direction, he would have very easily found that the problem could be reduced to that of finding the volume of a sphere or an ellipsoid, both of which were well known to him and in fact also treated in the *Method*. He would have had no need to become mixed up in the complicated reasoning that he gives for the 'hoof'. This suggests that after having completed *on Conoids and Spheroids*—and again when he wrote the *Method*, his last work—Archimedes did not think he had established a method for determining the volume (or area) of a figure. Each new figure represented a new challenge for which had to be invented a new technique for measuring. Reading Archimedes, what catches our eye are the elements that were developed later on, and we are inclined to underline his modern aspects. His works have provided indispensable starting points for the development of mathematics in the sixteenth and seventeenth centuries, points of departure that are among the roots of the invention of integral calculus. But the variety of reasoning used by Archimedes also testifies to the distance between him and the moderns, and leads us to consider as far from negligible the contributions of the mathematicians of the sixteenth and seventeenth centuries, who paved the way for something essentially new.

Translated from the Italian by Kim Williams.

Acknowledgments I wish to thank Colin McKinney for help with linguistics.

Appendix: Proposition 4 of the *Method*: the paraboloid and the cylinder

Let BAC be a segment of paraboloid and BEFC the cylinder that circumscribes it (Fig. 7). If the two solids are cut by plane MN perpendicular to axis AD, the sections obtained are respectively the circles of diameter PO and MN (we will indicate them as cir(PO) and cir(MN)).

On the extension of axis DA, take point H such that DA = AH, and imagine a lever DH whose fulcrum is A. By the properties of the parabola (the squares of the ordinates are proportional to the abscissas) and of circles (they are to one another as the squares of their radii), we have:

$$\text{cir(PO)} : \text{cir(MN)} = \text{SA:AH}.$$

By the law of the lever (if the distances between two magnitudes are inversely proportional to them, then the magnitudes are in equilibrium), the section of the paraboloid (cir(PO)), moved to point H, is in equilibrium to the section of the cylinder (cir(MN)) left where it is.

That relation (and thus the equilibrium) is valid for all sections MN and PO. Supposing that it is also valid for all sections taken together and that the cylinder and the segment of paraboloid are filled by such sections, the cylinder BF (which remains were it is) is in equilibrium with the segment of paraboloid transposed such that its centre of gravity is point H (Fig. 8).

Applying once again the law of the lever (because there is equilibrium, the distances are inversely proportional to the magnitudes), we have—bearing in mind that the centre of gravity of the cylinder is K (the midpoint of AD):

$$\text{(paraboloid)} : \text{(cylinder)} = \text{KA} : \text{AH} = 1 : 2.$$

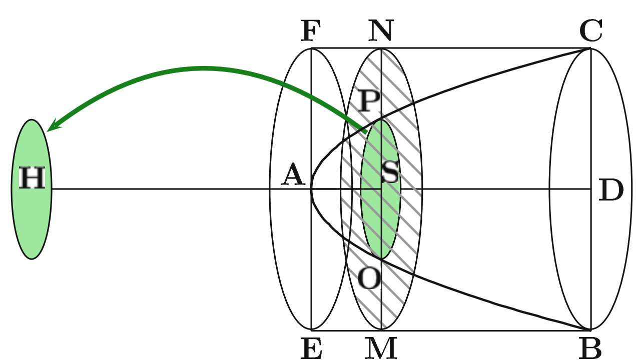

Fig. 7 *Method*, proposition 4. Balance of a section of the paraboloid

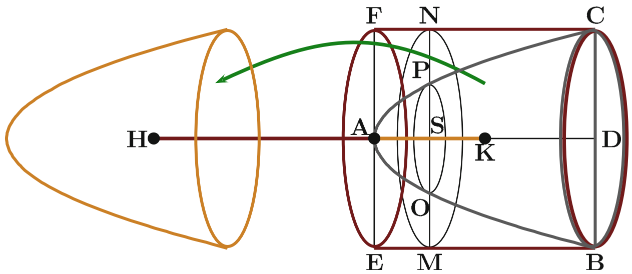

Fig. 8 *Method*, proposition 4. Balance of the whole figure

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Author Biography

Ken Saito (b. 1958) studied history of science (especially history of mathematics) at University of Tokyo, and at Università di Roma 'La Sapienza'. His main interest is the history of Greek mathematics, where he tries to understand the challenges and efforts of the mathematicians by reconstructing the "tool-box" (a concept of his own invention), the set of the theorems and the techniques to which the mathematicians had recourse. He is one of the founding editors of the journal *SCIAMVS*, dedicated to unpublished sources of pre-modern exact sciences. His recent research includes that of diagrams in the mathematical manuscripts, which have not always been printed faithfully in authoritative critical editions.