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Asymptotic preserving methods for quasilinear hyperbolic systems with stiff relaxation: a review

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Abstract
Hyperbolic systems with stiff relaxation constitute a wide class of evolutionary partial differential equations which describe several physical phenomena, ranging from gas dynamics to kinetic theory, from semiconductor modeling to traffic flow. Peculiar features of such systems is the presence of a small parameter that determines the smallest time scale of the system. As such parameter vanishes, the system relaxes to a different one with a smaller number of equations, and sometime of a different mathematical nature. The numerical solution of such systems may present some challenges, in particular if one is interested in capturing all regimes with the same numerical method, including the one in which the small parameter vanishes (*relaxed system*). The design, analysis and application of numerical schemes which are robust enough to solve this class of systems for arbitrary value of the small parameter is the subject of the current paper. We start presenting different classes of hyperbolic systems with relaxation, illustrate the properties of implicit–explicit (IMEX) Runge–Kutta schemes which are adopted for the construction of efficient methods for the numerical solution of the systems, and then illustrate how to apply IMEX schemes for the construction of *asymptotic preserving schemes*, i.e. scheme which correctly capture the behavior of the systems even when the relaxation parameter vanishes.

Keywords Hyperbolic systems with relaxation · semi-implicit scheme · IMEX Runge-Kutta methods · Asymptotic-preserving methods

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1 Introduction

Several physical systems are governed by hyperbolic systems of balance or conservation laws. Typical example of conservation laws are found in gas dynamic dynamics, where the

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fundamental laws of mass, momentum and energy, together with the constitutive laws proper of each gas, determines the motion. In presence of a source term, the system has the structure of a balance law: the rate of change of a density vector is given by the net inflow plus the rate of production.

Several systems of balance laws have the structure of *relaxation* systems: their behavior depends on a parameter (typically a relaxation time), and when such parameter vanishes the system relaxes to another one with a smaller number of equations. The most famous example comes from the kinetic theory of gas, where the population of a large collection of atoms is accurately described by the Boltzmann equation of rarefied gas dynamics [37]. The behavior of the gas strongly depends on the Knudsen number, equal to the ratio between the molecular mean free path and a typical macroscopic length scale: when this number is very small, the gas relaxes to local thermodynamical equilibrium and its behavior can be accurately described by compressible Euler or Navier-Stokes equations, with a drastic reduction in the complexity.

Other examples of systems with similar structure can be found in discrete velocity models [91], in gas with vibrational degrees of freedom [95], in hydrodynamical models for semiconductors [3], in shallow water equation with friction, in traffic flow model [9]. Other examples of this class of systems can be found in the classic book by Whitham [98], and in the book by Ruggeri and Sugiyama [85]. From a mathematical point of view, these systems have the structure of hyperbolic systems with relaxation (see Sect. 2, and have been extensively studied in recent years [13, 40, 44, 65, 74]).

Numerical solution of hyperbolic systems of conservation laws are typically obtained by *shock capturing schemes*, which are usually based on finite volume [69, 93], finite difference [89], or finite element (discontinuous Galerkin [12]) space discretization. All these schemes are based on three main ingredients: a numerical flux function, which is provided by an exact or approximated solution to the Riemann problem, a non-linear reconstruction, which allows to construct high order schemes in smooth regions, still avoiding the formation and amplification of spurious oscillations near discontinuities, and a suitable ODE solver that does not introduce oscillations when discretizing time.

When a source term is present on the right hand side, new problems arise. The two main issues that have attracted a lot of attention concern the preservation of equilibria and the treatment of stiff sources.

The first problem is very relevant when computing solutions which are small deviation from stationary ones. This problem has led to the construction of the so called *well-balanced schemes* [33].

The second problem is usually faced by adopting implicit schemes for the treatment of the stiff source. In the case of hyperbolic systems with stiff source, in general the hyperbolic term is not stiff, in the sense that stability and accuracy requirement on the time step are usually similar. For such a reason the hyperbolic term is normally treated explicitly. The natural way to deal with a hyperbolic systems with relaxation is therefore to make use of implicit–explicit (IMEX) schemes, so that the hyperbolic term is treated explicitly, while the stiff source is treated implicitly, thus maximizing efficiency.

If the numerical scheme is sufficiently robust, it should be able to solve the system over a wide range of the relaxation parameter, and to capture all the regimes, including the one in which the system relaxes to the reduced one. A numerical scheme which is able to capture the relaxed limit is called *Asymptotic preserving* (AP) [63].

In [27] the authors construct a scheme which is able to capture the relaxed limit, with particular application to the Broadwell model of rarefied gas dynamics. In [70] the authors couple a staggered central scheme for the hyperbolic part, with implicit scheme for the stiff relaxation, and construct effective schemes which are able to solve stiff relaxation problems

even when the solution to the Riemann problem is not known. After such pioneering work, it has been realized that the natural time discretization that has to be adopted for the construction of AP schemes for hyperbolic systems with stiff relaxation is provided by Implicit–Explicit (IMEX) schemes either Runge–Kutta [6] or multi-step [7].

The construction and analysis of AP schemes for hyperbolic system with relaxation is the focus of the present paper, whose plan is the following. After this introduction, in the next section we illustrate the main types of hyperbolic systems with relaxation, reviewing hyperbolic, parabolic, and multi-scale relaxation problems. In Sect. 3 we recall the main properties of IMEX-Runge Kutta schemes, which will then adopted in the next three sections for the construction of AP schemes for the three types of relaxation problems mentioned in Sect. 2. In the last section we draw some conclusions.

A more thorough and detailed description of Asymptotic Preserving schemes for hyperbolic systems with relaxation, together with several other applications, can be found in the forthcoming book “”Implicit–explicit methods for evolutionary partial differential equations”, currently in progress and scheduled for publication on SIAM in November 2024.

2 Relaxation problems

In this section we describe the different types of relaxations that are commonly encountered, according to the time scales of the hyperbolic and relaxation terms.

We start from the hyperbolic-hyperbolic relaxation, move to the hyperbolic to parabolic relaxation and then describe the general multiscale relaxation.

2.1 Hyperbolic systems with relaxation

Probably the simplest and most popular example of hyperbolic system with relaxation is represented by the 2×2 Jin-Xin system [65].

$$\begin{aligned} \partial_t u + \partial_x v &= 0 \\ \partial_t v + a^2 \partial_x u &= -\frac{1}{\varepsilon}(v - f(u)) \end{aligned} \tag{1}$$

where $a > 0$ defines the characteristic speed of the system. Indeed, the eigenvalues of the matrix defining the hyperbolic systems are given by $\lambda = \pm a$. Formally, as $\varepsilon \to 0$, the second equation gives the equilibrium state $v = f(u)$ which, inserted into the first equation, yields the scalar conservation law

$$\partial_u + \partial_x f(u) = 0. \tag{2}$$

Is this formal limit also a rigorous one? And what happens if ε is small but not zero? To answer such questions we can observe that from the second equation we have

$$v = f(u) - \varepsilon(v_t + a^2 u_x) = f(u) + O(\varepsilon), \tag{3}$$

from which it follows

$$\begin{aligned} v_t &= f(u)_t + O(\varepsilon) = f'(u)u_t + O(\varepsilon) = -f'(u)v_x + O(\varepsilon) \\ &= -f'(u)f(u)_x + O(\varepsilon) = -f'(u)^2 u_x + O(\varepsilon). \end{aligned}$$

Using this relation in Eq. (3), we obtain

$$v = f(u) - \varepsilon(a^2 - f'(u)^2)u_x + O(\varepsilon^2)$$

and, substituting this expression in the first equation of system (1), such equation becomes

$$u_t + f(u)_x = \varepsilon((a^2 - f'(u)^2)u_x)_x + O(\varepsilon^2). \tag{4}$$

Neglecting second order terms in the small parameter ε, the above expression represents a non-linear convection-diffusion equation, in which the diffusion coefficient is

$$\nu = \varepsilon(a^2 - f'(u)^2).$$

The for small values of ε, The well-posedness of the initial value problem for Eq. (4) requires $\nu \geq 0$, which means that the following *subcharacteristic condition*

$$|f'(u)| \leq a \tag{5}$$

has to be satisfied. This condition is therefore necessary for the well-posedness of system (1), at least for sufficiently small values of ε. For a more rigorous derivation of the subcharacteristic condition we refer to [71]. The above procedure is called Chapman-Enskog expansion, and has been adopted to formally derive Navier–Stokes equation from the Boltzmann equation in the limit of small Knudsen number [39].

The 2×2 system has the structure of a semilinear hyperbolic system with stiff relaxation, which, in the limit of vanishing relaxation time ε, relaxes to a single quasilinear convection equation.

Of course one can consider the relaxation of a quasilinear system to a quasilinear equation. For example, by replacing $a^2 u$ by $p(u)$ in (1) one has the system

$$\begin{aligned} \partial_t u + \partial_x v &= 0 \\ \partial_t v + \partial_x p(u) &= -\frac{1}{\varepsilon}(v - f(u)) \end{aligned} \tag{6}$$

In this case hyperbolicity of the system is guaranteed it $p'(u) \geq 0$, and the subcharacteristic condition becomes

$$f'(u)^2 \leq p'(u).$$

The convergence property has been used by Jin and Xin for numerical purposes: since the solution u of system (1) converges to the solution of the quasilinear equation (11), then one can exploit this property and propose numerical methods for the numerical solution of the latter based on numerical solution fo the former, which may be somehow simpler in some cases, given that system (1) is semilinear while (11) is quasilinear. The class of schemes based on this idea are called relaxation schemes, and have been widely studied in the literature (see for example [36] for relaxation schemes based on discontinuous Galerkin discretization, [35] for application to chemotaxis, and [4], where relaxation schemes have been applied to parabolic systems).

The concept can be generalized to larger systems. In one space dimension, $x \in \Omega \subset \mathbb{R}$, hyperbolic systems with relaxation take the form

$$\partial_t U + \partial_x F(U) = \frac{1}{\varepsilon} R(U), \tag{7}$$

where ε is called the *relaxation parameter* and $F : \mathbb{R}^N \to \mathbb{R}^N$ is the flux function. System (7) is said to be hyperbolic if the $N \times N$ Jacobian matrix $F'(U) = (\partial F_i / \partial U_j)$ has real eigenvalues and admits a basis of eigenvectors $\forall\, U \in \mathbb{R}^N$.

We use the notion of relaxation system in the sense of Whitham [98] and Liu [71], i.e., the operator $R : \mathbb{R}^N \to \mathbb{R}^N$ is said a *relaxation operator*, and consequently (7) defines a relaxation system, if there exists a constant $N_1 \times N$ matrix Q with rank $N_1 < N$ such that

$$QR(U) = 0, \quad \forall U \in \mathbb{R}^N. \tag{8}$$

This induces the existence of N_1 independent conserved quantities $u = QU \in \mathbb{R}^{N_1}$. In addition we assume that each such u uniquely determines a local equilibrium value, i.e. there exists a function

$$U = \mathcal{E}(u) \quad \text{such that} \quad R(\mathcal{E}(u)) = 0 \quad \text{and} \quad Q\mathcal{E}(u) = u \quad \forall u \in \mathbb{R}^{N_1}. \tag{9}$$

The function $\mathcal{E}$ represents the manifold of local equilibria, i.e. it spans the kernel of the relaxation function R.

Using (8) in (7), associated with Q, we obtain a system of N_1 conservation laws which is satisfied by every solution of (7)

$$\partial_t(QU) + \partial_x(QF(U)) = 0. \tag{10}$$

Note that the above system is not closed since it requires the knowledge of the full N-dimensional solution U. However, for vanishing small values of ε, Eq. (7) formally gives us $R(U) = 0$, which by (9) implies $U = \mathcal{E}(u)$ and in this case system (7) relaxes to the equilibrium system

$$\partial_t u + \partial_x f(u) = 0, \tag{11}$$

where $f(u) = QF(\mathcal{E}(u))$. In order for this limiting process to be stable in the sense of Chen, Levermore and Liu [40] a suitable sucharacteristic condition, linking the eigenvalues of the Jacobian of the flux in (7) to those of the flux in (11), must be satisfied.

Remark 2.1 It is interesting to observe that if speed a is constant, system (1) is semilinear, while the relaxed equation is quasilinear. Since it is easier to study, both analytically than numerically, semilinear systems than quasilinear system, Jin and Xin proposed this technique as a numerical tool for the solution of quasilinear hyperbolic systems: approximate a quasilinear system of order N_1 by a larger semilinear system, and solve it with a scheme that is able to capture the behavior of the original quasilinear system in the limit of vanishing ε. A scheme with such a property is called *Asymptotic Preserving* (AP), as explained in Sec. 4.1

In the following we report some examples of such system of particular interest in applications.

2.1.1 Broadwell model

A simple discrete velocity kinetic model for a gas was introduced by Broadwell [26]. It describes a three-dimensional (3-D), respectively two-dimensional (2-D) gas composed of particles with only six, respectively four, velocities with a binary collision law and spatial variation in only one direction. It represents a simple model of the Boltzmann equation and can qualitatively describe some properties of a rarefied gas and the corresponding fluid dynamic limit. When looking for one-dimensional (1-D) solutions of the two-dimensional gas, the evolution equations of the model are given by

$$\begin{aligned} \partial_t f + \partial_x f &= \frac{1}{\varepsilon}\left(h^2 - fg\right), \\ \partial_t h &= -\frac{1}{\varepsilon}\left(h^2 - fg\right), \\ \partial_t g - \partial_x g &= \frac{1}{\varepsilon}\left(h^2 - fg\right) \end{aligned} \tag{12}$$

where ε is the mean free path, f, h, and g denote the mass densities of gas particles with velocities 1, 0, and -1, respectively, in position x at time t. Similar equations are obtained for 1-D solutions of a 3-D gas [28].

Introducing the fluid dynamic variables: density ρ, momentum m, and energy z

$$\rho = f + 2h + g, \quad m = f - g, \quad z = f + g,$$

the kinetic model (12) can be rewritten as

$$\begin{aligned} &\partial_t \rho + \partial_x m = 0, \\ &\partial_t m + \partial_x z = 0, \\ &\partial_t z + \partial_x m = \frac{1}{2\varepsilon}\left(\rho^2 + m^2 - 2\rho z\right). \end{aligned} \tag{13}$$

Define $U = (\rho, m, z)^T$, $F(U) = (m, z, m)^T$ and $R(U) = (0, 0, (\rho^2 + m^2)/2 - \rho z)^T$, then (13) falls into the general form of the hyperbolic system with relaxation (7). Note that, we could have also written the system in the form (7) using the original kinetic variables in (12) that are obtained from the fluid dynamic variables using relations

$$f = \frac{z + m}{2}, \quad g = \frac{z - m}{2}, \quad h = \frac{\rho - z}{2}.$$

Since the only conserved quantities are the density ρ and the momentum m the matrix Q in the case of system (13) takes the form

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

which gives $QU = u$ with $u = (\rho, m)^T$. In the fluid-dynamic limit $\varepsilon \to 0$, one can see that formally z goes to the local equilibrium

$$z = z_E(\rho, m) \equiv \frac{1}{2}\left(\rho + \frac{m^2}{\rho}\right), \tag{14}$$

corresponding to $\mathcal{E}(u) = (\rho, m, (\rho + m^2/\rho)/2))^T$, and the Broadwell system (13) is approximated by the reduced Euler system

$$\begin{aligned} &\partial_t \rho + \partial_x m = 0, \\ &\partial_t m + \partial_x \left(\frac{1}{2}\left(\rho + \frac{m^2}{\rho}\right)\right) = 0. \end{aligned} \tag{15}$$

The above system can be written in the form (11) with $u = (\rho,\ m)^T$, $f(u) = QF(\mathcal{E}(u)) = (m,\ (\rho + m^2/\rho)/2))^T$. We refer to [27, 70, 77] for numerical examples on such a limiting process.

2.1.2 Traffic flow models

Traffic flow models are mathematical models of real-world traffic, usually, but not restricted to, road traffic. The quantity that is modeled and measured is the flow of mobile units (e.g., vehicles) per unit time and capacity of the transport medium (e.g., road or lane width). Models are used by researchers and engineers for various problems, such as ensuring optimal flow with a minimum number of traffic jams.

Among the various models, we focus on a second order model that consists of a continuity equation for the density ρ of vehicles together with an additional velocity equation that describes the mass flux variations due to the road conditions in front of the driver. The model can be written in conservative form as follows [8]

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0, \\ \partial_t(\rho w) + \partial_x(v\rho w) &= \frac{\rho}{\varepsilon}(V(\rho) - v), \end{aligned} \tag{16}$$

where $w = v + P(\rho)$ with $P(\rho)$ a given function describing the anticipation of road conditions in front of the drivers and $V(\rho)$ describing the dependence of the velocity with respect to the density in an equilibrium situation. System (16) can be cast into the form (7) taking $U = (\rho, \rho v)^T$, $F(U) = (\rho v, v\rho w)^T$ and $R(U) = (0, \rho(V(\rho) - v))^T$. The parameter ε is the relaxation time, and as $\varepsilon \to 0$, if the functions $V(\rho)$ and $P(\rho)$ satisfy the subcharacteristic condition

$$-P'(\rho) \le V'(\rho) \le 0, \quad \rho > 0,$$

we obtain the traffic equilibrium condition $v = V(\rho)$ that substituted into the first equation in system (16) yields the Lighthill-Whitham model [98]

$$\partial_t \rho + \partial_x(\rho V(\rho)) = 0. \tag{17}$$

A typical choice for the function $P(\rho)$ is given by [8]

$$P(\rho) = \begin{cases} \dfrac{c_v}{\gamma}\left(\dfrac{\rho}{\rho_m}\right)^{\gamma} & \gamma > 0, \\ c_v \log\left(\dfrac{\rho}{\rho_m}\right) & \gamma = 0, \end{cases}$$

where ρ_m is a given maximal density and c_v a constant velocity that represents the propagation speed of the backward wave at the maximum density. Whereas $V(\rho)$, which defines the concave function $\rho V(\rho)$ referred to as the fundamental diagram, is taken as [8]

$$V(\rho) = c(P(\rho_m) - P(\rho)), \qquad 0 \le c \le 1.$$

2.2 Diffusive relaxation

When looking for small effects under a very long time, the behavior of the solutions to relaxation problem appear quite different. In order to capture such behavior it is necessary to adopt a different scaling. Let us start from the Jin-Xin system (1), and let us look for long time behavior of the solution by scaling time as $t = \tilde{t}/\varepsilon$. The rescaled system becomes

$$\begin{aligned} \varepsilon \partial_{\tilde{t}} u + \partial_x v &= 0 \\ \varepsilon \partial_{\tilde{t}} v + a^2 \partial_x u &= -\frac{1}{\varepsilon}(\varepsilon v - f(u)) \end{aligned} \tag{18}$$

In order to have non trivial solutions, we need to look at small signals, for which also v and $f(u)$ are proportional to ε. Using the substitution $v = \varepsilon\tilde{v}$ and $f = \varepsilon\tilde{f}$ in (18) we obtain

$$\begin{aligned} \partial_{\tilde{t}} u + \partial_x \tilde{v} &= 0 \\ \varepsilon^2 \partial_{\tilde{t}} \tilde{v} + a^2 \partial_x u &= \tilde{f}(u) - \tilde{v} \end{aligned} \tag{19}$$

Formally, as $\varepsilon \to 0$, the system (19) reduces to

$$\begin{aligned} \partial_{\tilde{t}} u + \partial_x \tilde{f}(u)_x &= a^2 \partial_{xx} u \\ \tilde{v} &= \tilde{f}(u) - a^2 \partial_x u \end{aligned} \tag{20}$$

Notice that with the new scaling the propagation velocity of the hyperbolic part are now

$$\lambda_{\pm} = \pm \frac{a}{\varepsilon} \tag{21}$$

therefore they diverge as $\epsilon \to 0$.

2.2.1 Maxwell–Cattaneo model

Hyperbolic to parabolic diffusion has several concrete physical applications. Probably the best known model described by a similar system is the Maxwell–Cattaneo model for heat conduction. From a conceptual point of view classical heat equation based on the Fourier law is not satisfactory, since it implies infinite propagation speed: starting from a compact support initial condition, at any arbitrarily small time the solution to the heat equation has infinite support. To circumvent such a difficulty, in the late forties Carlo Cattaneo proposed the following model for heat conduction: he replaced the classical Fourier law in a solid [30, 34]

$$q = -\kappa \nabla T,$$

where $q = q(x,t) \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, denotes the heat flux, κ the thermal conductivity, and T the absolute temperature, with the relation

$$\tau \frac{\partial q}{\partial t} + q = -\kappa \nabla T, \tag{22}$$

where τ is a (small) relaxation time. When coupling this relation with energy balance in a solid (assuming the only energy exchange is due to heat flow!) one obtains the system

$$\begin{aligned} \rho c_v \frac{\partial T}{\partial t} + \nabla \cdot q &= 0 \\ \frac{\partial q}{\partial t} + \frac{\kappa}{\tau} \nabla T &= -\frac{q}{\tau} \end{aligned} \tag{23}$$

where ρ is the mass density of the solid, and c_v its specific heat per unit mass at constant volume. Equation (23) represents a hyperbolic system with relaxation composed of $d+1$ equations in d dimensions. In dimension $d \geq 1$ there are $d-1$ zero eigenvalues, and the remaining two are given by

$$\lambda^2 = \kappa/(\tau \rho c_v) \tag{24}$$

independently on the propagation direction $\vec{n}$.

As $\tau \to 0$, the largest characteristic speed diverges, while the equation reduces to the standard heat equation. For constant coefficient physical parameters, the limit equation for the temperature reduces to

$$\frac{\partial T}{\partial t} = \nu \Delta T$$

where $\nu = \kappa/(\rho c_v)$ is the diffusion coefficient. An interesting physical interpretation of Cattaneo model comes from kinetic theory. From the Boltzmann equation it is possible to derive a relation between the heat flux and temperature gradient of the form (see [96])

$$q(x, t + \tau) = -\kappa \nabla T(x, t),$$

i.e. the heat flux is proportional to the temperature gradient, but with a delay τ. Cattaneo relation (22) can be derived as first order Taylor expansion in τ of the above relation.

2.2.2 Goldstein–Taylor model

Another interesting model related to the diffusion limit is the one-dimensional Goldstein–Taylor model [52, 92]

$$\begin{aligned} \partial_t u + \frac{1}{\varepsilon}\partial_x u &= \frac{1}{\varepsilon^2}(u - v), \\ \partial_t v + \frac{1}{\varepsilon}\partial_x v &= \frac{1}{\varepsilon^2}(u - v). \end{aligned} \tag{25}$$

The macroscopic (fluid) variables for this model are the mass density ρ and the flux j, defined by

$$\rho = u + v, \quad j = \varepsilon^{-1}(u - v).$$

Expressing u and v in terms of ρ and j, system (25) is equivalent to the following system

$$\begin{aligned} \partial_t \rho + \partial_x j &= 0 \\ \partial_t j + \frac{1}{\varepsilon^2}\partial_x v &= -\frac{2}{\varepsilon^2} j. \end{aligned} \tag{26}$$

In the diffusive limit, i.e., $\varepsilon \to 0$, system (26) can be approximated to the leading order by

$$\partial_t \rho - \frac{1}{2}\partial_{xx}\rho = 0, \quad j = -\frac{1}{2}\partial_x \rho,$$

so we obtain the heat equation for the density ρ and the so-called *local equilibrium* for the flux j.

2.3 Multiscale relaxation

Hyperbolic systems with relaxation [40, 71, 74] often contain multiple space-time scales which may differ by several orders of magnitude. Indeed, the various parameters characterizing the models permit to describe several physical situations, such as flows which change regime from compressible to incompressible, or flows which range from rarefied to dense states. This is the case, for example, of kinetic equations close to the hydrodynamic limits [10, 37, 38, 94]. Near the fluid dynamic limit these systems can be more conveniently described in terms of macroscopic models such as Euler or Navier–Stokes equations, which are more

amenable to numerical treatment because of the drastic reduction of the dimensionality of the problem [94]. However, such macroscopic models can not handle all the possible regimes one is frequently interested in. For such reason sometimes one has to resort to the full kinetic models, which provide a more detailed physical description, but are computationally more expensive and limited by the stiffness induced by the scaling under consideration [47].

In this section we deal with hyperbolic systems with multiscale relaxation and as a prototype example, that we use to illustrate the subsequent theory, we consider the following simple 2×2 system

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \\ \partial_t v + \dfrac{1}{\varepsilon^{2\alpha}} \partial_x p(u) = -\dfrac{1}{\varepsilon^{1+\alpha}} \left(v - f(u) \right), \quad \alpha \in [0, 1] \end{cases} \tag{27}$$

where $p'(u) > 0$. System (27) is hyperbolic with two distinct real characteristics velocities $\pm\sqrt{p'(u)}/\varepsilon^{\alpha}$. The scaling in such system depends on an additional parameter α which modifies the nature of the asymptotic behavior which can be either hyperbolic or parabolic.

Note that the following scaling

$$t = \varepsilon^{\alpha}\tau, \ \xi = x, \ v(x,t) = V(\xi,\tau)/\varepsilon^{\alpha}, \text{ and } f(u) = F(u)/\varepsilon^{\alpha}$$

system (27) corresponds to the study of the limiting behavior of the solution for the usual hyperbolic system with a singular perturbation source

$$\begin{cases} \partial_\tau u + \partial_\xi V = 0, \\ \\ \partial_\tau V + \partial_\xi p(u) = -\dfrac{1}{\varepsilon} \left(V - F(u) \right). \end{cases} \tag{28}$$

For $\alpha = 0$, the system reduces to the usual hyperbolic scaling (6). Positive values of α correspond to looking for long time behavior of the solution. In particular, for small values of ε, using the Chapman-Enskog expansion, the behavior of the solution to (27) is, at least formally, governed by the following nonlinear parabolic system

$$\begin{cases} v = f(u) - \varepsilon^{1-\alpha} \partial_x p(u) + \varepsilon^{1+\alpha} f'(u)^2 \partial_x u + \mathcal{O}(\varepsilon^2), \\ \\ \partial_t u + \partial_x f(u) = \varepsilon^{1+\alpha} \partial_x \left[\left(\dfrac{p'(u)}{\varepsilon^{2\alpha}} - f'(u)^2 \right) \partial_x u \right] + \mathcal{O}(\varepsilon^2). \end{cases} \tag{29}$$

Therefore, as $\varepsilon \to 0$ when $\alpha \in [0, 1)$ we obtain the scalar conservation law

$$\begin{cases} v = f(u), \\ \\ \partial_t u + \partial_x f(u) = 0. \end{cases} \tag{30}$$

Note that, the main stability condition [40, 71] for system (29) corresponds to

$$f'(u)^2 < \frac{p'(u)}{\varepsilon^{2\alpha}}, \tag{31}$$

and it is always satisfied in the limit $\varepsilon \to 0$ when $\alpha > 0$, whereas for $\alpha = 0$ it requires suitable assumptions (21) on the functions $f(u)$ and $p(u)$.

In classical kinetic theory the space-time scaling just discussed leads the so-called hydrodynamical limits of the Boltzmann equation (see [38], chapter 11). For $\alpha = 0$ this corresponds to the *compressible Euler limit*, because the speed of the acoustic waves remains unbounded,

whereas for $\alpha \in (0, 1)$ the *incompressible Euler limit* is obtained, because acoustic wave speed diverges.

Something special happens when $\alpha = 1$. In this case, in fact, to leading order in ε, we obtain the convection-diffusion equation

$$\begin{cases} v = f(u) - \partial_x p(u), \\ \\ \partial_t u + \partial_x f(u) = \partial_{xx} p(u). \end{cases} \tag{32}$$

In other words, considering times larger than those typical for Euler dynamics, dissipative effects become non-negligible. This behavior characterizes the *incompressible Navier–Stokes limit* in classical kinetic theory.

The development of numerical methods to solve hyperbolic systems with stiff source terms in the case $\alpha = 0$ has been an active area of research in the past three decades [16, 27, 49, 59, 76, 79, 80, 84]. Another series of works is concerned with the construction of robust schemes for $\alpha = 1$ when a diffusion limit is obtained [20, 23, 62, 64, 67]. However, very few papers have considered the general multiscale problem of type (27) for the various possible values of α [63, 72].

The common goal of this general class methods, often referred to as *asymptotic-preserving* (AP) schemes, was to obtain the macroscopic behavior described by the equilibrium system by solving the original relaxation system (27) with coarse grids $\Delta t,\ \Delta x \gg O(\varepsilon)$, with Δt and Δx are respectively the time step and the mesh size. Note that, since the characteristic speeds of the hyperbolic part of system (27) are of order $1/\varepsilon^\alpha$, most of the popular methods [27, 59, 79], for the solution to hyperbolic conservation laws with stiff relaxation present several limitations when considering the whole range of $\alpha \in [0, 1]$ and fail to capture the right behavior of the limit equilibrium equation unless the small relaxation rate is numerically resolved, leading to a stability condition of the form $\Delta t \sim \varepsilon^\alpha \Delta x$. Clearly, this *hyperbolic stiffness* becomes very restrictive when $\alpha > 0$, and for $\alpha = 1$ in the parabolic regime $\varepsilon \ll O(\Delta x)$, where for an explicit scheme a parabolic time step restriction of the type $\Delta t \sim \Delta x^2$ is expected.

In Sect. 5, following the approach recently introduced in the papers [1, 21], we analyze the construction of high-order IMEX-RK schemes for a system like (27) in the stiff regime which work uniformly, independently of the choices of ε and the scaling parameter α. By this, we mean that the schemes are designed in such a way as to be stable for all different ranges of the scaling parameters independently of the time step. At the same time, they should ensure high order in space and time and should be able to accurately describe the various asymptotic limits. Moreover, whenever possible, the above described properties must be achieved without the need of an iterative solver for non linear equations.

In particular we require that, in the parabolic regime, $\alpha = 1$, our approach gives a scheme which is not only consistent with (32) without resolving the small ε scale, but is also capable to avoid the *parabolic stiffness* [20, 23].

Then we introduce the new approach with the aim to avoid the stiffness induced by the characteristic speeds of system (27). First we present the simple first order scheme and then using the IMEX formalism, we construct high order methods.

3 IMEX methods

Implicit–explicit (IMEX) schemes have been introduced with the precise purpose of integrating evolutionary systems of differential equations that contain both stiff and non stiff terms.

The presence of stiff terms require the use of implicit methods in order to avoid restrictive stability time step constraints. On the other hand, in several cases, not all terms in a system are stiff, and therefore a fully implicit treatment of such systems may be unnecessary expensive.

IMEX methods were somehow known in the ODE literature. For example additive and partitioned Runge–Kutta are mentioned even in the classic books on systems of ODES [56, 97]. In the PDE literature IMEX Runge–Kutta methods were introduced in [6], while IMEX Linear Multistep (IMEX-LM) have been adopted in [7].

Nowadays there is a vast literature on IMEX schemes: searching for "IMEX schemes" just on google scholar finds almost 1400 entries as of December 2023, so it would be impossible to give an exhaustive account of it. We shall limit ourselves to cite the main papers directly related to the systems we plan to solve.

We start the section by describing the different form the stiffness may appear in some classes of evolutionary PDE's, and then we provide a brief summary of IMEX Runge–Kutta schemes, which is the class of IMEX schemes we shall consider throughout the paper.

In the rest of the section we can assume we are dealing with systems of ordinary differential equations. Even in the case of evolutionary PDE's, we can imagine we first discretize space with a suitable finite difference technique, after that the evolutionary system can be considered as a large sparse system of ODE's (method of lines).

3.1 Additive and partitioned systems

Additive systems. In this section we consider a particular class of initial value problems for ODE's called *additive* systems of the form

$$U'(t) = F(U) + \frac{1}{\varepsilon} G(U), \quad U(t_0) = U_0, \tag{33}$$

where $U(t) \in \mathbb{R}^m$ and we assume that the vector functions F and $G/\varepsilon : \mathbb{R}^m \to \mathbb{R}^m$ have different stiffness properties, i.e., the time scale induced by the two terms may be considerably different. To emphasize this, the second term in the right hand side has a factor $1/\varepsilon$, where ε is supposed to be small.

Partitioned systems In this class of problems, the stiffness is associated to some of the variables, rather than to the right hand side. We consider autonomous differential equations in the partitioned form,

$$\begin{aligned} y(t)' &= f(y(t), z(t)), \\ \varepsilon z(t)' &= g(y(t), z(t)). \end{aligned} \tag{34}$$

where $y(t)$ and $z(t)$ may be vectors of different dimensions and $y(t_0) = y^0$, $z(t_0) = z^0$ are the initial conditions. We say that the variable y is non stiff, while z is stiff. Observe that a partitioned system of the form (34) can be written as an additive system (33) Indeed, by defining

$$U = \begin{pmatrix} y \\ z \end{pmatrix}, \quad F(U) = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad G(U) = \begin{pmatrix} 0 \\ g \end{pmatrix},$$

system (34) assumes the form (33). On the other hand, given an additive system (33), we can always define a partition of the solution of the form $U = y + z$, and use the extra degree of

freedom to define

$$y' = F(U) = F(y+z) =: f(y,z)$$
$$z' = G(U) = G(y+z) =: g(y,z)$$

thus showing that additive systems can in principle be written as partitioned systems, although at the price of doubling the number of unknowns.

The formal equivalence is very useful, since it allows to analyze the systems and the properties of the corresponding numerical methods for their solution choosing the more convenient form.

3.2 IMEX-RK methods

Runge–Kutta (RK) methods and their additive and partitioned variations [41, 42, 54, 56, 57, 83, 97] found extensive application in numerical PDEs, [6, 29, 32, 66] and offered several advantages in terms efficiency, high-order accuracy and good stability properties. IMEX-RK methods are a particular class of additive and partitioned RK methods.

To introduce IMEX-RK methods, we consider the ODE initial value problem (33). An s-stage IMEX-RK method applied to system (33) consists of applying an implicit discretization to the term G, called *stiff*-term, and an explicit one to the term F, called *non-stiff* term, [15, 16, 78, 79].

IMEX-RK methods have been investigated in the context of PDEs for: hyperbolic systems with relaxations, in [78, 79], convection–diffusion equations [6] and convection–diffusion–reaction equations [29, 32].

An s stage IMEX-RK scheme applied to system (33) takes the form:

$$U^{(i)} = U^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} F(U^{(j)}) + \Delta t \sum_{j=1}^{s} a_{ij} G(U^{(j)}), \quad i = 1, ..., s \tag{35a}$$

$$U^{n+1} = U^n + \Delta t \sum_{i=1}^{s} \tilde{b}_i F(U^{(i)}) + \Delta t \sum_{i=1}^{s} b_i G(U^{(i)}), \tag{35b}$$

where the quantities $U^{(i)}$ for $i = 1, ..., s$, are called *internal stages* and approximate the exact solution $U(t)$ at time $t = t^n + c_i \Delta t$, whereas U^{n+1}, called *numerical solution*, approximate the exact solution $U(t)$ at time $t = t^n + \Delta t$. At variance with multistep methods, which store and adopt the numerical solution at several time steps, Runge–Kutta are *one step methods*, i.e. the solution at time $t^n + \Delta t$ is computed using only information at time t^n [56, 97]. An s-stage IMEX-RK method is defined by two $s \times s$ real matrices $\tilde{A} = (\tilde{a}_{ij})$, and $A = (a_{ij})$ where the matrix corresponding to the explicit method, $\tilde{A}$, is a lower triangular matrix with zero diagonal, i.e. $\tilde{a}_{ij} = 0$ for $j \geq i$, while A is the one corresponding to the implicit scheme. The matrices are constructed in such a way that the resulting method is explicit in F, and implicit in G. This property is guaranteed if A is a lower triangular matrix, i.e. if it corresponds to a diagonally implicit Runge–Kutta method (DIRK). The use of non-zero coefficients above the diagonal, in addition to making the implicit step more complex, imposes further restrictions on the structure of matrix $\tilde{A}$. For such a reason, in most, in not all, IMEX methods available in the literature, A is lower triangular.

The method is also characterized by the quadrature nodes $\tilde{\mathbf{c}} = (c_1, \cdots, c_s)^\top$, $\mathbf{c} = (c_1, \cdots, c_s)^\top$, given by the usual relation [56, 97]:

$$\tilde{c}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij}, \quad c_i = \sum_{j=1}^{i} a_{ij}, \tag{36}$$

and by the weights: $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_s)^\top, \mathbf{b} = (b_1, b_2, \cdots, b_s)^\top$ in $\mathbb{R}^s$. IMEX-RK methods can be represented by the a double *tableau* in the usual Butcher notation with

$$\begin{array}{c|c} \tilde{\mathbf{c}} & \tilde{A} \\ \hline & \tilde{\mathbf{b}}^\top \end{array} \qquad \begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^\top \end{array}. \tag{37}$$

From now to identify the IMEX-RK methods, we shall use the notation NAME(s_E, s_I, p) with the triplet (s_E, s_I, p), where s_E, s_I, and p represent, respectively, the number of stages of the implicit part, the number of the stages of the explicit one, and p the order of the IMEX-RK method. By number of stages we mean the number of function evaluations for the explicit scheme and the number of implicit solvers for the implicit scheme. We denote by $s \geq \max(s_E, s_I)$ the *apparent* number of stages, so both implicit and explicit schemes appear to have s stages, but in practice the effective number of stages can be lower for one of the two schemes (or for both of them).

Below we report the two double Butcher tables corresponding to first order IMEX schemes:

- SP(1,1,1). It is a type I scheme with $s = s_E = s_I = 1$ and $p = 1$:

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}. \tag{38}$$

- ARS(1,1,1). It is a type II scheme with $s = 2$, $s_E = s_I = 1$ and $s = 2$ with $p = 1$:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1 & 0 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline & 0 & 1 \end{array}. \tag{39}$$

In the double *tableau* of the IMEX-RK scheme, the explicit part is usually displayed before the implicit part, mainly because we follow the order in which the terms appear in hyperbolic systems with relaxation.

Without loss of generality, we can restrict to consider autonomous systems. Indeed, a non autonomous system with m equations is equivalent to an autonomous system with $m + 1$ unknowns, in which time is considered as an extra unknown satisfying the equation $t' = 1$. Indeed, consider the following non-autonomous system

$$U'(t) = F(t, U) + G(t, U), \quad U(t_0) = U_0 \tag{40}$$

It is equivalent to the following autonomous system

$$\tilde{U}' = \tilde{F}(\tilde{U}) + \tilde{G}(\tilde{U}) \tag{41}$$

where $\tilde{U}^\top = (U^\top, t)$, $\tilde{F}^\top = (F^\top, 1)$, $\tilde{G}^\top = (G^\top, 0)$, and $\tilde{U}^\top(t_0) = (U^\top, 0)$.

An IMEX-RK methods (35a)–(35b) applied to the system reads

$$U^{(i)} = U^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} F(t_n + \tilde{c}_j \Delta t, U^{(j)}) + \Delta t \sum_{j=1}^{i} a_{ij} G(t_n + c_j \Delta t, U^{(j)}), \quad i = 1, .., s$$

$$U^{n+1} = U^n + \Delta t \sum_{i=1}^{s} \tilde{b}_i F(t_n + \tilde{c}_i \Delta t, U^{(i)}) + \Delta t \sum_{i=1}^{s} b_i G(t_n + c_i \Delta t, U^{(i)}).$$

Under the assumptions

$$\sum_{i=1}^{s} \tilde{b}_i = 1, \quad \sum_{i=1}^{s} b_i = 1,$$

and

$$\sum_{j=1}^{i-1} \tilde{a}_{i,j} = \tilde{c}_i, \quad \sum_{j=1}^{i} a_{i,j} = c_i, \ \forall\, i$$

applying the IMEX-RK method to both problem (40) and (41), we get the same results.

It is useful to characterize the different IMEX-RK methods presented in the literature in two different types accordingly to the structure of the matrix of the DIRK method. Following [14] we have

Definition 3.1 An IMEX-RK method is said of type I (called also type A) if the matrix $A \in \mathbb{R}^{s\times s}$ is invertible, or equivalently $a_{ii} \neq 0, i = 1, \ldots, s$. An IMEX-RK method of type II (called also type CK) if the matrix A can be written as

$$A = \begin{pmatrix} 0 & 0 \\ a & \hat{A} \end{pmatrix}, \tag{42}$$

with $a = (a_{21}, ..., a_{s1})^\top \in \mathbb{R}^{s-1}$ and the sub-matrix $\hat{A} \in \mathbb{R}^{(s-1)\times(s-1)}$ with $\hat{a}_{ij} = a_{ij}$, $i, j = 2, ..., s$, is invertible, or equivalently $a_{ii} \neq 0, i = 2, \ldots, s$. In the special case $a = 0$, $b_1 = 0$, the scheme is said of type ARS (see [6]) and the DIRK method is reducible to a method using $s - 1$ stages. Note that for type II IMEX schemes one has $s_I \leq s - 1$.

We will make use of the following representation of the matrix $\tilde{A}$:

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ \tilde{a} & \hat{\tilde{A}} \end{pmatrix}, \tag{43}$$

where $\tilde{a} = (\tilde{a}_{21}, \ldots, \tilde{a}_{s1})^\top \in \mathbb{R}^{s-1}$ and $\hat{\tilde{A}} \in \mathbb{R}^{(s-1)\times(s-1)}$. This representation of matrix $\tilde{A}$ is useful for the analysis of IMEX-RK methods of type II and ARS.

The following definition will also be useful to characterize the properties of the methods in the sequel.

Definition 3.2 We call an IMEX-RK method *stiffly accurate* (SA) if the corresponding DIRK method is *stiffly accurate*, namely:

$$a_{si} = b_i, \ i = 1, ..., s. \tag{44}$$

If in addition the explicit method is *First Same As Last (FSAL)*, namely if

$$\tilde{a}_{si} = \tilde{b}_i, \ i = 1, ..., s - 1, \tag{45}$$

the IMEX-RK method is said to be *globally stiffly accurate* (GSA).

The above definitions follow naturally from the combination of the SA definition for s-stages implicit R–K methods, [97] and the FSAL property for s-stage explicit RK methods, (see [56] for details).

Note that FSAL methods have the advantage that they require only $s-1$ function evaluations per time step, because the last stage of step n coincides with the first stage of the step $n+1$, see [56, 97] for details.

For GSA IMEX-RK methods the numerical solution is the same as the last stage value, namely $U^{n+1} = U^{(s)}$.

Finally, is noteworthy the following result [97]

Proposition 3.3 *If an A-stable implicit RK method is SA then the method is also L-stable, i.e.,* $\lim_{z\to\infty} R(z) = 0$*, where* $R(z)$ *is the absolute stability function associated to the implicit method.*

L-stability of the implicit Runge–Kutta integrator is a highly desirable property, in fact, for the so called asymptotic preservation property for hyperbolic systems with relaxation (133), i.e. consistency of the scheme in the stiff limit $\varepsilon \to 0$. For an IMEX-RK scheme this property is guaranteed if the implicit step is solved by a L-stable scheme, [97].

Partitioned Runge–Kutta method. In this class of methods the two components in system (34) are discretized by different formulas [55, 56, 97].

Although we showed the formal equivalence between additive and partitioned systems, implementation of IMEX-RK for partitioned systems is more efficiently obtained as follows.

Consider an IMEX method defined by the double tableau (37), and apply it to system (34) where we treat the first variable $\mathbf{y}(t)$ with the first (explicit) method, and the second variable $\mathbf{z}(t)$ with the second (implicit) one:

$$\begin{aligned} k_i &= \mathbf{f}\Big(\mathbf{y}^n + \Delta t \sum_{j=1}^{i-1} \hat{a}_{ij} k_j, \mathbf{z}^n + \Delta t \sum_{j=1}^{i} a_{ij} \ell_j\Big), \\ \ell_i &= \mathbf{g}\Big(\mathbf{y}^n + \Delta t \sum_{j=1}^{i-1} \hat{a}_{ij} k_j, \mathbf{z}^n + \Delta t \sum_{j=1}^{i} a_{ij} \ell_j\Big), \\ \mathbf{y}^{n+1} &= \mathbf{y}^n + \Delta t \sum_{i=1}^{s} \hat{b}_i k_i, \quad \mathbf{z}^{n+1} = \mathbf{z}^n + \Delta t \sum_{i=1}^{s} b_i \ell_i, \end{aligned} \tag{46}$$

Sometimes it is more appropriate to use internal stages rather than function evaluations. In this case the scheme can be written as

$$\begin{aligned} Y^{(i)} &= \mathbf{y}^n + \Delta t \sum_{j=1}^{i-1} \hat{a}_{ij} \mathbf{f}(Y^{(j)}, Z^{(j)}), \qquad Z^{(i)} = \mathbf{z}^n + \Delta t \sum_{j=1}^{i} a_{ij} \mathbf{g}(Y^{(j)}, Z^{(j)}), \\ \mathbf{y}^{n+1} &= \mathbf{y}^n + \Delta t \sum_{i=1}^{s} \hat{b}_i \mathbf{f}(Y^{(j)}, Z^{(j)}), \qquad \mathbf{z}^{n+1} = \mathbf{z}^n + \Delta t \sum_{i=1}^{s} b_i \mathbf{g}(Y^{(j)}, Z^{(j)}). \end{aligned} \tag{47}$$

Generalized IMEX-RK methods In Sect. 3.1 we introduced systems of *additive type* (33) in which the right hand side is the sum of two terms, a non stiff one and a stiff one which can be treated by an IMEX RK method.

In other cases the stiffness can be associated to some variables. For example, in a system of the *partitioned* form (34), the stiffness is associated to variable $\mathbf{z}$, and the corresponding

equation will be treated implicitly, while the equation for $\mathbf{y}$ is treated explicitly and an IMEX Runge–Kutta method can be used, as illustrated above.

However, not all systems that involve stiff terms can be expressed in a partitioned or additive form. Consequently, using IMEX schemes in the standard way is not always straightforward. It is however possible to use IMEX methods also for such cases, adopting the approach presented in [17]. We consider a more general class of problems of the form

$$\begin{cases} \dfrac{du}{dt}(t) = \mathcal{H}(u(t), u(t)), \ \forall\ t \geq t_0, \\ \\ u(t_0) = u_0, \end{cases} \tag{48}$$

where the function $\mathcal{H}$: $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is sufficiently differentiable and the right hand side has a stiff dependence only on the last argument. Hereafter, we denote by *semi-implicit (SI) schemes* numerical methods which solve problems of the form (48) in which the variable u appearing as the first argument of $\mathcal{H}$ is treated explicitly, while u appearing as second argument is treated implicitly.

IMEX-RK schemes can be applied to system (48) by adopting two families of internal stages:

$$\begin{aligned} U_E^{(i)} &= u^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} \mathcal{H}(U_E^{(j)}, U_I^{(j)}), \\ U_I^{(i)} &= u^n + \Delta t \sum_{j=1}^{i} a_{ij} \mathcal{H}(U_E^{(j)}, U_I^{(j)}) \end{aligned} \tag{49}$$

and the numerical solution is given by

$$u^{n+1} = u^n + \Delta t \sum_{i=1}^{s} b_i \mathcal{H}(U_E^{(j)}, U_I^{(j)}) \tag{50}$$

Note that although two families of stage values are needed, the number of function evaluations (and the corresponding solution of the implicit step) is just s.

As an example of how this SI IMEX-RK methods work, we consider the following strongly degenerate parabolic convection–diffusion equation [68]:

$$u_t + \partial_x f(u) = \epsilon \partial_x (a(u) \partial_x u), \quad \epsilon a(u) \geq 0. \tag{51}$$

We take $\epsilon = 0.1$, $f(u) = u^2$ and

$$a(u) = \begin{cases} 0, & |u| \leq 0.25, \\ 1, & |u| > 0.25. \end{cases} \tag{52}$$

The equation is of hyperbolic nature when $u \in [-0.25, 0.25]$ and parabolic elsewhere. Using a uniform grid in space, system (51) can be discretized by finite difference (method of lines), on a grid with N_x points, resulting in a ODE system of the form

$$\frac{dU}{dt} = \mathcal{F}(U) + \mathcal{B}(U)U, \quad U(t_0) = U_0, \tag{53}$$

where $U = (U_1(t), ..., U_{Nx}(t))^\top$, $U_j(t) \approx U(x_j, t)$, $j = 1, ..., N_x$, $\mathcal{F}$ represents the conservative discretization of $\partial_x f(u)$, so the spectral radius of the corresponding Jacobian matrix scales as $1/\Delta x$, with Δx is the uniform grid spacing, while $\mathcal{B}(U) \in \mathbb{R}^{N_x \times N_x}$ is a tridiagonal

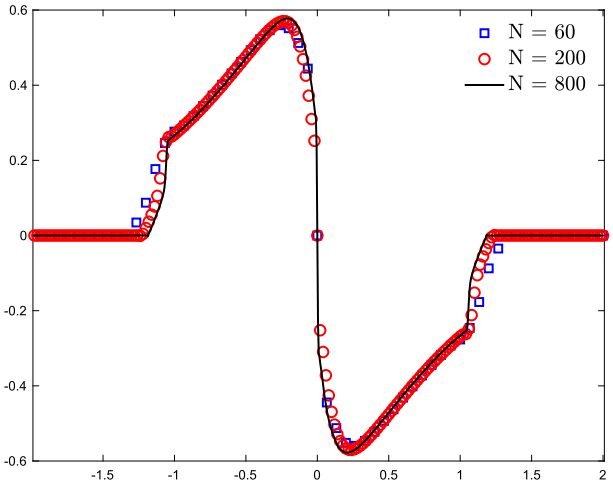

Fig. 1 Riemann problem (51), (52), and (54). T = 0.7

matrix arising form the discretization of the right hand side of Eq. (51), so the spectral radius of $\mathcal{B}(U)$ scales like $1/\Delta x^2$. Clearly If we want to apply an IMEX method to system (53), we have to distinguish in (53) between stiff and non-stiff terms and choose the time discretization by an IMEX-RK method accordingly. The SI IMEX-RK approach, is based on carefully distinguishing in (53) between stiff and non-stiff dependence on the solution vector **U**. Then in the product $\mathcal{B}(U)U$ the occurrence of the solution U within $\mathcal{B}(U)$ is considered non-stiff, while that of the factor U is considered stiff. Thus, the implicit treatment is applied only to the second factor U while the term $\mathcal{B}(U)$ is treated explicitly, i.e.,

$$\mathcal{H}(U_E, U_I) = \mathcal{F}(U_E) + \mathcal{B}(U_E)U_I.$$

This approach does not require the solution of nonlinear systems since the SI method solves a discretized convection-diffusion equation with a linear diffusion term in which the matrix $B(U)$ is given.
Then, now, we solve problem (51) considering the following initial data

$$u(x,0) = \begin{cases} 1, & -\frac{1}{\sqrt{2}} - 0.4 < x < -\frac{1}{\sqrt{2}} + 0.4 \\ -1, & \frac{1}{\sqrt{2}} - 0.4 < x < \frac{1}{\sqrt{2}} + 0.4 \\ 0, & \text{otherwise} \end{cases} \tag{54}$$

Here we used the third order I-IMEX(3,4,3) scheme for the integration in time, the coefficient are given in the appendix. We consider the classical hyperbolic CFL condition

$$\max_u |f'(u)| \frac{\Delta t}{\Delta x} = 1.0, \tag{55}$$

to set Δt. In Fig. 1, the numerical simulations for different numbers of grid points $N = 60, 200, 800$. The scheme provides the high resolution of discontinuities and the accurate transition between the hyperbolic and parabolic regions, comparable with the numerical resolutions reported in [68]. An application of this approach to hyperbolic systems with diffusive relaxation can be found in [18].

4 Hyperbolic relaxation

Iin this section we show how to apply IMEX methods to hyperbolic problem with relaxation of the form (7), requiring that the method is able to capture the relaxed limit. If a scheme has this property we say that it is Asymptotic Preserving (AP).

4.1 Asymptotic preserving scheme

Asymptotic-Preserving (AP) property is of paramount importance in the construction of numerical methods for multiscale problems and has been used for a long time in the physics and mathematics literature [43, 45, 65, 81, 82].

As an illustrative example to introduce the Asymptotic Preserving (AP) property, we consider the hyperbolic relaxation system (1).

Let Δt and Δx denote the discretization parameters in time and space. As the relaxation term becomes stiff, i.e. for very small values of ε, such term has to be treated implicitly in order to avoid stability restrictions of the form $\Delta t = O(\varepsilon)$. On the other hand, since the convection term is not stiff (the characteristic speeds are $\pm a = \mathcal{O}(1)$) one can use an explicit scheme for the hyperbolic term.

Even if problem (1) is linear in the advection part, since for small values of ε the limit problem (2) has a nonlinear flux that can give rise to shocks, high-order spatial discretizations are typically nonlinear [88]. A fully implicit treatment of the hyperbolic term with nonlinear reconstruction would introduce an unnecessary complication which leads to inefficiency

The second aspect concerns the AP property, corresponding to the request that as $\varepsilon \to 0$, keeping Δt and Δx fixed, the scheme mimics correctly the behavior of the solution u, i.e. the scheme for the system (1) becomes a consistent and a stable discretization of the limit equation (2).

The formal definition of AP schemes was introduced first by Jin, in the framework of kinetic models in a diffusive regime [60] (see also the review on AP schemes, [61]).

The notion of AP is summarized by the diagram in Fig. 2. Consider a continuous problem P^ε which depends on parameter ε that characterizes the small scale. For example the simple Jin-Xin system (1). As $\varepsilon \to 0$ the model is approximated by a limit continuous model P^0, which is independent of ε. In our example, the scalar conservation law (2). Now, denote with P^ε_δ the numerical discretization of P^ε, with δ a numerical parameter such as mesh size and/or

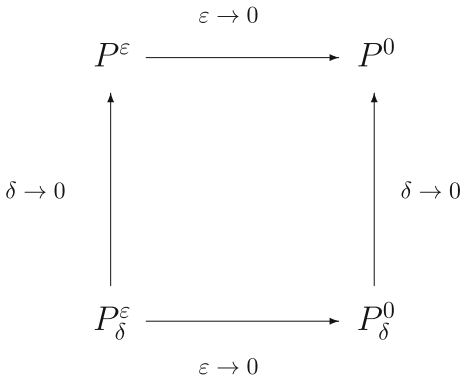

Fig. 2 A schematic illustration of the AP property

time step $\delta = (\Delta t, \Delta x)$. We denoted by P^0_δ the asymptotic limit, if exists, of P^ε_δ. Then we have the following definition:

Definition 4.1 The numerical discretization P^ε_δ is said to be Asymptotic-Preserving (AP) if in the limit $\varepsilon \to 0$ the limit discretization P^0_δ is a consistent and stable discretization of the limit continuous model P^0, and the stability constraints on δ are independent of ε.

The basic idea is to develop numerical schemes that preserve the asymptotic limits from P^ε to P^0 models, in the discrete setting. Note that in an AP framework we expct that the same numerical scheme is used for the discretization of P^ε as well as for P^0, and this means that such schemes mimic, at the discrete level, the asymptotic behaviour of problem guaranteeing an automatic transition from P^ε to P^0.

Finally, we note that definition (4.1) does not imply in general that the scheme preserves the order of accuracy in time in the stiff limit $\varepsilon \to 0$. In the latter case the scheme is said *asymptotically accurate* [79].

Definition 4.2 The numerical discretization P^ε_δ is said to be Asymptotically-Accurate (AA) if in the limit $\varepsilon \to 0$ it maintains its temporal order of accuracy for the limit discretization P^0_δ.

The design of an AP scheme needs special care for both time and spatial discretization. Often, however, temporal discretization is far more crucial, and for this reason most of the research activity has been focused on the properties of the time discretization.

A first order IMEX scheme and the AP property

Let us consider the simplest IMEX scheme (142) as applied to (1). It is a first order implicit–explicit Euler scheme, based on taking the fluxes explicitly and the stiff term implicitly

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} &= -v^n_x, \\ \frac{v^{n+1} - v^n}{\Delta t} &= -a^2 u^n_x - \frac{1}{\varepsilon}(v^{n+1} - f(u^{n+1})). \end{aligned} \tag{56}$$

We kept the space continuous, so we can focus on time discretization, and choose later the space discretization most suitable for each term. Solving for u^{n+1} and v^{n+1} one obtains

$$\begin{aligned} u^{n+1} &= u^n - \Delta t\, v^n_x \\ v^{n+1} &= \frac{\varepsilon(v^n - \Delta t\, a^2\, u^n_x) + \Delta t\, f(u^{n+1})}{\varepsilon + \Delta t} \end{aligned}$$

Now as $\varepsilon \to 0$, the expression for v^{n+1} becomes

$$v^{n+1} = f(u^{n+1}),$$

and substituting it into the first equation we get

$$u^{n+1} = u^n - \Delta t f(u^n)_x,$$

i.e. a consistent and stable explicit discretization of the limit equation (2), hence the AP property is satisfied.

A condition for the AP property of a scheme applied (1) is that the solution must be driven to the local equilibrium when $\varepsilon \to 0$, namely, for $\Delta t \gg \varepsilon$,

$$v^n - f(u^n) = \mathcal{O}(\varepsilon) \tag{57}$$

for any initial data v^0 and for some $n \geq n_0$. The value of n_0 depends essentially on the initial condition. If (57) is satisfied for $n = 0$, then we say that the initial condition is *well prepared*. 'In such a case $n_0 = 0$, and (57) is always satisfied. If, on the other hand, the initial condition does not satisfy (57), then there will be an *initial layer*, i.e. the solution will be far from the manifold $v = f(u)$ for a short time $O(\varepsilon)$ and condition (57) will be valid again after $n_0 = O(\varepsilon/\Delta t)$ time steps. Since the initial layer lasts only for a relatively short time, in the analysis of schemes it is common to assume that the initial condition is well prepared.

More formally, we introduce the following definition

Definition 4.3 The initial data (u^0, v^0) for system (1) are said Well-Prepared (WP) if

$$v^0 = f(u^0) + \mathcal{O}(\varepsilon). \tag{58}$$

Roughly speaking, well prepared initial data prevent the solution from forming an initial layer for $\varepsilon \ll 1$. We refer to [28, 59] for numerical and analytical results on the initial layer problem for the Broadwell model.

We now explore the $\mathcal{O}(\varepsilon)$ behavior of the first order scheme (56). Then, let us consider the expansion $v^{n+1} = f(u^{n+1}) + \varepsilon v_1^{n+1}$ which from (56) it gives

$$v_1^{n+1} = -\frac{f(u^{n+1}) - v^n}{\Delta t} - a^2 u_x^n - \frac{\varepsilon}{\Delta t} v_1^{n+1}. \tag{59}$$

If we assume the initial data to be well prepared, i.e. $v^n = f(u^n) + \varepsilon v_1^n$, from (59) we obtain

$$\begin{aligned}
v_1^{n+1} &= -\frac{f(u^{n+1}) - f(u^n)}{\Delta t} - a u_x^n - \varepsilon \frac{v_1^{n+1} - v_1^n}{\Delta t} \\
&= f'(u^n)(u^n)_t - a^2 u_x^n - \varepsilon \frac{v_1^{n+1} - v_1^n}{\Delta t} + O(\Delta t) \\
&= -f'(u^n)(v^n)_x - a^2 u_x^n - \varepsilon \frac{v_1^{n+1} - v_1^n}{\Delta t} + O(\Delta t) \\
&= -f'(u^n)(f(u^n) + \varepsilon v_1^n)_x - a^2 u_x^n - \varepsilon \frac{v_1^{n+1} - v_1^n}{\Delta t} + O(\Delta t) \\
&= f'(u^n) f(u^n)_x - a^2 u_x^n - \varepsilon \left(\frac{v_1^{n+1} - v_1^n}{\Delta t} - f'(u^n)(v_1^n)_x \right) + O(\Delta t) \\
&= \left(f'(u_n)^2 - a^2 \right) u_x^n + \mathcal{O}(\varepsilon, \Delta t).
\end{aligned}$$

Thus from $v^{n+1} = f(u^{n+1} + \varepsilon v_1^{n+1})$, we can write

$$v^{n+1} = f(u^{n+1}) - \varepsilon \left(a^2 - f'(u_n)^2 \right) u_x^n + \mathcal{O}(\varepsilon^2, \varepsilon \Delta t), \tag{60}$$

where we assume v_1 to have bounded first derivatives in space and time. We finally obtain the following scheme for Eq. (4)

$$\frac{u^{n+1} - u^n}{\Delta t} + f(u^n)_x = \varepsilon \left(a^2 - f'(u^{n-1})^2 \right) u_x^{n-1} + O(\varepsilon^2, \varepsilon \Delta t). \tag{61}$$

Therefore, we have a two level scheme (in time) which is consistent to equation (4) up to first order in ε. Although (61) is a consistent time discretization of (4), the presence of the time level $n-1$ may degrade the accuracy of the resulting scheme. In general the AP property guarantees only the consistency of the scheme, which is sufficient for first order schemes. However, for higher order schemes, in the limit as $\varepsilon \to 0$, the order of accuracy may drop to lower orders. In the literature of hyperbolic system with stiff relaxation this order reduction phenomenon has been extensively studied, see for example [14–16].

4.2 IMEX-RK methods for hyperbolic systems with stiff relaxation

We now turn our attention to analyze IMEX-RK schemes applied to general hyperbolic systems with relaxation. Consider system (6). We shall treat the hyperbolic term explicitly, while the stiff relaxation term will be treated implicitly.

An IMEX-RK scheme (35a)–(35b) applied to the system (7) reads:

$$\begin{aligned} U^{(i)} &= U^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} \partial_x F(U^{(j)}) + \frac{\Delta t}{\varepsilon} \sum_{j=1}^{i} a_{ij} R(U^{(j)}), \quad i = 1, ..., s \\ U^{n+1} &= U^n - \Delta t \sum_{j=1}^{s} \tilde{b}_i \partial_x F(U^{(i)}) + \frac{\Delta t}{\varepsilon} \sum_{i=1}^{s} b_i R(U^{(j)}). \end{aligned} \tag{62}$$

Here we focus on time discretization. Space discretization of the flux derivative may be obtained by conservative finite difference schemes [89], while the source term is simply computed on the grid nodes. We shall discuss issues about space discretization when needed.

In this section we give sufficient conditions for asymptotic preservation and asymptotic accuracy properties of IMEX-RK schemes which are strongly related to L-stability of the implicit part of the scheme.

Asymptotic preservation properties of IMEX-RK methods have been analyzed and studied in several papers [20, 46, 79], while IMEX-LM methods have been treated in [48, 58]. Here we quickly recall some results and derive conditions for asymptotic preservation in the case of hyperbolic systems with stiff relaxation and examples.

First we recall the main result in [79]:

Theorem 4.4 *If the IMEX method applied to (7) is of type I then in the limit $\varepsilon \to 0$ it becomes the explicit RK scheme characterized by the pair $(\tilde{A}, \tilde{b})$ applied to the limit equilibrium system (11).*

Proof It is easy to verify that in the limit $\varepsilon \to 0$ the IMEX-RK scheme (62) collapses to the system

$$\sum_{j=1}^{i} a_{ij} R(U^{(i)}) = 0 \quad \text{for } i = 1, \ldots, s.$$

Since A is invertible, this relation corresponds to the set of algebraic equations $R(U^{(i)}) = 0$, for $i = 1, ..., s$, which, from property (9) of the relaxation operator, implies $U^{(i)} = \mathcal{E}(u^{(i)})$ with $u^{(i)} = QU^{(i)}$ for $i = 1, ..., s$, as unique solution.

By (62) we obtain

$$u^{(i)} = u^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} f(u^{(j)})_x,$$

and

$$u^{n+1} = u^n - \Delta t \sum_{j=1}^{s} \tilde{b}_i f(u^{(i)})_x,$$

where $f(u) = QF(\mathcal{E}(u))$, i.e., the desired scheme for (11). □

For IMEX-RK schemes of type II the proof requires one more step. It is useful to introduce the notion of *well prepared* initial data with the limit problem.

Definition 4.5 The initial data for equation (7) is said *well prepared* if

$$U_0(x) = \mathcal{E}(u_0) + O(\varepsilon).$$

We can now show the following

Theorem 4.6 *If the IMEX method applied to (7) is of type II and GSA then for well-prepared initial data, in the limit $\varepsilon \to 0$, it becomes the explicit RK scheme characterized by the pair $(\tilde{A}, \tilde{b})$ applied to the limit equilibrium system (11).*

Proof To prove this result it is enough to observe that as $\varepsilon \to 0$ from scheme (62) we get

$$a_{i1} R(U^{(1)}) + \sum_{j=2}^{i} \hat{a}_{ij} R(U^{(j)}) = 0, \quad i = 2, \ldots, s \tag{63}$$

where we used notation (42) for the matrix A.

If the initial data are well prepared we have at the first step $R(U^{(1)}) = 0$ since $U^{(1)} = U_0$ and for well prepared initial data, as $\varepsilon \to 0$, it is $R(U_0) = 0$. Now from the invertibility of $\hat{A}$ and (63) we obtain $U^{(i)} = \mathcal{E}(u^{(i)})$ where $u^{(i)} = QU^{(i)}$ for $i = 2, \cdots, s$. Then, following the same idea in Theorem 4.4, the IMEX-RK scheme of type II in the limit coincides with the explicit Runge–Kutta method applied to (11). □

From Theorems 4.4 and 4.6 we get $R(U^{(i)}) = 0$ for all i for both types of schemes. Note that if the IMEX-RK scheme is GSA then the solution U^{n+1} lies on the equilibrium manifold, namely $R(U^{n+1}) = 0$. For GSA schemes the numerical solution is the same as the last stage value, namely, $U^{n+1} = U^{(s)}$. Then we get $R(U^{n+1}) = R(U^{(s)}) = 0$ and $R(\mathcal{E}(u^{n+1})) = 0$, with u^{n+1} being obtained by the explicit scheme applied to the the equilibrium system (11), as proved in Theorem (4.4).

Remark 4.7 (i) There is a close analogy between hyperbolic systems with stiff relaxation and differential algebraic equations (DAE) [5, 14]. The limit system as $\varepsilon \to 0$ is the analog of an index 1 DAE, in which the algebraic equation is explicitly solved in terms of the differential variable.

(ii) This result does not guarantee, in general, the accuracy of the solution for the $N_2 = N - N_1$ non conserved quantities (sometimes refereed as the *algebraic variable*, in analogy with differential algebraic systems). Indeed, since the very last step in the scheme is not a projection toward the local equilibrium, a final layer occurs. The use of globally stiffly accurate schemes may serve as a remedy to this problem, since in this case the field variable U is projected onto the manifold $R(U) = 0$ also at a numerical level. In order to obtain an asymptotically accurate scheme, also for the algebraic variable, even for non GSA methods, more order conditions have to be imposed on the implicit scheme, which matches the numerical solution and the exact solution at various order in an expansion in the stiffness parameter ε. A detailed analysis of this problem for IMEX-RK schemes is reported in [14].

(iii) The theorems guarantee that in the stiff limit the numerical scheme becomes the explicit RK scheme applied to the equilibrium system, and therefore the order of accuracy of the limiting scheme is greater or equal to the order of accuracy of the original IMEX-RK scheme (for the differential variables.)

Example 4.8 To illustrate the limit behavior of IMEX-RK schemes we consider the rescaled problem (1), with $f(u) = \beta u$.

$$\begin{aligned} &\partial_t u + \partial_x v = 0, \\ &\partial_t v + \alpha^2 \partial_x u = -\frac{1}{\varepsilon}(v - \beta u). \end{aligned} \tag{64}$$

As $\varepsilon \to 0$ we get, formally, the local equilibrium $v = \beta u$ while u satisfies the linear advection equation

$$\partial_t u + \beta \partial_x u = 0. \tag{65}$$

In vector form a general IMEX-RK scheme of type I for (64) can be written as

$$\mathbf{u} = u^n \mathbf{e} - \Delta t\, \tilde{A}\, \partial_x \mathbf{v}$$

$$\mathbf{v} = v^n \mathbf{e} - \Delta t\, A_*\, \alpha^2\, \partial_x \mathbf{u} - \Delta t\, A\, \frac{1}{\varepsilon}(\mathbf{v} - \beta \mathbf{u}) \tag{66}$$

$$u^{n+1} = u^n - \Delta t\, \tilde{b}^\top\, \partial_x \mathbf{v},$$

$$v^{n+1} = v^n - \Delta t\, b_*^\top\, \alpha^2\, \partial_x \mathbf{u} - \Delta t\, b^\top\, \frac{1}{\varepsilon}(\mathbf{v} - \beta \mathbf{u}), \tag{67}$$

where $\mathbf{e} = (1, 1, .., 1)^T \in \mathbb{R}^s$, $A_* = \tilde{A}$, $b_* = \tilde{b}$ for the additive approach and $A_* = A$, $b_* = b$ for the partitioned case.

In the case $\varepsilon = 0$, we obtain $\mathbf{v} = \beta \mathbf{u}$. Then this yields for the $\mathbf{u}$ component:

$$\begin{aligned} \mathbf{u} &= u^n \mathbf{e} - \Delta t\, \tilde{A}\, \beta\, \partial_x \mathbf{u} \\ u^{n+1} &= u^n - \Delta t\, \tilde{b}^\top \beta\, \partial_x \mathbf{u}, \end{aligned}$$

and it is exactly the tableau of the explicit part of the IMEX-RK method of type I applied to (65). In order to obtain an accurate scheme even for the algebraic variable v, using the invertibility of the matrix A we get, from the second equation of (67),

$$-\Delta t(\mathbf{v} - \beta \mathbf{u}) = \varepsilon A^{-1} \left(\mathbf{v} - v^n \mathbf{e}\right) + \varepsilon \alpha^2 \Delta t A^{-1} A_* \partial_x \mathbf{u},$$

and substituting in the expression for the numerical solution v^{n+1} we obtain

$$v^{n+1} = (1 - b^T A^{-1} \mathbf{e}) v^n - \Delta t \left(b_*^T - b^T A^{-1} A^*\right) \alpha^2 \partial_x \mathbf{u} + b^T A^{-1} \mathbf{v}. \tag{68}$$

Now if the IMEX-RK scheme of type I in additive or partitioned form is GSA, then we get: $(1 - b^T A^{-1} \mathbf{e}) = 0$, $\left(b_*^T - b^T A^{-1} A_*\right) = 0$. Thus we have

$$v^{n+1} = b^T A^{-1} \mathbf{v} = v^{(s)}$$

and from $v^{(s)} = \beta u^{(s)}$ and FSAL property (Def. (3.2)) for which $u^{n+1} = u^{(s)}$, it follows: $v^{n+1} = \beta u^{n+1}$.

Finally, the limit scheme reads

$$\begin{aligned} \mathbf{u} &= u^n \mathbf{e} - \Delta t\, \tilde{A}\, \beta\, \partial_x \mathbf{u} \\ u^{n+1} &= u^n - \Delta t\, \tilde{b}^T \beta\, \partial_x \mathbf{u}, \\ \mathbf{v} &= \beta \mathbf{u}, \quad v^{n+1} = \beta u^{n+1}, \end{aligned}$$

Table 1 Additional order conditions up to order three for $O(\varepsilon)$ accuracy

Combined order	Additional $O(\varepsilon)$ conditions
First order	$w^T \mathbf{e} = 1$
Second order	$\tilde{b}^T d = 1/2$, $w^T \tilde{c} = 1/2$
Third order	$w^T \tilde{c}^2 = 1/3$, $\tilde{b}^T \tilde{c} d = 1/3$, $\tilde{b}^T d \tilde{c} = 1/3$
	$w^T \tilde{A} \tilde{c} = 1/6$, $w^T B \tilde{c} = 1/6$

so we obtain a uniformly accurate scheme that in the limit is AP and asymptotic accurate. A similar analysis can be carried on using type II IMEX methods, see [19].

Remark 4.9 A natural question that arises from the above analysis concerns the behavior of the schemes for small but non zero values of ε. In this case, which physically corresponds to the compressible Navier–Stokes limit, degradation of accuracy is expected unless additional conditions are satisfied by the IMEX-RK method (see [14–16, 79]). In [19] the authors analyzed and derived several IMEX-RK schemes under some additional order conditions to avoid accuracy degradation in the Navier–Stokes limit. Here we report the following result for the type I IMEX-RK methods:

Theorem 4.10 *An IMEX-RK method applied to system* (1)*, in additive or partitioned form, for small values of* ε *and well-prepared initial data, i.e.,* $v^n = f(u^n) + \mathcal{O}(\varepsilon)$*, yields an explicit additive Runge–Kutta method for the* $O(\varepsilon)$ *limit* (2) *characterized by the pairs* $(\tilde{A}, \tilde{b})$ *and* (B, w) *satisfying several additional order conditions* (1) *for the type I.*

Here B and w are given by

$$B = \tilde{A} A^{-1} \tilde{A}, \quad w^T = \tilde{b}^T A^{-1} \tilde{A}, \tag{69}$$

In Table 1 we report the conditions required to reach third order accuracy where $d = B\mathbf{e}$ and we use the notation d^2 to denote the vector $(d_1^2, \ldots, d_s^2)^\top$. A similar analysis can be carried out using type II IMEX methods, in which case it is possible to derive analogue additional order conditions (1) with different definition of B and w (see [19] for details).

5 Parabolic relaxation

In this section we study system (18) when the classical IMEX-RK approaches presented in Sect. 3.2, i.e., *partitioned* and *additive* approaches, are applied, and focus on schemes which, in the limit of infinite stiffness, i.e., $\varepsilon \to 0$, become consistent explicit schemes for the limit diffusion equation ($\tilde{f}(u) = 0$) or convection-diffusion equation ($\tilde{f}(u) \neq 0$).

5.1 Partitioned IMEX-RK approach

We consider system (18) and observe that the stiffness is naturally associated to the variable v rather than to some operator, therefore it has the structure of a singularly perturbed system, in which the first equation is can be treated explicitly and the second one implicitly, i.e.,

$$\begin{aligned} u_t &= -v_x && \text{(Explicit)} \\ \varepsilon^2 v_t &= -(p(u)_x + v - q(u)), && \text{(Implicit)} \end{aligned} \tag{70}$$

and it is solved by a partitioned Runge–Kutta method (47). This approach has been used in [67, 72].

By applying ARS(1,1,1) (142) to system (70) we obtain

$$\begin{aligned} u^{n+1} &= u^n - \Delta t v_x^n, \\ \varepsilon^2 v^{n+1} &= \varepsilon^2 v^n + \Delta t (q(u^{n+1}) - p(u^{n+1})_x - v^{n+1}). \end{aligned}$$

Solving the second equation for v^{n+1} gives

$$v^{n+1} = \frac{\varepsilon^2 v^n + \Delta t (q(u^{n+1}) - p(u^{n+1})_x)}{\varepsilon^2 + \Delta t}$$

As $\varepsilon \to 0$ one has

$$v^{n+1} = q(u^{n+1}) - p(u^{n+1})_x$$

therefore, for $n \geq 1$ the same relation applies to n, and we can write

$$u^{n+1} = u^n - \Delta t q(u^n)_x - p(u^n)_{xx},$$

which is Explicit Euler applied to the limit system given by the first equation of system (20) with $q(u) = \tilde{f}(u)$ and $p(u) = a^2 u$.

System (70) has the the structure of a partitioned system of the form

$$U_t = F(U, V), \quad \varepsilon^2 V_t = G(U) - V. \tag{71}$$

and can be thought of as a system of ordinary differential equations, where the unknown variables U and V correspond to the unknown functions discretized on a grid, and the functions F and G are obtained from the space discretization of the differential operators that define the PDE system (method of lines).

In the limit $\varepsilon \to 0$ the system formally relaxes to the limit system

$$U_t = F(U, G(U)), \quad V = G(U). \tag{72}$$

For simplicity of notation we assume that in the original system of PDE's the functions $u(x, t)$ and $v(x, t)$ are scalar functions of space and time. The formal extension to systems is straightforward.

Here we show that IMEX schemes of type I are AP for this class of systems. A similar analysis is also possible for IMEX-RK schemes of type II, see [20]. By applying an IMEX R-K scheme of type I to system (71) we obtain

$$\begin{aligned} \mathbf{U} &= U^n \mathbf{e} + \Delta t \tilde{A} F(\mathbf{V}), \\ \varepsilon^2 \mathbf{V} &= \varepsilon^2 V^n \mathbf{e} + \Delta t A (G(\mathbf{U}) - \mathbf{V}) \end{aligned} \tag{73}$$

for the internal stages and

$$\begin{aligned} U^{n+1} &= U^n + \Delta t \, \tilde{b}^\top F(\mathbf{V}), \\ \varepsilon^2 V^{n+1} &= \varepsilon^2 V^n + \Delta t \, b^\top (G(\mathbf{V}) - \mathbf{V}) \end{aligned} \tag{74}$$

for the numerical solution, where $\mathbf{e} \in \mathbb{R}^s$ is a vector whose components are all 1, $\mathbf{U} \in \mathbb{R}^s$ and $\mathbf{V} \in \mathbb{R}^s$ denote the stage values, and $F(\mathbf{U}, \mathbf{V}) \in \mathbb{R}^s$ denote a vector whose i-th component is $F(U^{(i)}, V^{(i)})$.

Since A is invertible, from the second equation of (73) we obtain

$$\Delta t (G(\mathbf{U}) - \mathbf{V}) = \varepsilon^2 A^{-1} (\mathbf{V} - V^n \mathbf{e}) \tag{75}$$

and inserting it in the numerical solution for V we get

$$V^{n+1} = (1 - b^T A^{-1}\mathbf{e})V^n + b^T A^{-1}\mathbf{V}, \tag{76}$$

As $\varepsilon \to 0$ we get from (75)

$$\mathbf{V} = G(\mathbf{U}), \tag{77}$$

and this yields for the U-component

$$U^{n+1} = U^n + \Delta t\, \tilde{b}^T \hat{F}(\mathbf{U}) \tag{78}$$

with $\mathbf{U} = U^n\mathbf{e} + \Delta t \tilde{A} F(\mathbf{U}, G(\mathbf{U}))$ for the internal stages. This represents the explicit scheme of the starting IMEX-RK one of type I applied to the limit equation $U_t = \hat{F}(U)$ in the case $\varepsilon = 0$ and furthermore, it maintains the same order of the starting scheme, i.e. the IMEX-RK scheme is AA. As a particular case, if $p(u) = u$ and $q(u) = 0$, from (73) and (78), this is the explicit scheme $(\tilde{A}, \tilde{b})$ applied to the limit diffusion equation under the usual parabolic stability restriction $\Delta t = \mathcal{O}(\Delta x^2)$.

Note that from system (71) in the case $\varepsilon = 0$ we get $V = G(U)$ and an interesting property for the numerical solutions is that $V^{n+1} = G(U^{n+1})$. Usually in general the numerical solution (U^{n+1}, V^{n+1}) does not lie on the manifold $V - G(U) = 0$, since $V^{n+1} - G(U^{n+1})$ is not necessarily zero. Indeed, if the scheme is only SA, then we have that $V^{n+1} = V^{(s)}$, and in the relaxed limit, $V^{(s)} = G(U^{(s)})$ however it is not guaranteed that $V^{n+1} = G(U^{n+1})$. If, on the other hand, the scheme is GSA, then $U^{n+1} = U^{(s)}$, and therefore condition $V^{n+1} = G(U^{n+1})$ is guaranteed.

Now, since U^{n+1} can be interpreted as the numerical solution of the relaxed system, it has the same order of accuracy p of the explicit scheme applied to (72). Therefore the quantity $V^{n+1} = G(U^{n+1})$ approximate the solution $V(t^{n+1})$ to order the same order p, and we can conclude that not only the variable U, but also V can be automatically computed with order or accuracy p. We conclude that a GSA IMEX scheme of order p is also Asymptotically Accurate (AA).

A similar analysis can be performed for the type II IMEX applied to the system (71), with slight modifications (for details see [20]).

A common feature of this approach is that in the limit case, schemes of type I and II relax to an explicit scheme under the usual parabolic restriction. In order to avoid such a restriction in the relaxed limit, a reformulation of the problem is necessary, as illustrated in Sect. 5.3 (see [20] for details). Therefore the idea in the paper [20] was to reformulate the problem (70) in order to avoid the parabolic restriction.

5.2 Additive IMEX-RK approach

In system (70) we require an implicit treatment of some hyperbolic component (see the second equation). Even if such implicit term can be explicitly computed, in most case this procedure requires a reformulation of the discretization of the whole system.

For practical applications it would be desirable to apply IMEX-RK schemes in an additive approach in which the whole term containing the flux is treated explicitly, while reserving the implicit treatment only to the source. At a first sight this seems a formidable task, because of the divergence of the characteristic speeds in the system (70).

Here we show that one can in fact have IMEX-RK schemes which are fully explicit in the hyperbolic term, and still AP and AA in the limit case $\varepsilon \to 0$.

We show how to do this in the Maxwell–Cattaneo model for diffusion, i.e. (70), with $p(u) = u$ and $q(u) = 0$. We treat the hyperbolic part explicitly and the source term implicitly by an additive approach, i.e.,

$$\begin{aligned} u_t &= -v_x \\ v_t &= \underbrace{-u_x/\varepsilon^2}_{\text{explicit}} \quad \underbrace{-\, v/\varepsilon^2}_{\text{implicit}}. \end{aligned} \tag{79}$$

Such approach should be easier to apply, since the fluxes retain their original interpretation [16, 79].

We apply IMEX RK schemes to system (79) and study their behavior as $\varepsilon \to 0$. Hereafter we shall restrict our analysis to the type I IMEX RK schemes with matrix A invertible, which are easier to analyze than type II schemes.

In vector form we have, for the numerical solution,

$$\begin{aligned} u_{n+1} &= u_n - \Delta t\, \tilde{b}^\top \mathbf{V}_x \\ v_{n+1} &= \hat{v}_n - \frac{\Delta t}{\varepsilon^2}\, \tilde{b}^\top \mathbf{U}_x - \frac{\Delta t}{\varepsilon^2}\, b^\top \mathbf{V}, \end{aligned} \tag{80}$$

while the internal stages are given by

$$\begin{aligned} \mathbf{U} &= \mathbf{e} u_n - \Delta t \tilde{A} \mathbf{V}_x \\ \mathbf{V} &= \mathbf{e} \hat{v}_n - \frac{\Delta t}{\varepsilon^2} \tilde{A} \mathbf{U}_x - \frac{\Delta t}{\varepsilon^2} A \mathbf{V}, \end{aligned} \tag{81}$$

From the second equation one has

$$\Delta t \mathbf{V} = -\varepsilon^2 A^{-1}(\mathbf{V} - v_n \mathbf{e}) - \Delta t A^{-1} \tilde{A} \mathbf{U}_x,$$

Now substituting $\mathbf{V}$ in the numerical solution we have

$$v_{n+1} = (1 - b^\top A^{-1}\mathbf{e}) v_n + \frac{\Delta t}{\varepsilon^2}(b^\top A^{-1} \tilde{A} - \tilde{b}^\top)\mathbf{U}_x + b^\top A^{-1} \mathbf{V}, \tag{82}$$

Consistency in (82) as $\varepsilon \to 0$ implies

$$b^\top A^{-1} \tilde{A} - \tilde{b}^\top = 0. \tag{83}$$

Furthermore, if the scheme is SA, (see Def. 3.2), one has $b^\top A^{-1} = e_s^\top$ therefore $1 - b^\top A^{-1}\mathbf{e} = 0$, and, as $\varepsilon \to 0$, one has

$$v_{n+1} = \mathbf{e}_s^\top \mathbf{V}, \tag{84}$$

where the internal stages are given by

$$\mathbf{V} = -A^{-1} \tilde{A} \mathbf{U}_x. \tag{85}$$

Note that a sufficient condition to guarantee that (83) is satisfied is that the IMEX RK of type I is GSA (Def. 3.2). This yields for the U-component

$$\begin{aligned} \mathbf{U} &= u^n \mathbf{e} + \Delta t\, \bar{A} \mathbf{U}_{xx} \\ u_{n+1} &= u_n + \Delta t\, \bar{b}^T \mathbf{U}_{xx}, \end{aligned} \tag{86}$$

where $\bar{b}^T = \tilde{b}^T A^{-1} \tilde{A}$ and $\bar{A} = \tilde{A} A^{-1} \tilde{A}$. This is the limit scheme $(\bar{A}, \bar{b})$ applied to the limit diffusion equation under the usual parabolic stability restriction $\Delta t = \mathcal{O}(\Delta x^2)$. Note that in

order to guarantee that the IMEX-RK scheme is AP, and also AA, we need to introduce the following extra conditions to guarantee that the scheme is second order in the u component:

$$\bar{b}^T \mathbf{e} = 1, \qquad \bar{b}^T \bar{A} \mathbf{e} = \frac{1}{2}. \tag{87}$$

The derivation of such conditions, as well as the analysis for the v component, is reported in the Appendix of [23]. In the same paper, a general theory is developed to derive new additional order conditions on the IMEX-RK schemes of type I up to a fixed order p even for the v-component.

In the general case $q(u) \neq 0$ we get

$$\mathbf{V} = -A^{-1} \tilde{A} \mathbf{U}_x + q(\mathbf{U}),$$

After some algebra, this yields for the u-component

$$\frac{\mathbf{U} - u^n \mathbf{e}}{\Delta t} + \tilde{A} q(\mathbf{U})_x = \Delta t \bar{A} \mathbf{U}_{xx}, \tag{88}$$

$$\frac{u_{n+1} - u_n}{\Delta t} + \tilde{b}^T q(\mathbf{U})_x = \bar{b}^T \mathbf{U}_{xx}. \tag{89}$$

Scheme (88) is an explicit scheme characterized by two pairs of explicit tableau: $(\tilde{A}, \tilde{b})$, $(\bar{A}, \bar{b})$. Of course this means that here we have to use the usual parabolic stability restriction $\Delta t = \mathcal{O}(\Delta x^2)$ and if we require that the limit (88) scheme is at least second order we require classical and additional order conditions (87).

A detailed analysis of this type of schemes is reported in [23], where a lower bound on the number of stages for a second order GSA scheme has been derived.

The results are summarized by the following two theorems.

Theorem 5.1 *There are no second order globally stiffly accurate IMEX R-K schemes of type A with three stages.*

Theorem 5.2 *There are no second order GSA IMEX-RK schemes of type I with four stages where the implicit part is singly diagonally implicit (SDIRK).*

The proof of these theorems are reported in [23].

A consequence of the above theorems is that second order GSA schemes of type I require at least 4 stages. At this point, with 4 stages there have several free parameters, which can be used to improve the stability properties of the schemes. In particular, we can remove the parabolic stiffness of the schemes that naturally appears when in the relaxed limit the schemes reduces to an explicit method for the parabolic equation. How this is obtained is discussed in the next section.

A similar analysis (although more technical) can be performed in the case of type II IMEX schemes, see [20] for details.

5.3 Removing parabolic stiffness

The procedure outlined in the previous section provides schemes that converge to explicit ones for the parabolic equation and convection-diffusion equations in the limit case $\varepsilon \to 0$. Such schemes suffer from the standard CFL restriction $\Delta t = \mathcal{O}(\Delta x^2)$, In order to remove this restriction, we adopt a penalization technique similar to the one described in [20, 23].

The idea is to reformulate problem (70) by adding and subtracting the same term $p(u)_x$ on the right-hand side of the first equation, obtaining

$$\begin{aligned} u_t &= \underbrace{-(v + p(u)_x)_x}_{\text{non stiff}} + \underbrace{p(u)_{xx}}_{\text{stiff}}, \\ v_t &= -\frac{1}{\varepsilon^2}\left(p(u)_x + v - q(u)\right). \end{aligned} \tag{90}$$

The idea is that, since the quantity $v + p(u)_x$ is close to $q(u)$ as $\varepsilon \to 0$, the first term on the right-hand side can be treated explicitly in the first equation, while the term $p(u)_{xx}$ will be treated implicitly. This can be done naturally by using an IMEX-RK approach.

This reformulation allows a hyperbolic CFL condition $\Delta t = \mathcal{O}(\Delta x)$ independent of ε in the stiff limit.

Note that if we want the scheme to be accurate also in the cases in which ε is not too small, then no term should be added when not needed, for example when ε is not small, since additional terms may degrade the accuracy. This can be achieved by replacing $p(u)$ by $\mu(\varepsilon)p(u)$ in the first equation of (90). The quantity $\mu(\varepsilon)$ should decrease as ε increases.

Here $\mu = \mu(\varepsilon) \in [0, 1]$ and will be chosen such that $\mu(0) = 1$. A possible expression for μ is, for example,

$$\mu(\varepsilon) = \exp(-\alpha\varepsilon^2/\Delta x), \tag{91}$$

where α is a tuned dimensional parameter that makes the argument of the exponential non-dimensional. If ε is nondimensioanl α will be a lengthscale typical of the problem.

This choice guarantees $\mu(0) = 1$, and $\mu(1) = \exp(-\alpha/\Delta x) \approx 0$ for small Δx.

In the papers [20, 23], the authors introduced two different approaches to treat system (90) according to whether the hyperbolic term $p(u)_x$ in the second equation is treated implicitly or explicitly.

The first approach, as analyzed in [20], takes into account the following time discretizations for the terms in (90),

$$\begin{aligned} u_t &= \underbrace{-(v + \mu(\varepsilon)p(u)_x)_x}_{\text{explicit}} + \underbrace{\mu(\varepsilon)p(u)_{xx}}_{\text{implicit}} \\ \varepsilon^2 v_t &= \underbrace{-(p(u)_x + v - q(u))}_{\text{implicit}} \end{aligned}, \tag{92}$$

and the corresponding IMEX-RK schemes are called **IMEX-I** RK to remind that the term containing $p(u)_x$ in the second equation is implicit, in the sense that it appears at the new time level. Observe that the term $p(u)_x + v - q(u)$ appearing in the second equation is formally treated implicitly, however, since the term $p(u)_x$ is computed at the new time from the first equation, it is in fact explicitly computed.

The previous approach requires an implicit treatment of some hyperbolic component even if such implicit term can be explicitly computed in most cases. This requires a reformulation of the discretization of the whole system. It would be desirable to apply IMEX R-K schemes in which the whole hyperbolic part is treated explicitly because this allows the use of well tested space discretization with no modification of the original system. This approach is

analyzed in [23] and it is given by

$$\begin{aligned} u_t &= \underbrace{-(v+\mu(\varepsilon)p(u)_x)_x}_{\text{explicit}} + \underbrace{\mu(\varepsilon)p(u)_{xx}}_{\text{implicit}} \\ \varepsilon^2 v_t &= \underbrace{-p(u)_x}_{\text{explicit}} - \underbrace{(v-q(u))}_{\text{implicit}}, \end{aligned} \tag{93}$$

and the corresponding schemes are called **IMEX-E** RK. However, the development of such methods presents the difficulty that the characteristic speeds diverge in the diffusive limit making the hyperbolic part very stiff. We show how to overcome these difficulties with the introduction of particular conditions on the coefficients of the IMEX R-K scheme such that we can treat the stiff component on the hyperbolic part explicitly without any manipulation of the original system.

Note that in both approaches the schemes, with the introduction of particular conditions on the coefficients, relax, in the diffusion limit, to an IMEX R-K scheme for the convection-diffusion equation, in which the convection term is treated explicitly and the diffusion one is treated implicitly, thus overcoming the classical parabolic CFL restriction.

5.3.1 Analysis of the IMEX-I and IMEX-E RK schemes

Let us start to consider the system

$$\begin{aligned} u_t &= -(v+\mu p(u)_x)_x + \mu p(u)_{xx}, \\ \varepsilon^2 v_t &= -p(u)_x - (v-q(u)). \end{aligned} \tag{94}$$

In the analysis of the schemes we adopt a notation similar to the one adopted in Sect. 5.1. Here we consider IMEX schemes of type I.

IMEX-I RK approach. System (94) can be written as

$$\begin{aligned} U' &= \underbrace{f_1(U,V)}_{\text{explicit}} + \underbrace{f_2(U)}_{\text{implicit}}, \\ \varepsilon^2 V' &= \underbrace{g(U,V)}_{\text{implicit}} \end{aligned} \tag{95}$$

where the primes denote the time derivatives and

$$\begin{aligned} f_1(U,V) &= -D_x(V+\mu D_x p(U)), \quad f_2(U) = -\mu D_x^2 D p(U), \\ g(U,V) &= -D_x p(U) - V + q(U). \end{aligned}$$

Here we denote D_x and D_x^2 suitable space discretization of the first and second derivative, while with capital letter we indicate the discretization of the the unknown functions on a grid.

In the limit $\varepsilon \to 0$ from (95) we obtain a differential algebraic system (DAE),

$$\begin{aligned} U' &= f_1(U,V) + f_2(U), \\ 0 &= g(U,V). \end{aligned} \tag{96}$$

In order to guarantee the solvability of system (96) we should require that $g(U,V) = 0$ can always be solved as $V = G(U)$, for seme function G. Note that this is always the case for relaxation problems. In this specific case $g(U,V) = 0$ gives $V = q(U) - D_x p(u)$. When replaced in the first equation of (95) we obtain the limit equation for U, namely

$$U' = \hat{f}_1(U) + f_2(U) \tag{97}$$

where $\hat{f}_1(U) = f(U, G(U))$ and $V = G(U)$.

Applying an IMEX R-K scheme of type I to system (95) we obtain

$$\begin{aligned} u^{n+1} &= u^n + \Delta t \tilde{b}^\top f_1(\mathbf{U}, \mathbf{V}) + \Delta t b^\top f_2(\mathbf{U}), \\ \varepsilon^2 v^{n+1} &= \varepsilon^2 v^n + \Delta t b^\top g(\mathbf{U}, \mathbf{V}), \end{aligned} \tag{98}$$

for the numerical solution and

$$\begin{aligned} \mathbf{U} &= u^n \mathbf{e} + \Delta t \tilde{A} f_1(\mathbf{U}, \mathbf{V}) + \Delta t A f_2(\mathbf{U}), \\ \varepsilon^2 \mathbf{V} &= \varepsilon^2 v^n \mathbf{e} + \Delta t A g(\mathbf{U}, \mathbf{V}), \end{aligned} \tag{99}$$

Starting from the stage value (99), since A is invertible, from the second equation we obtain

$$\Delta t g(\mathbf{U}, \mathbf{V}) = \varepsilon^2 A^{-1}(\mathbf{V} - \mathbf{e} v^n). \tag{100}$$

Inserting this into the numerical solution v^{n+1}, we make v^{n+1} independent of ε^2

$$v^{n+1} = (1 - b^\top A^{-1}\mathbf{e}) v^n + b^\top A^{-1} \mathbf{V}$$

In the limit $\varepsilon \to 0$, we get $\Delta t g(\mathbf{U}, \mathbf{V}) = 0$, i.e., $\mathbf{V} = G(\mathbf{U})$, therefore the scheme gives

$$U^{n+1} = U^n + \Delta t \tilde{b}^\top \hat{f}_1(\mathbf{U}) + \Delta t b^\top f_2(\mathbf{U}), \tag{101}$$

for the numerical solution and

$$\mathbf{U} = u^n + \Delta t \tilde{A} \hat{f}_1(\mathbf{U}) + \Delta t A f_2(\mathbf{U}), \tag{102}$$

where $\hat{f}_1(U) \equiv f_1(U, G(U))$, for the internal stages. Therefore, the IMEX-RK scheme of type I in the limit $\varepsilon \to 0$ is the same IMEX-RK one applied to the limit equation $U' = \hat{f}_1(U) + f_2(U)$, and hence this means that no stability restriction on the time step is required in the convection-diffusion limit equation $u_t + q(u)_x = p(u)_{xx}$, i.e., we get an unconditionally stable method.

Note that if an IMEX-RK method is GSA, then we get $U^{n+1} = U^{(s)}$, $V^{n+1} = V^{(s)}$, and therefore $\lim_{\varepsilon \to 0} g(U^{n+1}, V^{n+1}) = 0$, because from (100) for $\varepsilon = 0$, $g(U^{(s)}, V^{(s)}) = 0$, and for GSA schemes one has $U^{n+1} = U^{(s)}$ and $V^{n+1} = V^{(s)}$, therefore $g(U^{n+1}, V^{n+1}) = 0$.

If the scheme is SA, but not GSA, then the numerical solution for U is correctly captured, but the numerical solution for V does not lie on the manifold $G(U, V) = 0$, since in general $v^{n+1} \neq V^{(s)}$. Similar consideration can be re-proposed for the type II when applied to the system (95), with slight modifications, [20].

IMEX-E RK approach. System (94) can be written in the following form

$$\begin{aligned} U' &= f_1(U, V) + f_2(U), \\ \varepsilon^2 V' &= g_1(U) - g_2(U, V), \end{aligned} \tag{103}$$

where the primes denote the time derivatives and

$$\begin{aligned} f_1(U, V) &= -D_x(V + \mu D_x p(U)), \quad \mu \hat{f}_2(U) = \mu D_x^2 p(U), \\ g_1(U) &= -D_x p(U), \quad g_2(U, V) = V - q(U). \end{aligned}$$

In the limit $\varepsilon \to 0$, i.e. $\mu = 1$, we obtain from (103)

$$g_2(U, V) = g_1(U), \tag{104}$$

or explicitly $V = g_1(U) + q(U)$ that gives

$$f_1(U, V) = -D_x q(U), \tag{105}$$

Then, setting $\hat{f}_1(U) = -D_x q(U)$ and with $\hat{f}_2(U) = D_x^2 p(U)$, system (103) becomes

$$U' = \hat{f}_1(U) + \hat{f}_2(U). \tag{106}$$

Now applying an IMEX Runge–Kutta GSDA of type I to system (103) we get in vector form for the numerical solution

$$\begin{aligned} U^{n+1} &= U^n + \Delta t\, \tilde{b}^\top f_1(\mathbf{U}, \mathbf{V}) + \Delta t\, b^\top \hat{f}_2(\mathbf{U}), \\ \varepsilon^2 V^{n+1} &= \varepsilon^2 V^n + \Delta t\, (\tilde{b}^\top g_1(\mathbf{U}) - b^\top g_2(\mathbf{U}, \mathbf{V})) \end{aligned} \tag{107}$$

and for the internal stages

$$\begin{aligned} \mathbf{U} &= U^n \mathbf{e} + \Delta t \tilde{A} f_1(\mathbf{U}, \mathbf{V}) + \Delta t A \hat{f}_2(\mathbf{U}), \\ \varepsilon^2 \mathbf{V} &= \varepsilon^2 V^n \mathbf{e} + \Delta t \tilde{A} g_1(\mathbf{U}) - \Delta t A g_2(\mathbf{U}, \mathbf{V}) \end{aligned} \tag{108}$$

As usual we have denoted by $\mathbf{e} = (1, \dots, 1)^\top$.

Starting from (108), and by the invertibility of the matrix A, from the second equation in (108) we have $\Delta t\, g_2(\mathbf{U}, \mathbf{V}) = \varepsilon^2 A^{-1}(V^n \mathbf{e} - \mathbf{V}) + \Delta t\, A^{-1} \tilde{A} g_1(\mathbf{U})$. Now substituting this expression in the numerical solution V^{n+1} we have

$$\varepsilon^2 V^{n+1} = \varepsilon^2 (1 - b^T A^{-1} \mathbf{e}) V^n + \varepsilon^2 b^\top A^{-1} \mathbf{V} + \Delta t (\tilde{b}^\top - b^\top A^{-1} \tilde{A}) g_1(\mathbf{U}). \tag{109}$$

By the GSA property of the scheme we get $\tilde{b}^\top - b^\top A^{-1} \tilde{A} \mathbf{e} = 0$ and $1 - b^T A^{-1} \mathbf{e} = 0$. Then we obtain $V^{n+1} = \mathbf{e}_s^\top \mathbf{V} = V^{(s)}$.

Concerning the second equation in (108), as $\varepsilon^2 \to 0$, we get $g_2(\mathbf{U}, \mathbf{V}) = A^{-1} \tilde{A} g_1(\mathbf{U})$ and from the first equation in (108) by (104)–(105) we get

$$\mathbf{U} = U_n \mathbf{e} + \Delta t \tilde{A} \hat{f}_1(\mathbf{U}) + \Delta t \bar{A} \hat{f}_2(\mathbf{U}), \tag{110}$$

where $\bar{A} = (\tilde{A}\mathcal{C} - A)$, $\mathcal{C} = I - A^{-1}\tilde{A}$, $\hat{f}_1(\mathbf{U}) = -D_x q(\mathbf{U})$, and $\hat{f}_2(\mathbf{U}) = D_x^2 p(\mathbf{U})$, and for the numerical solution

$$U_{n+1} = U_n + \Delta t \tilde{b}^\top \hat{f}_1(\mathbf{U}) + \Delta t (b^\top - \tilde{b}^\top \mathcal{C}) \hat{f}_2(\mathbf{U}). \tag{111}$$

Then as in the previous section in order to guarantee that the IMEX-RK scheme is AP, and also AA, we need to introduce extra conditions [23]. Here we report the extra order conditions for u-component up to second order (refer to the computations in the Appendix of [23]):

$$(b^T - \tilde{b}^T \mathcal{C}) \mathbf{e} = 1, \quad (\tilde{b}^T \mathcal{C} - b^T) \bar{A} \mathbf{e} = \frac{1}{2}. \tag{112}$$

The first condition guarantees that the scheme is AP, the second one that the scheme is at least second order in time. These conditions have to be satisfied in addition to the classical ones to achieve the expected order in the limit case, i.e. $\varepsilon \to 0$.

In [23] an IMEX-RK scheme of type I that satisfies additional conditions (112) has been introduced. The scheme, called called AGSA(4,3,2), is reported in the Appendix (147).

Example 5.3 (*Non-linear diffusion system*)

We apply the schemes to a nonlinear diffusion problem. We consider the generalized Carleman model. The Carleman model describes the time evolution of a one dimensional gas composed of two species of particles that move at a constant speed $c > 0$ in the x-direction

[31]. The number density at time t and position x of particles moving at speed $+c$ is denoted by $u = u(x,t)$ while that of particles moving at speed $-c$ is denoted $v = v(x,t)$. Carleman's system is

$$\begin{aligned} u_t + cv_x &= (u+v)(v-u) \\ v_t - cv_x &= (u+v)(u-v) \end{aligned} \tag{113}$$

An extension of the model, called the generalized Carleman model, involves a collision frequency that is proportional to some power of the macroscopic density $\rho = u + v$, as follows:

$$\begin{aligned} u_t + cv_x &= (u+v)^m(v-u) \\ v_t - cv_x &= (u+v)^m(u-v) \end{aligned} \tag{114}$$

For $m = 1$ the original Carleman system is recovered, whereas $m = 0$ gives another known system, the Goldstein–Taylor model [52], that can be reduced to a damped wave equation (the telegrapher's equation). The only significant hydrodynamic limits of the Carleman model, generalized or not, are diffusion limits. Indeed, local equilibria for those models are those densities for which $(u-v)(u+v)^m = 0$, which implies $u = v$. Hence all local equilibria for those models have mean velocity 0, since the number density of particles with velocity $+c$ is equal of the number density of particles with velocity $-c$, and the density $u + v = 2u$ is constant in time. Therefore a non trivial limiting dynamics of the system can be observed only in the diffusion scaling.

After setting $c = 1$ without loss of generality, to understand such a diffusive limit we consider the generalized Carleman model, under the diffusive scaling

$$\begin{aligned} u_t + \frac{1}{\varepsilon}cv_x &= \frac{1}{\varepsilon^2}(u+v)^m(v-u) \\ v_t - \frac{1}{\varepsilon}cv_x &= \frac{1}{\varepsilon^2}(u+v)^m(u-v) \end{aligned} \tag{115}$$

Defining the macroscopic mass density $\rho = u + v$ and the current $j = (u-v)/\varepsilon$ by we put the system (115) in the form

This model relaxes to the porous media equation when $\varepsilon \to 0$ where the local equilibrium is given by

$$j = -\frac{\partial_x \rho}{2\rho^m} = -\frac{\partial_x(\rho^{1-m})}{2(1-m)},$$

and the system relaxes to the non linear parabolic equation

$$\partial_t \rho = \frac{\partial_{xx}(\rho^{1-m})}{2(1-m)}.$$

Note that for $m = 0$ we get the Goldstein–Taylor model. Now we apply our scheme to the equivalent system

$$\begin{aligned} \rho_t &= -j_x - \mu(\varepsilon)\frac{\partial_{xx}(\rho^{1-m})}{2(1-m)} + \mu(\varepsilon)\frac{\partial_{xx}(\rho^{1-m})}{2(1-m)} \\ \varepsilon^2 j_t &= -\rho_x - 2\rho^m j. \end{aligned} \tag{116}$$

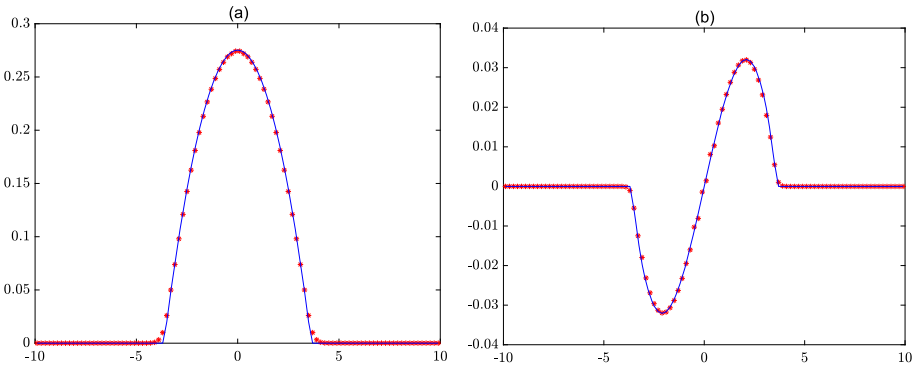

Fig. 3 Numerical solution at time $t = 3.0$ for parabolic regime $\varepsilon = 10^{-6}$ with $\Delta x = 0.2$, $CFL = 0.5$ and $\Delta t = 0.1$. On the left panel **a** the mass density ρ and on the right panel **b** the flow j. The blue line is the Barenblatt problem (116) (color figure online)

Here we choose $m = 1$ and we compare the numerical solution for $\varepsilon \to 1$ with the exact Barenblatt [11] for the porous media equation,

$$\rho(x,t) = \frac{1}{R(t)}\left[1 - \left(\frac{x}{R(t)}\right)^2\right], \quad j(x,t) = \rho\frac{2x}{R(t)^3}, \quad |x| < R(t),$$
$$\rho(x,t) = 0, \quad j(x,t) = 0, \quad |x| > R(t), \tag{117}$$

where $R(t) = [12(t+1)]^{1/3}$, $t \geq 0$. We take $\Delta x = 0.2$ and $x \in [-10, 10]$. In Fig. 3 the scheme AGSA(3,4,2) captures well the correct behavior of the exact solution.

6 Multiscale relaxation

In this section we analyze system (27) from a numerical point of view. We focus on time discretization. Space discretization can be performed later, choosing suitable discretization of the differential operators.

6.1 First-order time discretization

We start with first order implicit–explicit time-discretization of the relaxation system (27) and we analyze its relationship with a reformulated system in which the eigenvalues are bounded for any value of the scaling parameter ε. To this aim, following [1, 21], we consider the following implicit–explicit first order method applied to the system (27) given by

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} &= -\partial_x v^{n+1}, \\ \varepsilon^{1+\alpha}\frac{v^{n+1} - v^n}{\Delta t} &= -\left(\varepsilon^{1-\alpha}\partial_x p(u^n) + v^{n+1} - f(u^n)\right). \end{aligned} \tag{118}$$

Notice that the second equation can be explicitly solved for v^{n+1}:

$$v^{n+1} = \frac{\varepsilon^{1+\alpha}}{d(\varepsilon, \Delta t)}v^n - \frac{\Delta t}{d(\varepsilon, \Delta t)}\left(\varepsilon^{1-\alpha}\partial_x p(u^n) - f(u^n)\right). \tag{119}$$

where $d(\varepsilon, \Delta t) \equiv \varepsilon^{1+\alpha} + \Delta t$. Then, inserting the above relation in the first equation, one gets

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{\varepsilon^{1+\alpha}}{d(\varepsilon, \Delta t)} v_x^n + \frac{\Delta t}{d(\varepsilon, \Delta t)} \partial_x f(u^n) = \frac{\Delta t\, \varepsilon^{1-\alpha}}{d(\varepsilon, \Delta t)} \partial_{xx} p(u^n). \tag{120}$$

Making use of Eq. (119), the second equation of (118) can be written as

$$\frac{v^{n+1} - v^n}{\Delta t} + \frac{\varepsilon^{1-\alpha}}{d(\varepsilon, \Delta t)} \partial_x p(u^n) = -\frac{1}{d(\varepsilon, \Delta t)} \left(v^n - f(u^n)\right). \tag{121}$$

Therefore, the IMEX scheme can be recast in an equivalent fully explicit form. Similarly to the continuous case, depending on the choice of α, as $\varepsilon \to 0$, we have different limit behaviors. For $\alpha \in [0, 1)$ we obtain, as $\varepsilon \to 0$,

$$\frac{u^{n+1} - u^n}{\Delta t} + \partial_x f(u^n) = 0, \tag{122}$$

whereas in the case $\alpha = 1$ we get

$$\frac{u^{n+1} - u^n}{\Delta t} + \partial_x f(u^n) = \partial_{xx} p(u^n), \tag{123}$$

and this makes the scheme (118) asymptotic preserving.

For small values of Δt, scheme (120)–(121) corresponds, up to first order in time, to the system

$$\begin{aligned} &\partial_t u + \frac{\varepsilon^{1+\alpha}}{d(\varepsilon, \Delta t)} \partial_x v + \frac{\Delta t}{d(\varepsilon, \Delta t)} \partial_x f(u) = \frac{\Delta t\, \varepsilon^{1-\alpha}}{d(\varepsilon, \Delta t)} \partial_{xx} p(u) + \mathcal{O}(\Delta t), \\ &\partial_t v + \frac{\varepsilon^{1-\alpha}}{d(\varepsilon, \Delta t)} \partial_x p(u) = -\frac{1}{d(\varepsilon, \Delta t)} \left(v - f(u)\right) + \mathcal{O}(\Delta t). \end{aligned} \tag{124}$$

The main feature of system (124) is that its left-hand side has bounded characteristic speeds. These are given by

$$\lambda_{\pm}^{\alpha}(\Delta t, \varepsilon) = \frac{\xi_\alpha}{2} \left(c \pm \sqrt{c^2 + \frac{4\varepsilon^2}{\Delta t}} \right), \tag{125}$$

with

$$\xi_\alpha(\Delta t, \varepsilon) = \frac{\Delta t}{\varepsilon^{1+\alpha} + \Delta t},$$

and where, for simplicity, we considered $f'(u) = c$, $c \in \mathbb{R}$ and $p'(u) = 1$ so that

$$\partial_x f(u) = f'(u) \partial_x u = c \partial_x u, \qquad \partial_x p(u) = p'(u) \partial_x u = \partial_x u. \tag{126}$$

If we fix ε and send $\Delta t \to 0$ we obtain the usual characteristic speeds of the original hyperbolic system, i.e.

$$\lambda_{\pm}^{\alpha}(0, \varepsilon) = \pm \frac{1}{\varepsilon^{\alpha}},$$

while for a fixed Δt, the characteristic speeds λ_+^α and λ_-^α are respectively decreasing and increasing functions of ε and, as $\varepsilon \to 0$, they converge to

$$\lambda_{\pm}^{\alpha}(\Delta t, 0) = \frac{1}{2} \left(c \pm |c|\right). \tag{127}$$

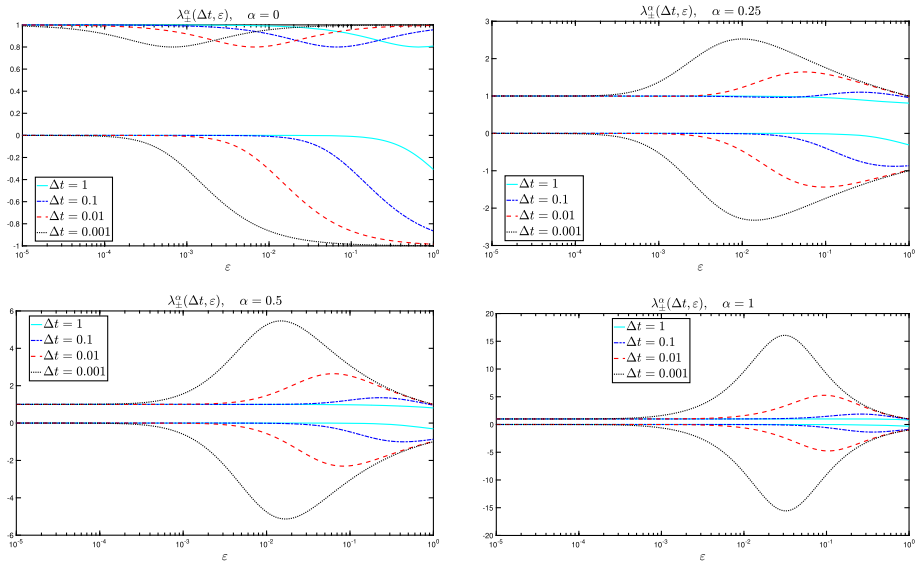

Fig. 4 Eigenvalues of the modified system (124) as a function of ε for different values of the time step Δt and choices of α

In Fig. 4, we show the shape of the eigenvalues (125) for $c = 1$ and different values of the scaling parameter α and the time step Δt. We observe that the absolute value of the eigenvalues is always bounded and achieves its maximum when $\Delta t = \mathcal{O}(\varepsilon)$.

Thus, for a given Δt, if we denote by Δx the space discretization parameter, from the left hand side of (124) we expect the hyperbolic CFL condition $\Delta t \leq \Delta x/|c|$ in the limit $\varepsilon \to 0$. On the other hand, the stability restriction coming from the parabolic term requires $\Delta t = \mathcal{O}(\Delta x^2)$ when $\alpha = 1$.

Now we generalize the above arguments to the case of high order IMEX-RK methods. For clarity of presentation, we separate the discussion of the diffusive case $\alpha = 1$ from the general case $\alpha \in [0, 1)$.

6.2 Removing the parabolic stiffness. AP-implicit IMEX R-K methods

In the next section, we discuss IMEX discretization dealing with the stiffness caused by the parabolic term in the asymptotic limit, and how to remove the parabolic restriction appearing for small ε. We call these *AP-implicit* methods.

A similar technique has been developed for the construction and analysis of IMEX multistep methods, see [2] for details.

Here the idea is to treat the term $p(u)_x$ in (27) implicitly. First order IMEX reads

$$
\begin{aligned}
u^{n+1} &= u^n - \Delta t v_x^{n+1} \\
v^{n+1} &= v^n + \frac{\Delta t}{\varepsilon^{2\alpha}} f(u^n) - \frac{\Delta t}{\varepsilon^{1+\alpha}} (v^{n+1} + p(u^{n+1})_x).
\end{aligned}
$$

so as $\varepsilon \to 0$, in the case $\alpha = 1$ we obtain a first order discretization of the limit convection-diffusion equation (32)

$$\frac{u^{n+1} - u^n e}{\Delta t} + f(u^n)_x = p(u^{n+1})_{xx},$$

with the diffusive term treated implicitly, while for $\alpha \in [0, 1)$ we obtain the first order explicit scheme for the limit scalar conservation law

$$u^{n+1} = u^n - \Delta t f(u^n)_x.$$

Now we consider high-order IMEX-RK schemes applied to the (27), as

$$\begin{aligned} \mathbf{u} &= u^n e - \Delta t A \mathbf{v}_x \\ \mathbf{v} &= v^n e + \frac{\Delta t}{\varepsilon^{2\alpha}} \tilde{A} f(\mathbf{u}) - \frac{\Delta t}{\varepsilon^{1+\alpha}} A(\mathbf{v} + p(\mathbf{u})_x). \end{aligned} \tag{128}$$

and

$$\begin{aligned} u^{n+1} &= u^n - \Delta t\, b^\top \mathbf{v}_x \\ v^{n+1} &= v^n + \frac{\Delta t}{\varepsilon^{2\alpha}} \tilde{b}^\top f(\mathbf{u}) - \frac{\Delta t}{\varepsilon^{1+\alpha}} b^\top (\mathbf{v} + p(\mathbf{u})_x). \end{aligned} \tag{129}$$

Therefore we state the following

Theorem 6.1 [21] *If the IMEX-RK scheme* (128)–(129)*, applied to* (27) *for* $\alpha = 1$*, satisfies the GSA property then, as* $\varepsilon \to 0$*, it becomes the IMEX-RK method characterized by the pairs* $(\tilde{A}, \tilde{b})$ *and* (A, b) *for the limit convection–diffusion equation* (32)*. Otherwise for* $\alpha \in [0, 1)$ *the IMEX-RK scheme as* $\varepsilon \to 0$ *yields the explict RK method characterized by the pair* $(\tilde{A}, \tilde{b})$ *for the limit scalar conservation law* (30)*.*

After some algebraic manipulation (see [21] for details), in the case $\alpha = 1$, we get for the internal stages

$$\begin{aligned} \frac{\mathbf{u} - u^n e}{\Delta t} + \zeta A(\zeta I + A)^{-1} v_x^n e &= A(\zeta I + A)^{-1}(Ap(\mathbf{u})_x - \tilde{A} f(\mathbf{u}))_x, \\ \frac{\mathbf{v} - v^n e}{\Delta t} + \frac{1}{\varepsilon^2} \tilde{A} f(\mathbf{u}) &= -\frac{1}{\varepsilon^2} A(\mathbf{v} + p(\mathbf{u})_x), \end{aligned} \tag{130}$$

and similarly for the numerical solution

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} + \zeta b^\top (\zeta I + A)^{-1} v_x^n e &= b^\top (\zeta I + A)^{-1}(Ap(\mathbf{u})_x - \tilde{A} f(\mathbf{u}))_x, \\ \frac{v^{n+1} - v^n}{\Delta t} + \frac{1}{\varepsilon^2} \tilde{b}^\top f(\mathbf{u}) &= -\frac{1}{\varepsilon^2} b^\top (\mathbf{v} + p(\mathbf{u})_x). \end{aligned} \tag{131}$$

where, for shortness, we set $\zeta = \varepsilon^2 / \Delta t$. We call this scheme (130)–(131): *AP-implicit IMEX-RK*. Concerning the AP and AA property, in the limit for $\varepsilon \to 0$. i.e. $\zeta \to 0$, the scheme (130)–(131), becomes an IMEX-RK scheme for the convection–diffusion equation (32)

$$\begin{aligned} \frac{\mathbf{u} - u^n e}{\Delta t} + \tilde{A} f(\mathbf{u})_x &= Ap(\mathbf{u})_{xx}, \\ \frac{u^{n+1} - u^n}{\Delta t} + \tilde{b}^\top f(\mathbf{u})_x &= b^\top p(\mathbf{u})_{xx}. \end{aligned} \tag{132}$$

Clearly, scheme (132) is a p-order approximation, with p the order of the IMEX-RK scheme, of the limit equation (32) where now the diffusion term is evaluated implicitly, therefore the

CFL condition of such scheme is uniquely determined by the hyperbolic restriction $\Delta t \sim \Delta x$. A similar analysis in the case $\alpha \in [0, 1)$ for scheme (128)–(129) under the GSA assumption produces the explicit Runge–Kutta scheme

$$\begin{aligned} \mathbf{u} &= u^n - \Delta t \tilde{A} f(\mathbf{u})_x, \\ u^{n+1} &= u^n - \Delta t \tilde{b}^\top \tilde{A} f(\mathbf{u})_x. \end{aligned} \tag{133}$$

Note that a similar analysis for the AP and AA property for GSA methods of type II can be found in [21] and we direct the reader to it for more details.

Test: a discrete kinetic model

As a mathematical model for our numerical experiment we consider the so-called Ruijgrook–Wu model of the discrete kinetic theory of rarefied gas dynamic [21, 50, 51, 64, 86]. The model describes a two-speed gas in one space dimension and corresponds to the system

$$\begin{aligned} M\partial_t f^+ + \partial_x f^+ &= -\frac{1}{\mathrm{Kn}}(af^+ - bf^- - cf^+ f^-), \\ M\partial_t f^- - \partial_x f^- &= \frac{1}{\mathrm{Kn}}(af^+ - bf^- - cf^+ f^-), \end{aligned} \tag{134}$$

where f^+ and f^- denote the particle density distribution at time t, position x and with velocity $+1$ and -1 respectively. Here Kn is the Knudsen number, M is the Mach number of the system and a,b and c are positive constants which characterize the microscopic interactions. The local (Maxwellian) equilibrium $f_\infty^\pm$ is obtained by equating to zero the collision term, and is characterized by

$$f_\infty^+ = \frac{bf_\infty^-}{a - cf_\infty^-}, \quad \text{or, equivalently,} \quad f_\infty^- = \frac{af_\infty^+}{b + cf_\infty^+}. \tag{135}$$

The macroscopic variables for the model are the density ρ and momentum v defined by

$$u = f^+ + f^-, \quad v = (f^+ - f^-)/M. \tag{136}$$

The non dimensional multiscale problem is obtained taking $M = \varepsilon^\alpha$ and $\mathrm{Kn} = \varepsilon$

In macroscopic variables taking $a = b = 1/2$, $c = M = \varepsilon^\alpha$ this can be written as [51]

$$\begin{aligned} &\partial_t u + \partial_x v = 0\,, \\ &\partial_t v + \frac{1}{\varepsilon^{2\alpha}}\partial_x u = -\frac{1}{\varepsilon^{1+\alpha}}\left[v - \frac{1}{2}\left(v^2 - \varepsilon^{2\alpha}v^2\right)\right]. \end{aligned} \tag{137}$$

The model provides nontrivial limit behaviors for several values of α including the corresponding compressible Euler ($\alpha = 0$) limit and the incompressible Euler ($\alpha \in (0, 1)$) and Navier–Stokes ($\alpha = 1$) limits (see [50, 51]). For the Ruijgrook–Wu model (137) it can be shown, via Chapman-Enskog expansion, that for $\alpha \in (1/3, 1]$ so that $2\alpha > 1 - \alpha$, and small values of ε, we have

$$v = \frac{1}{2}u^2 - \varepsilon^{1-\alpha}\partial_x u + \mathcal{O}(\varepsilon^{2\alpha}). \tag{138}$$

Then, the solution behavior is characterized by the viscous Burgers equation

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = \varepsilon^{1-\alpha}\partial_{xx}u + \mathcal{O}(\varepsilon^{2\alpha}). \tag{139}$$

Now in order to test the AP-implict IMEX-RK method (128)–(129), we consider the Ruijgrook–Wu model (137), in the space interval $[-0.5, 0.5]$ with initial data defined as follows

$$u_0(x) = \begin{cases} 1 & \text{if } |x| \le 1/8 \\ 0 & \text{otherwise,} \end{cases} \qquad v_0(x) = 0, \tag{140}$$

where we account for reflecting boundary conditions, i.e. $v = 0$, $\partial_x u = 0$ on the boundaries $x = \pm 0.5$. This problem has been previously studied in [1, 21, 73].[1]

We study the solution to (137) for three different regimes of the parameters α and ε and we solve the model with $N = 100$ space grid points using the BPR(3,5,3) (146) IMEX-RK scheme given in the Appendix.

In Fig. 5 we report the evolution of the solution for the density u and the momentum v, respectively on the top and bottom rows. The first column represents the rarefied regime, with $\varepsilon = 0.7, \alpha = 1$ at time $T = 0.2$. In this regime, the transport part dominates and the initial data propagates in the directions of the particles. This behavior is well described by the method without spurious numerical oscillations. In the second column we have the hyperbolic limit for $\varepsilon = 10^{-12}$, $\alpha = 2/3$, at time $T = 0.5$ corresponding to the inviscid Burger equation with a shock propagating in the right direction. Even in this case the shock profile is well captured. Finally, in the last column we report the parabolic limit for $\varepsilon = 10^{-10}$, $\alpha = 4/5$, at time $T = 0.5$, corresponding to a viscous Burger equation. As expected, the shock profile is regularized by the presence of the diffusive term

The numerical solutions for the mass density u and momentum v are computed in the rarefied regime and in the diffusive regime with $\Delta t = 0.5\Delta x$ and are depicted with a reference solution obtained using fine grids with $\Delta x = 0.001$.

7 Conclusions

The paper gives a brief account on the development and analysis of Asymptotic Preserving schemes for hyperbolic systems of conservation laws with stiff relaxation source. Because of the stiffness of the source, one cannot use explicit schemes, because they their stability requires strong restrictions on the time step, making such schemes very inefficient when the time scale that controls stiffness is very small. On the other hand, in most cases the hyperbolic term is not stiff, therefore it would not benefit from an implicit treatment. Furthermore, high order schemes for quasilinear hyperbolic systems of balance laws are nonlinear both in the flux term and in the non-oscillatory reconstructions, which are adopted to prevent formation of spurious oscillations. For such reasons, it appears natural to treat the stiff source by an implicit scheme, and the hyperbolic term by an explicit one, thus optimally guaranteeing both stability and efficiency. The special structure of the relaxation terms guarantees that the solution of the system obeys a reduced system as the relaxation parameter vanishes. It is therefore tempting to construct schemes which solve the problem for all stiffness regimes, and in particular, that become consistent schemes for the limit relaxed problem as the relaxation

[1] Originally, this test has been taken from [73], and, because of the different notation, the parameters ε and α adopted in [73], which we denote here by $\tilde{\alpha}, \tilde{\varepsilon}$, have a different meaning in this paper. The correspondence between $\tilde{\alpha}, \tilde{\varepsilon}$ and α and ε adopted in the present paper is

$$\tilde{\alpha} = \frac{1-\alpha}{\alpha}, \qquad \tilde{\varepsilon} = \varepsilon^{\alpha}, \qquad \alpha \neq 0. \tag{141}$$

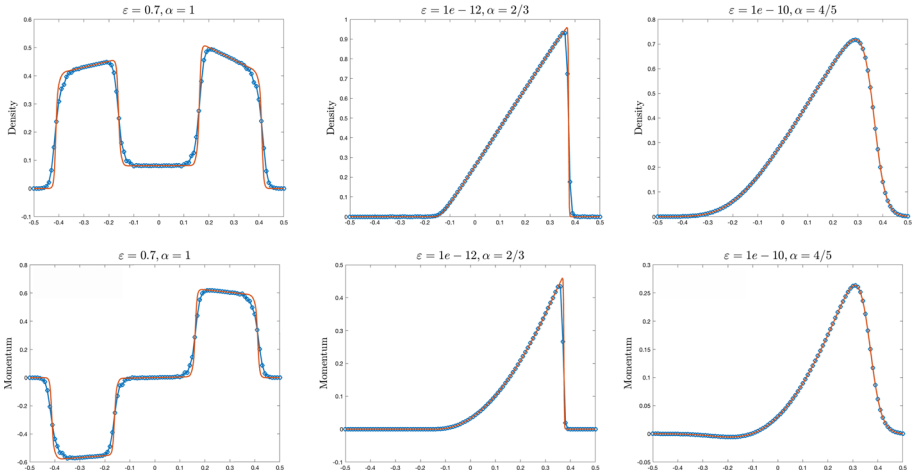

Fig. 5 Numerical solutions of system (137) for the AP-implicit BPR(3,5,3) scheme with initial conditions (140) for the mass density (top) and the momentum (bottom) in the rarefied regime (left column)

parameter vanishes. Such schemes are said *asymptotic preserving*, and are particularly useful for this class of problems, since they are constructed to work for all regimes. Several kinds of relaxation are possible. The two most common cases are the hyperbolic-hyperbolic and the hyperbolic-parabolic relaxation, which are deeply treated, respectively, in Sects. 4 and 5.

A class of problems that has attracted a great attention in the last decade consists in hyperbolic problems in which there is a large variation of characteristic speeds. Even without a stiff relaxation, such problems are stiff, since the speeds and therefore the typical evolution time scales, may change by several orders of magnitude. There are cases in which high speed waves carry a small signal one is not particularly interested in. In such cases IMEX schemes can provide effective tools to tackle the problem: one may adopt explicit schemes for the slow waves one is interested in, while allowing an implicit treatment of the fast waves.

The recent literature on the topic is too vast for an exhaustive citation here. We just mention the following papers which deal with the development and analysis of the so called *all Mach number* schemes [22, 24, 25, 75].

Another related problem concerns the development of well-balanced schemes for hyperbolic systems with source. In this case the interest consists in constructing schemes which are able to preserve, exactly or with great accuracy, the stationary solutions of the system. This will allow accurate calculation of time dependent solution which are a small perturbation of a stationary solution, see [33] and references therein. This problem is considered, for example, in [53], where preliminary result on the topic are presented. An interesting problem that connects AP schemes with WB scheme concerns the construction and analysis of asymptotic-preserving well-balanced schemes for systems with stiff relaxation which relax to a reduced system of balance laws with non trivial source. This, among other related topic, is subject of current investigation.

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A Appendix

In this appendix we report some of the most common IMEX schemes of order 2 and 3 which are adopted for the numerical solution of hyperbolic systems with stiff relaxation.

- HT(1,1,2) scheme of type II with $s_E, s_I = 1$ and $s = 2$ with $p = 2$:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}. \tag{142}$$

  This method combines Heun scheme with trapezoidal rule. The implicit part is A-stable [97], but not L-stable. Notice that scheme HT(2,2,2) is a natural choice when dealing with a convection-diffusion equation, since the Heun method has good property for the explicit part and the trapezoidal rule (also known as the Crank-Nicolson scheme) is A-stable and widely used for diffusion problems.
- SSP2-IMEX(2,2,2) scheme, [79] of type I with $s_E, S_I = 2$ and $s = 2$ with $p = 2$, $\gamma = 1 - 1/\sqrt{2}$:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} 0 & \gamma & 0 \\ 1-\gamma & 1-2\gamma & \gamma \\ \hline & 1/2 & 1/2 \end{array}. \tag{143}$$

  This scheme is a combination of the Heun method for the explicit part coupled with a SDIRK method with $a_{ii} = \gamma$ for $i = 1, 2$. Note that the explicit method is an SSP (Strong Stability Preserving [87, 90]) suitable for systems of conservation laws. The choice of $\gamma = (2 - \sqrt{2})/2$ in the implicit part guarantees that the implicit part of the scheme is L-stable, [97].
- SSP2-IMEX(3,3,2) [79]:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 \\ \hline & 1/3 & 1/3 & 1/3 \end{array} \qquad \begin{array}{c|ccc} 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 \\ 1 & 1/3 & 1/3 & 1/3 \\ \hline & 1/3 & 1/3 & 1/3 \end{array} \tag{144}$$

This method of type I is obtained by combining a 2nd order SSP explicit RK method $s_E = 2$ with a 2nd order DIRK method. The implicit part is SA according to the Definition 3.2, therefore by the proposition 3.2 it is L-stable.

- SSP3-IMEX(3,4,3) [79]:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 & 0 \\ \hline & 0 & 1/6 & 1/6 & 2/3 \end{array} \qquad \begin{array}{c|cccc} \alpha & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 1 & 1/3 & 1/3 & \alpha & 0 \\ 1/2 & \beta & \eta & 1/2-\beta-\eta-\alpha & \alpha \\ \hline & 0 & 1/6 & 1/6 & 2/3 \end{array} \tag{145}$$

$\alpha = 0.241694260788$, $\beta = 0.0604235651970$, $\eta = 0.12915286960590$.
This method of type I is a combination of an SSP explicit method of order 3 and an third order L-stable SDIRK method, i.e. $a_{ii} = \alpha$ for all i. Note that the implicit part is not SA.
This method and methods (143), (144) were introduced in the context of hyperbolic systems with hyperbolic relaxation, [79].

- BPR(3,4,3) scheme [20]:

$$\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2/3 & 4/9 & 2/9 & 0 & 0 & 0 \\ 1 & 1/4 & 0 & 3/4 & 0 & 0 \\ 1 & 1/4 & 0 & 3/4 & 0 & 0 \\ \hline & 1/4 & 0 & 3/4 & 0 & 0 \end{array} \qquad \begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 2/3 & 5/18 & -1/9 & 1/2 & 0 & 0 \\ 1 & 1/2 & 0 & 0 & 1/2 & 0 \\ 1 & 1/4 & 0 & 3/4 & -1/2 & 1/2 \\ \hline & 1/4 & 0 & 3/4 & -1/2 & 1/2 \end{array} \tag{146}$$

This scheme is a third order IMEX R-K scheme of type II and GSA according to Definition (3.2). This method was introduced in the context of hyperbolic systems with diffusive relaxation, [20].

- AGSA(3,4,2) [23]:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \tilde{c}_2 & \tilde{c}_2 & 0 & 0 & 0 \\ \tilde{c}_3 & \tilde{a}_{31} & \tilde{a}_{32} & 0 & 0 \\ 1 & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & 0 \\ \hline & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & 0 \end{array} \qquad \begin{array}{c|cccc} c_1 & c_1 & 0 & 0 & 0 \\ c_2 & a_{21} & a_{22} & 0 & 0 \\ c_3 & a_{31} & a_{32} & a_{33} & 0 \\ 1 & b_1 & b_2 & b_3 & \gamma \\ \hline & b_1 & b_2 & b_3 & \gamma \end{array} \tag{147}$$

$\tilde{c}_2 = \tilde{a}_{21} = (-139833537)/38613965$, $c_1 = 168999711/74248304$,
$\tilde{a}_{31} = 85870407/49798258$, $\gamma = a_{22} = 202439144/118586105$,
$\tilde{a}_{32} = (-121251843)/1756367063$, $a_{33} = 12015439/183058594$,
$\tilde{b}_2 = 1/6$, $\tilde{b}_3 = 2/3$, $a_{31} = (-6418119)/169001713$,
$a_{21} = 44004295/24775207$, $a_{32} = (-748951821)/1043823139$,
$b_2 = 1/3$, $b_3 = 0$, $\tilde{b}_1 = 1 - \tilde{b}_2 - \tilde{b}_3$ $b_1 = 1 - \gamma - b_2 - b_3$,

This scheme is a second order IMEX R-K scheme of type I and GSA according to Definition (3.2) satisfying the additional order conditions (112). The implicit part is a DIRK method. This method was introduced in the context of hyperbolic systems with diffusive relaxation, [23].

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