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Sharp upper bounds for Steklov eigenvalues of a hypersurface of revolution with two boundary components in Euclidean space

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Abstract
We investigate the question of sharp upper bounds for the Steklov eigenvalues of a hypersurface of revolution in Euclidean space with two boundary components, each isometric to $\mathbb{S}^{n-1}$. For the case of the first non zero Steklov eigenvalue, we give a sharp upper bound $B_n(L)$ (that depends only on the dimension $n \geq 3$ and the meridian length $L > 0$) which is reached by a degenerated metric g^* that we compute explicitly. We also give a sharp upper bound B_n which depends only on n. Our method also permits us to prove some stability properties of these upper bounds.

Résumé
Nous étudions la question des bornes supérieures optimales pour les valeurs propres de Steklov d'une hypersurface de révolution de l'espace euclidien avec deux composantes connexes du bord, chacune isométrique à $\mathbb{S}^{n-1}$. Dans le cas de la première valeur propre de Steklov non nulle, nous donnons une borne supérieure optimale $B_n(L)$ (qui ne dépend que de la dimension n et de la longueur d'un méridien $L > 0$) qui est atteinte par une métrique dégénérée g^* que l'on calcule explicitement. Nous donnons aussi une borne supérieure optimale B_n qui ne dépend que de n. Notre méthode nous permet également de prouver des propriétés de stabilité que possèdent ces bornes supérieures.

Keywords Spectral geometry · Steklov problem · Hypersurfaces of revolution · Sharp upper bounds

Mathematics Subject Classification 58J50

1 Introduction

Let (M, g) be a smooth compact connected Riemannian manifold of dimension $n \geq 2$ with smooth boundary Σ. The Steklov problem on (M, g) consists of finding the real numbers σ and the harmonic functions $f : M \longrightarrow \mathbb{R}$ such that $\partial_\nu f = \sigma f$ on Σ, where ν denotes the

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outward normal on Σ. Such a σ is called a Steklov eigenvalue of (M, g). It is well known that the Steklov spectrum forms a discrete sequence $0 = \sigma_0(M, g) < \sigma_1(M, g) \leq \sigma_2(M, g) \leq \cdots \nearrow \infty$. Each eigenvalue is repeated with its multiplicity, which is finite. If the context is clear, then we simply write $\sigma_k(M)$ for $\sigma_k(M, g)$.

It is known [3, Thm. 1.1] that for any connected compact manifold (M, g) of dimension $n \geq 3$, there exists a family (g_ε) of Riemannian metrics conformal to g which coincide with g on the boundary of M, such that

$$\sigma_1(M, g_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \infty.$$

Therefore, to obtain upper bounds for the Steklov eigenvalues, it is necessary to study manifolds that satisfy certain additional constraints. We refer to [6] for an overview of the current state-of-the-art on geometric upper bounds for the Steklov eigenvalues.

Recently, authors investigated the Steklov problem on manifolds of revolution [9, 10, 12, 13]. A natural constraint for the manifolds is that they are (hyper)surfaces of revolution in Euclidean space. Some work has already been done on these kinds of manifolds, see for example [4, 5]. We refer to [4, Sect. 3.1] for a review about what these manifolds are, and consider a particular case in this paper that we define below (see Definition 1).

This work led to the discovery of lower and upper bounds for the Steklov eigenvalues of a hypersurface of revolution. We begin by recalling some recent results.

We first consider results for hypersurfaces of revolution with one boundary component that is isometric to $\mathbb{S}^{n-1}$. In dimension $n = 2$, it is proved in [4, Prop. 1.10] that each surface of revolution $M \subset \mathbb{R}^3$ with boundary $\mathbb{S}^1 \subset \mathbb{R}^2 \times \{0\}$ is Steklov isospectral to the unit disk. In dimension $n \geq 3$, many bounds were given. It is proved that each hypersurface of revolution $M \subset \mathbb{R}^{n+1}$ with one boundary component isometric to $\mathbb{S}^{n-1}$ satisfies $\sigma_k(M) \geq \sigma_k(\mathbb{B}^n)$, where $\mathbb{B}^n$ is the Euclidean ball and equality holds if and only if $M = \mathbb{B}^n \times \{0\}$, see [4, Thm. 1.8]. In [5, Thm. 1], the authors show the following upper bound: if $M \subset \mathbb{R}^{n+1}$ is a hypersurface of revolution with one boundary component isometric to $\mathbb{S}^{n-1}$, then for each $k \geq 1$, we have

$$\sigma_{(k)}(M) < k + n - 2,$$

where $\sigma_{(k)}(M)$ is the kth distinct Steklov eigenvalue of M. Although there exists no equality case within the collection of hypersurfaces of revolution, this upper bound is sharp. Indeed, for each $\varepsilon > 0$ and each $k \geq 1$, there exists a hypersurface of revolution M_ε such that $\sigma_{(k)}(M_\varepsilon) > k + n - 2 - \varepsilon$.

These results concern hypersurfaces of revolution that have one boundary component isometric to $\mathbb{S}^{n-1}$. Therefore, the goal of this paper is to investigate the Steklov problem on a hypersurface of revolution with two boundary components. As was already done in [4] and in [5], we will consider hypersurfaces with boundary components isometric to $\mathbb{S}^{n-1}$. We begin by defining the context.

Definition 1 An n-dimensional compact hypersurface of revolution (M, g) in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ is the warped product $M = [0, L] \times \mathbb{S}^{n-1}$ endowed with the Riemannian metric

$$g(r, p) = dr^2 + h^2(r) g_0(p),$$

where $(r, p) \in [0, L] \times \mathbb{S}^{n-1}$, g_0 is the canonical metric of the $(n-1)$-sphere of radius one and $h : [0, L] \longrightarrow \mathbb{R}_+^*$ is a smooth function which satisfies:

(1) $|h'(r)| \leq 1$ for all $r \in [0, L]$;

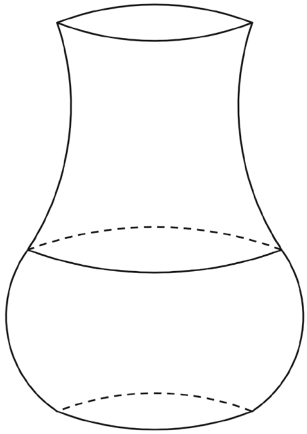

Fig. 1 Since $h(0) = h(L) = 1$, the boundary of M consists of two copies of $\mathbb{S}^{n-1}$

(2) $h(0) = h(L) = 1$.

Assumption (1) comes from the fact that (M, g) is a hypersurface in Euclidean space $\mathbb{R}^{n+1}$, see [4, Sect. 3.1] for more details. Assumption (2) implies that each component of the boundary is isometric to $\mathbb{S}^{n-1}$, as commented in Fig. 1.

We now make some remarks on the terminology used throughout this paper. If $M = [0, L] \times \mathbb{S}^{n-1}$ and $h : [0, L] \longrightarrow \mathbb{R}^*_+$ satisfies the properties above, we say that M is a *hypersurface of revolution*, we say that $g(r, p) = dr^2 + h^2(r)g_0(p)$ is a *metric of revolution on M induced by h* and we call the number L the *meridian length* of M.

Some lower bounds have already been obtained is this case. Indeed, [4, Thm. 1.11] states that if $M \subset \mathbb{R}^{n+1}$, $n \geq 3$, is a hypersurface of revolution (in the sense of Definition 1), and $L > 2$ is the meridian length of M, then for each $k \geq 1$,

$$\sigma_k(M) \geq \sigma_k(\mathbb{B}^n \sqcup \mathbb{B}^n).$$

Moreover, this inequality is sharp. In the case $0 < L \leq 2$, a lower bound is also obtained:

$$\sigma_k(M) \geq \left(1 - \frac{L}{2}\right)^{n-1} \sigma_k(C_L, dr^2 + g_0),$$

However, this inequality does not appear to be sharp.

In this paper, we will look for *upper bounds* for the Steklov eigenvalues of hypersurfaces of revolution. First, we recall that there exists a bound $B_n^k(L)$ such that for all metrics of revolution g on M, we have $\sigma_k(M, g) < B_n^k(L)$. Indeed, Proposition 3.3 of [4] states that if $M = [0, L] \times \mathbb{S}^{n-1}$ is a hypersurface of revolution, then we have

$$\sigma_k(M) \leq \left(1 + \frac{L}{2}\right)^{n-1} \sigma_k(C_L, dr^2 + g_0).$$

As such, a natural question is the following:

Given the dimension $n \geq 3$ and the meridian length L of M, does a metric of revolution g^* on M exist, such that $\sigma_k(M, g) \leq \sigma_k(M, g^*)$ for all metrics of revolution g on M?

Our investigations show that the answer is negative. Indeed, a sharp upper bound $B_n^k(L)$ exists, but no metric of revolution on $M = [0, L] \times \mathbb{S}^{n-1}$ achieves the equality case. However, there exists a non-smooth metric g^*, that we will call a *degenerated maximizing metric*, which maximizes the kth Steklov eigenvalue, for each $k \in \mathbb{N}$. This metric is non-smooth, therefore g^* is not a metric of revolution on M in the sense of Definition 1. Endowed with this metric, (M, g^*) can be seen as two annuli glued together; we provide more information about this degenerated maximizing metric g^* and the geometric representation of (M, g^*) in Sect. 3.

We state our first result:

Theorem 2 *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g_1)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and meridian length L. We suppose $n \geq 3$. Then there exists a metric of revolution g_2 on M such that for each $k \geq 1$,*

$$\sigma_k(M, g_1) < \sigma_k(M, g_2).$$

This result implies that among all metrics of revolution on M, none maximizes the kth non zero Steklov eigenvalue. Nevertheless, given any metric of revolution g_1 on M, we can iterate Theorem 2 to generate a sequence of metrics $(g_i)_{i=1}^{\infty}$ on M. This sequence converges to a unique non-smooth metric g^* on M, which is quite simple (see Sect. 3) and which maximizes the kth Steklov eigenvalue. That is why we call g^* the degenerated maximizing metric. Hence, as we search for the optimal bounds $B_n^k(L)$, we must use information contained in g^*.

We start by studying the case $k = 1$. We fix $n \geq 3$ and $L > 0$ and search for a sharp upper bound $B_n(L)$ for $\sigma_1(M, g)$. In this case, we are able to calculate an expression for $B_n(L)$:

Theorem 3 *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and dimension $n \geq 3$. Then the first non trivial Steklov eigenvalue $\sigma_1(M, g)$ is bounded above, by a bound that depends only on the dimension n and the meridian length L of M:*

$$\sigma_1(M, g) < B_n(L) := \min \left\{ \frac{(n-2)\,(1+L/2)^{n-2}}{(1+L/2)^{n-2}-1}, \frac{(n-1)\left((1+L/2)^n - 1\right)}{(1+L/2)^n + n - 1} \right\}.$$

Moreover, this bound is sharp: for each $\varepsilon > 0$, there exists a metric of revolution g_ε on M such that $\sigma_1(M, g_\varepsilon) > B_n(L) - \varepsilon$.

We have the following asymptotic behaviour:

$$B_n(L) \underset{L\to\infty}{\longrightarrow} n-2$$
$$B_n(L) \underset{L\to 0}{\longrightarrow} 0,$$

see Fig. 4.

We also study the function $L \longmapsto B_n(L)$. This allows us to find a sharp upper bound B_n such that for all meridian lengths $L > 0$ and metrics of revolution g on M, we have $\sigma_1(M, g) < B_n$:

Corollary 4 *Let $n \geq 3$. Then there exists a bound $B_n < \infty$ such that for all hypersurfaces of revolution (M, g) in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$, we have*

$$\sigma_1(M, g) < B_n := \frac{(n-2)\left(1+\frac{L_1}{2}\right)^{n-2}}{\left(1+\frac{L_1}{2}\right)^{n-2} - 1},$$

where L_1 is the unique real positive solution of the equation

$$(1+L/2)^{2n-2} - (n-1)(1+L/2)^n - (n-1)^2(1+L/2)^{n-2} + n - 1 = 0.$$

Moreover, this bound is sharp: for each $\varepsilon > 0$, there exists a hypersurface of revolution with two boundary components each isometric to a unit sphere $(M_\varepsilon, g_\varepsilon)$ such that $\sigma_1(M_\varepsilon, g_\varepsilon) > B_n - \varepsilon$.

We say that L_1 is a *critical length* associated with $k = 1$, see Definition 8.

Proposition 5 *Let $n \geq 3$, and let $L_1 = L_1(n)$ be the critical length associated with $k = 1$. Then we have:*

$$\lim_{n\to\infty} L_1(n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} B_n = \infty.$$

Note that the behaviour of L_1 is surprising since we know that when n is fixed, then $L \ll 1$ implies $\sigma_1(M, g) \ll 1$. Indeed, by [4, Prop. 3.3], we have

$$\sigma_1(M) \leq \left(1 + \frac{L}{2}\right)^{n-1} \sigma_1(C_L) \underset{L\to 0}{\longrightarrow} 0.$$

Now that we have provided information about sharp upper bounds for $\sigma_1(M, g)$, it is natural to wonder what kind of stability properties the hypersurfaces of revolution possess. A first interesting question is the following:

Given the information that $\sigma_1(M = [0, L] \times \mathbb{S}^{n-1}, g)$ is close to the sharp upper bound B_n, can we conclude that the meridian length L of M is close to the critical length L_1?

The answer to this question is positive. Indeed we will prove that if L is not close to L_1, then $\sigma_1(M, g)$ is not close to B_n. Additionally, given the information that $\sigma_1(M, g)$ is δ-close to B_n, we will show that the distance between L and L_1 is less than δ, up to a constant of proportionality which depends only on the dimension n.

Theorem 6 *Let $M = [0, L] \times \mathbb{S}^{n-1}$, with $L > 0$ and $n \geq 3$. We suppose $L \neq L_1$. Then there exists a constant $C(n, L) > 0$ such that for all metrics of revolution g on M, we have*

$$B_n - \sigma_1(M, g) \geq C(n, L).$$

Moreover, there exists a constant $C(n) > 0$ such that for all $0 < \delta < \frac{B_n - (n-2)}{2}$, we have

$$|B_n - \sigma_1(M, g)| < \delta \implies |L_1 - L| < C(n) \cdot \delta.$$

We also consider the following question about stability properties:

Given the information that $\sigma_1(M, g)$ is close to the sharp upper bound $B_n(L)$, can we conclude that the metric of revolution g is close (in a sense that is defined below) to the degenerated maximizing metric g^*?

We prove that if g is not close to g^*, then $\sigma_1(M, g)$ is not close to $B_n(L)$. For this purpose, given $m \in [1, 1 + L/2)$, we define

$$\mathcal{M}_m := \{\text{metrics of revolution } g \text{ on } M \\ \text{induced by a function } h \text{ such that } \max_{r\in[0,L]} \{h(r)\} \leq m\}.$$

The collection $\mathcal{M}_m$ can be thought of the set of all metrics of revolution that are not close to the degenerated maximizing metric g^*, where the qualitative appreciation of the word "close" is given by the parameter m. The larger m is, the closer to g^* the metrics in $\mathcal{M}_m$ can be. We get the following result:

Theorem 7 *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and dimension $n \geq 3$. Let $m \in [1, 1 + L/2)$ and $\mathcal{M}_m$ as above. Then there exists a constant $C(n, L, m) > 0$ such that for all $g \in \mathcal{M}_m$, we have*

$$B_n(L) - \sigma_1(M, g) \geq C(n, L, m).$$

These results solve the case $k = 1$. Therefore, it would be interesting to find the same kind of results for any $k \geq 1$. After having calculated sharp upper bounds for some higher values of k in Sects. 6.1 and 6.2, we will see that in order to get an expression for $B_n^k(L)$, we need to distinguish between many cases. As such, giving a general formula for $B_n^k(L)$ or $B_n^k := \sup_{L \in \mathbb{R}_+^*} \{B_n^k(L)\}$ via this method seems difficult. We discuss this in Remark 20.

Definition 8 We say that $L_k \in \mathbb{R}_+^*$ is a finite critical length associated with k if we have $B_n^k = B_n^k(L_k)$. We say that k has a critical length at infinity if it satisfies $B_n^k = \lim_{L \to \infty} B_n^k(L)$.

These lengths are critical in the following sense: if $L_k \in \mathbb{R}_+^*$ is a finite critical length for a certain $k \in \mathbb{N}$ and if we write g^* the degenerated maximizing metric on $M_k = [0, L_k] \times \mathbb{S}^{n-1}$, then

$$B_n^k = \sigma_k(M_k, g^*).$$

Given $n \geq 3$, there exist some k which have a finite critical length associated with them. Indeed, thanks to Corollary 4, we know that $k = 1$ has this property. Moreover, we know that there exist some k which have a critical length at infinity, see Sect. 6.1.

Since we want to study upper bounds for the Steklov eigenvalues, it is then natural to ask what qualitative and quantitative information we can provide about these critical lengths.

We get the following result:

Theorem 9 *Let $n \geq 3$. Then there exist infinitely many $k \in \mathbb{N}$ which have a finite critical length associated with them. Moreover, if we call $(k_i)_{i=1}^{\infty} \subset \mathbb{N}$ the increasing sequence of such k and if we call $(L_i)_{i=1}^{\infty}$ the associated sequence of finite critical lengths, then we have*

$$\lim_{i \to \infty} L_i = 0.$$

The existence of finite critical lengths is something surprising when we compare with what happens in the case of hypersurfaces of revolution with one boundary component. Indeed, using our vocabulary, we can state that in the case of hypersurfaces of revolution with one boundary component, each $k \in \mathbb{N}$ has a critical length at infinity, see [5, Prop. 7]. Nevertheless, in our case, Theorem 9 guarantees that there exist infinitely many $k \in \mathbb{N}$ which have a finite critical length associated with them. Moreover, we will show in Sect. 6.1 that there exist some k which have a critical length at infinity. However, we do not know if there are *infinitely many* of them. This consideration leads to the following open question (Question 22):

Given $n \geq 3$, are there finitely or infinitely many $k \in \mathbb{N}$ such that k has a critical length at infinity?

Plan of the paper. In Sect. 2, we recall the variational characterizations of the Steklov eigenvalues before giving the expression of eigenfunctions on hypersurfaces of revolution, and we introduce the notion of mixed Steklov–Dirichlet and Steklov–Neumann problems and state some propositions about them. We will then have enough information to prove Theorem 2 in Sect. 3. This will allow us to prove Theorem 3, Corollary 4 and Proposition 5 in Sect. 4. Then we prove the stability properties of hypersurfaces of revolution, i.e Theorem 6 and Theorem 7 in Sect. 5. We continue by performing some calculation for sharp upper bounds for higher eigenvalues in Sect. 6. We conclude by proving Theorem 9 in Sect. 7.

2 Variational characterization of the Steklov eigenvalues and mixed problems

We state some general facts about Steklov eigenfunctions and define the mixed Steklov–Dirichlet and Steklov–Neumann problems.

2.1 Variational characterization of the Steklov eigenvalues

Let (M, g) be a Riemannian manifold with smooth boundary Σ. Then we can characterize the kth Steklov eigenvalue of M by the following formula:

$$\sigma_k(M, g) = \min\left\{R_g(f) : f \in H^1(M),\ f \perp_\Sigma f_0, f_1, \dots, f_{k-1}\right\}, \tag{1}$$

where

$$R_g(f) = \frac{\int_M |\nabla f|^2 dV_g}{\int_\Sigma |f|^2 dV_\Sigma}$$

is called the Rayleigh quotient and

$$f \perp_\Sigma f_i \iff \int_\Sigma f f_i dV_\Sigma = 0.$$

Another way to characterize the kth eigenvalue of M is given by the Min-Max principle:

$$\sigma_k(M, g) = \min_{E \in \mathcal{H}_{k+1}(M)} \max_{0 \neq f \in E} R_g(f), \tag{2}$$

where $\mathcal{H}_{k+1}$ is the set of all $(k+1)$-dimensional subspaces in the Sobolev space $H^1(M)$.

We state now a proposition that provides us with information about the expression of the Steklov eigenfunctions of a hypersurface of revolution.

We denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow \infty$ the spectrum of the Laplacian on $(\mathbb{S}^{n-1}, g_0)$ and we consider $(S_j)_{j=0}^{\infty}$ an orthonormal basis of eigenfunctions associated to $(\lambda_j)_{j=0}^{\infty}$.

Proposition 10 *Let (M, g) be a hypersurface of revolution as in Definition 1. Then each eigenfunction on M can be written as $f_k(r, p) = u_l(r)S_j(p)$, where u_l is a smooth function on* $[0, L]$.

This property is well known for warped product manifolds (and thus for our case of hypersurfaces of revolution) and it is used often, see for example [7, Remark 1.1], [8, Lemma 3], [11, Prop. 3.16] or [12, Prop. 9].

2.2 Mixed problems and their variational characterizations

Let $(N, \partial N)$ be a smooth compact connected Riemannian manifold and $A \subset N$ be a domain which satisfies $\partial N \subset \partial A$. We suppose that ∂A is smooth and we call $\partial_{int} A$ the intersection of ∂A with the interior of N.

Definition 11 The Steklov–Dirichlet problem on A is the eigenvalue problem

$$\begin{cases} \Delta f = 0 & \text{in } A \\ \partial_\nu f = \sigma f & \text{on } \partial N \\ f = 0 & \text{on } \partial_{int} A. \end{cases}$$

It is well known that this mixed problem possesses solutions that form a discrete sequence

$$0 < \sigma_0^D(A) \leq \sigma_1^D(A) \leq \cdots \nearrow \infty.$$

The variational characterization of the kth Steklov–Dirichlet eigenvalue is the following:

$$\sigma_k^D(A) = \min_{E \in \mathcal{H}_{k+1,0}(A)} \max_{0 \neq f \in E} \frac{\int_A |\nabla f|^2 dV_A}{\int_\Sigma |f|^2 dV_\Sigma},$$

where $\mathcal{H}_{k+1,0}$ is the set of all $(k+1)$-dimensional subspaces in the Sobolev space

$$H_0^1(A) = \{f \in H^1(A) : f = 0 \text{ on } \partial_{int} A\}.$$

Definition 12 The Steklov–Neumann problem on A is the eigenvalue problem

$$\begin{cases} \Delta f = 0 & \text{in } A \\ \partial_\nu f = \sigma f & \text{on } \partial N \\ \partial_\nu f = 0 & \text{on } \partial_{int} A. \end{cases}$$

It is well known that this mixed problem possesses solutions that form a discrete sequence

$$0 = \sigma_0^N(A) \leq \sigma_1^N(A) \leq \cdots \nearrow \infty.$$

The variational characterization of the kth Steklov–Neumann eigenvalue is the following:

$$\sigma_k^N(A) = \min_{E \in \mathcal{H}_{k+1}(A)} \max_{0 \neq f \in E} \frac{\int_A |\nabla f|^2 dV_A}{\int_\Sigma |f|^2 dV_\Sigma},$$

where $\mathcal{H}_{k+1}$ is the set of all $(k+1)$-dimensional subspaces in the Sobolev space $H^1(A)$.

2.3 Mixed problems on annular domains

Let $\mathbb{B}_1$ and $\mathbb{B}_R$ be the balls in $\mathbb{R}^n$, $n \geq 3$, with radius 1 and $R > 1$ respectively centered at the origin. The annulus A_R is defined as follows: $A_R = \mathbb{B}_R \backslash \overline{\mathbb{B}}_1$. We say that this annulus is of inner radius 1 and outer radius R. This particular kind of domain shall be useful in this paper.

For such domains, it is possible to compute $\sigma_{(k)}^D(A_R)$ explicitly, which is the (k)th eigenvalue of the Steklov–Dirichlet problem on A_R, counted without multiplicity.

We state here Proposition 4 of [5]:

Proposition 13 *For A_R as above, consider the Steklov–Dirichlet problem*

$$\begin{cases} \Delta f = 0 & \textit{in } A_R \\ \partial_\nu f = \sigma f & \textit{on } \partial \mathbb{B}_1 \\ f = 0 & \textit{on } \partial \mathbb{B}_R. \end{cases}$$

Then, for $k \geq 0$, the (k)th eigenvalue (counted without multiplicity) of this problem is

$$\sigma^D_{(k)}(A_R) = \frac{(k+n-2)R^{2k+n-2}+k}{R^{2k+n-2}-1}.$$

By [5, Prop. 4], it is possible to get the expression of the eigenfunctions of the Steklov–Dirichlet problem on an annular domain.

Lemma 14 *Each eigenfunction φ_l of the Steklov–Dirichlet problem on the annulus A_R can be expressed as $\varphi_l(r,p) = \alpha_l(r)S_l(p)$, where S_l is an eigenfunction for the l^{th} harmonic of the sphere $\mathbb{S}^{n-1}$.*

It is possible to compute $\sigma^N_{(k)}(A_R)$ explicitly, which is the (k)th eigenvalue of the Steklov–Neumann problem on A_R, counted without multiplicity.

We state now Proposition 5 of [5]:

Proposition 15 *For A_R as above, consider the Steklov–Neumann problem*

$$\begin{cases} \Delta f = 0 & \text{in } A_R \\ \partial_\nu f = \sigma f & \text{on } \partial\mathbb{B}_1 \\ \partial_\nu f = 0 & \text{on } \partial\mathbb{B}_R. \end{cases}$$

Then, for $k \geq 0$, the (k)th eigenvalue (counted without multiplicity) of this problem is

$$\sigma^N_{(k)}(A_R) = k\frac{(k+n-2)(R^{2k+n-2}-1)}{kR^{2k+n-2}+k+n-2}.$$

In the same manner as before, we have the following expression for the Steklov–Neumann eigenvalues, see [5, Prop. 5].

Lemma 16 *Each eigenfunction ϕ_l of the Steklov–Neumann problem on the annulus A_R can be expressed as $\phi_l(r,p) = \beta_l(r)S_l(p)$, where S_l is an eigenfunction for the l^{th} harmonic of the sphere $\mathbb{S}^{n-1}$.*

3 The degenerated maximizing metric

A particular case of hypersurfaces of revolution is the following: let $M = [0, L] \times \mathbb{S}^{n-1}$ be endowed with a metric of revolution $g(r, p) = dr^2 + h^2(r)g_0(p)$. Let us suppose that there exists $\varepsilon > 0$ such that $h(r) = 1+r$ on $[0, \varepsilon]$. Let us consider the connected component of the boundary $\mathbb{S}_0$ associated with $h(0)$. Then the ε-neighborhood of $\mathbb{S}_0$ is an annulus with inner radius 1 and outer radius $1 + \varepsilon$ (Fig. 2).

This particular case is the key idea that we use to prove Theorem 2. We prove it now.

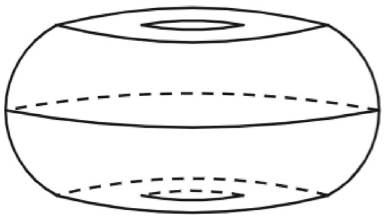

Fig. 2 On $[0, \varepsilon]$, we have $h(r) = 1 + r$ and on $[L - \varepsilon, L]$, we have $h(r) = -r + L + 1$. This implies that the ε-neighborhood of the boundary consists of two disjoint copies of an annulus with inner radius 1 and outer radius $1 + \varepsilon$

Proof We write $g_1(r, p) = dr^2 + h_1^2(r)g_0(p)$. Because h_1 is smooth and $|h_1'| \leq 1$, we have $h_1(r) < 1 + \frac{L}{2}$ for all $r \in [0, L]$. Since h_1 is continuous and $[0, L]$ is compact, h_1 reaches its maximum on $[0, L]$. We call

$$m := \max_{r\in[0,L]} \{h_1(r)\}.$$

Notice that $1 \leq m < 1 + \frac{L}{2}$.

We define a smooth function $h_2 : [0, L] \longrightarrow \mathbb{R}$ by

$$h_2(r) = \begin{cases} 1 + r & \text{if } 0 \leq r \leq m - 1 \\ 1 + L - r & \text{if } L - m + 1 \leq r \leq L. \end{cases}$$

For $r \in (m - 1, L - m + 1)$, we only require that $h_2(r) > m$, that $h_2(L/2) = \frac{1+L/2+m}{2}$ and that

$$g_2(r, p) := dr^2 + h_2^2(r)g_0(p)$$

is a symmetric metric of revolution on M, i.e for all $r \in [0, L]$, we have $h_2(r) = h_2(L - r)$. Note that we have $h_2 \geq h_1$ and that for $r \in (m - 1, L - m + 1)$ we have $h_2(r) > h_1(r)$.

Besides, for f a smooth function on M, we have

$$R_{g_1}(f) = \frac{\int_M |\nabla f|_{g_1}^2 dV_{g_1}}{\int_\Sigma |f|^2 dV_\Sigma} = \frac{\int_M \left((\partial_r f)^2 + \frac{1}{h_1^2}|\tilde{\nabla} f|_{g_0}^2\right) h_1^{n-1} dV_{g_0} dr}{\int_\Sigma |f|^2 dV_\Sigma}$$

and

$$R_{g_2}(f) = \frac{\int_M |\nabla f|_{g_2}^2 dV_{g_2}}{\int_\Sigma |f|^2 dV_\Sigma} = \frac{\int_M \left((\partial_r f)^2 + \frac{1}{h_2^2}|\tilde{\nabla} f|_{g_0}^2\right) h_2^{n-1} dV_{g_0} dr}{\int_\Sigma |f|^2 dV_\Sigma},$$

where $\tilde{\nabla} f$ is the gradient of f seen as a function of p.

Since $n \geq 3$, for all functions $f \in H^1(M)$, we have $R_{g_1}(f) \leq R_{g_2}(f)$. Using the Min-Max principle, we can conclude that for all $k \geq 1$, we have $\sigma_k(M, g_1) \leq \sigma_k(M, g_2)$. However, here we want to show a strict inequality.

Because of the existence of a continuum of points r for which $h_1(r) < h_2(r)$, if $\partial_r f$ does not vanish on any interval, then the inequality is strict.

Let $k \geq 1$ be an integer. Let $E_{k+1} := Span(f_{0,2}, \ldots, f_{k,2})$, where $f_{i,2}$ is a Steklov eigenfunction associated with $\sigma_i(M, g_2)$. We can choose these functions such that for all $i = 0, \ldots, k$, we have

$$\int_\Sigma (f_{i,2})^2 dV_\Sigma = 1,$$

and hence

$$\int_M |\nabla f_{i,2}|_{g_2}^2 dV_{g_2} = \sigma_i(M, g_2).$$

Let $f^* = \sum_{i=0}^k a_i f_{i,2} \in E_{k+1}$ be such that $\max_{f\in E_{k+1}} R_{g_1}(f) = R_{g_1}(f^*)$.

We now consider two cases:

1. Let us suppose $f^* = a_k f_{k,2}$ with $a_k \neq 0$, i.e f^* is an eigenfunction associated with $\sigma_k(M, g_2)$. Then by Proposition 10, we have $f^*(r, p) = u_j(r)S_j(p)$. Moreover, using [5, Prop. 2], we know that u_j is a non trivial solution of the ODE

$$\frac{1}{h^{n-1}}\frac{d}{dr}\left(h^{n-1}\frac{d}{dr}u_j\right) - \frac{1}{h_2^2}\lambda_j u_j = 0.$$

   (a) If $\lambda_j = 0$, which means $S_j = S_0 = const$, then u_j cannot be locally constant. Indeed, otherwise f^* would be locally constant, but since f^* is harmonic, this implies that f^* is constant, see [1]. That is not the case because $k \geq 1$.
   (b) If $\lambda_j \neq 0$, then u_j cannot be locally constant, otherwise the ODE is not satisfied.

   Hence u_j is not locally constant and then $\partial_r f^*$ does not vanish on any interval. Therefore, using the Min-Max principle (2), we have

$$\sigma_k(M, g_1) \leq \max_{f\in E_{k+1}} R_{g_1}(f) = R_{g_1}(f^*) < R_{g_2}(f^*) = \sigma_k(M, g_2).$$

2. Let us suppose $f^* = \sum_{i=0}^{k} a_i f_{i,2}$ such that there exists $0 \leq i < k$ such that $a_i \neq 0$. Then by the Min-Max principle (2), we have

$$\begin{aligned}\sigma_k(M, g_1) &\leq \max_{f\in E_{k+1}} R_{g_1}(f) = R_{g_1}(f^*) \leq R_{g_2}(f^*)\\ &= \frac{\int_M \sum_{i=0}^{k} a_i^2 |\nabla f_{i,2}|^2 dV_{g2}}{\int_\Sigma (\sum_{i=0}^{k} a_i f_{i,2})^2 dV_\Sigma}\\ &= \frac{\sum_{i=0}^{k} a_i^2 \sigma_i(M, g_2)}{\sum_{i=0}^{k} a_i^2} \quad \text{since } \int_\Sigma f_{i,2} f_{j,2} dV_\Sigma = \delta_{ij}\\ &< \sigma_k(M, g_2).\end{aligned}$$

   In both cases, we have

$$\sigma_k(M, g_1) < \sigma_k(M, g_2).$$

□

Remark 17 We never used the assumption that g_2 is a *symmetric* metric of revolution on M in the previous proof. However, it will be useful in the proofs of the theorems that follow.

The process that constructs the metric g_2 from g_1 can then be repeated to create a third metric g_3, and so on. This generates a sequence of metrics (g_i), obtained from a sequence of functions (h_i) (Fig. 3).

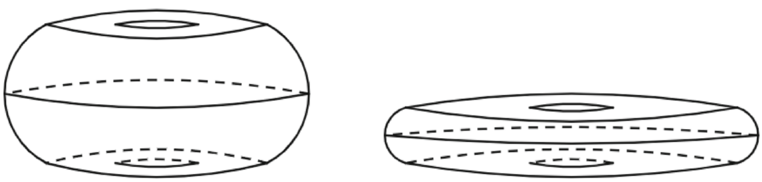

Fig. 3 On the left, $M = [0, L] \times \mathbb{S}^{n-1}$ is endowed with a metric g_i of the sequence. On the right, $M = [0, L] \times \mathbb{S}^{n-1}$ is endowed with another metric g_j of the sequence, $j > i$

The sequence (h_i) uniformly converges to the function

$$h^* : [0, L] \longrightarrow \mathbb{R}$$
$$r \longmapsto \begin{cases} 1 + r & \text{if } 0 \le r \le L/2 \\ 1 + L - r & \text{if } L/2 \le r \le L. \end{cases}$$

This function is not smooth. Hence (M, g^*), where $g^* = dr^2 + h^{*2}(r)g_0$, is not a hypersurface of revolution in the sense of Definition 1. In the limit, (M, g^*) can be seen as the gluing of two copies of an annulus of inner radius 1 and outer radius $1 + L/2$. The metric g^* is therefore a maximizing metric, but is degenerated since it is induced by the function h^* which is non-smooth. That is why, as already mentioned, we call g^* the *degenerated maximizing metric on* M.

4 The first non trivial eigenvalue

In this section, we prove Theorem 3. The idea consists of comparing $\sigma_1(M, g)$ with the Rayleigh quotient of a test function that is obtained from an eigenfunction for a mixed problem (Steklov–Dirichlet or Steklov–Neumann) introduced in Sect. 2.2. Then, to show that the upper bound $B_n(L)$ given is sharp, we take a metric of revolution g_ε on M that is close to the degenerated maximizing metric g^* and show that $\sigma_1(M, g_\varepsilon)$ is close to $B_n(L)$.

Proof Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution, where $L > 0$ is the meridian length of M. We recall that the boundary Σ of M consists of two disjoint copies of $\mathbb{S}^{n-1}$. We want to find a sharp upper bound $B_n(L)$ for $\sigma_1(M, g)$.

We consider $A_{1+L/2}$ the annulus of inner radius 1 and outer radius $1 + L/2$. Let φ_0 be an eigenfunction for the first eigenvalue of the Steklov–Dirichlet problem on $A_{1+L/2}$, i.e.

$$\sigma_0^D(A_{1+L/2}) = \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \varphi_0)^2 + \frac{1}{(1+r)^2} |\tilde{\nabla} \varphi_0|^2 \right) (1 + r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \varphi_0^2(0, p) dV_{g_0}}.$$

We define a new function

$$\tilde{\varphi_0} : [0, L] \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$$
$$(r, p) \longmapsto \begin{cases} \varphi_0(r, p) & \text{if } 0 \le r \le L/2 \\ -\varphi_0(L - r, p) & \text{if } L/2 \le r \le L. \end{cases} \tag{3}$$

The function $\tilde{\varphi_0}$ is continuous and we can check that

$$\begin{aligned} \int_\Sigma \tilde{\varphi_0}(r, p) dV_\Sigma &= \int_{\mathbb{S}^{n-1}} \tilde{\varphi_0}(0, p) dV_{g_0} + \int_{\mathbb{S}^{n-1}} \tilde{\varphi_0}(L, p) dV_{g_0} \\ &= \int_{\mathbb{S}^{n-1}} \varphi_0(0, p) dV_{g_0} - \int_{\mathbb{S}^{n-1}} \varphi_0(0, p) dV_{g_0} \\ &= 0. \end{aligned}$$

Hence, thanks to formula (1), the function $\tilde{\varphi_0}$ can be used as a test function for $\sigma_1(M, g)$.

We have

$$\begin{aligned} \sigma_1(M, g) &\le R_g(\tilde{\varphi_0}) \\ &< R_{\tilde{g}}(\tilde{\varphi_0}) \quad \text{where } \tilde{g} = dr^2 + \tilde{h}^2 g_0 \text{ comes from Theorem 2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\int_0^L \int_{\mathbb{S}^{n-1}} \left((\partial_r \tilde{\varphi_0})^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \tilde{\varphi_0}|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{\int_\Sigma \tilde{\varphi_0}^2(0,p) dV_\Sigma} \\
&= \frac{2 \times \int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \tilde{\varphi_0})^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \tilde{\varphi_0}|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{2 \times \int_{\mathbb{S}^{n-1}} \tilde{\varphi_0}^2(0,p) dV_{g_0}} \quad \text{since } \tilde{g} \text{ is symmetric} \\
&= \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \varphi_0)^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \varphi_0|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \varphi_0^2(0,p) dV_{g_0}} \\
&< \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \varphi_0)^2 + \frac{1}{(1+r)^2} |\tilde{\nabla} \varphi_0|^2 \right) (1+r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \varphi_0^2(0,p) dV_{g_0}} \\
&= \sigma_0^D(A_{1+L/2}), \qquad (4)
\end{aligned}
$$

where the second strict inequality comes from the existence of a continuum of points $r \in [0, L/2]$ such that $\tilde{h}(r) < 1 + r$.

If ϕ_1 is an eigenfunction for the first non trivial eigenvalue of the Steklov–Neumann problem on $A_{1+L/2}$, i.e

$$
\sigma_1^N(A_{1+L/2}) = \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \phi_1)^2 + \frac{1}{(1+r)^2} |\tilde{\nabla} \phi_1|^2 \right) (1+r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \phi_1^2(0,p) dV_{g_0}},
$$

then we define a new function

$$
\begin{aligned}
\tilde{\phi_1} : [0, L] \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{R} \\
(r, p) &\longmapsto \begin{cases} \phi_1(r,p) & \text{if } 0 \le r \le L/2 \\ \phi_1(L-r,p) & \text{if } L/2 \le r \le L. \end{cases}
\end{aligned}
$$

The function $\tilde{\phi_1}$ is continuous and we can check that

$$
\begin{aligned}
\int_\Sigma \tilde{\phi_1}(r,p) dV_\Sigma &= \int_{\mathbb{S}^{n-1}} \tilde{\phi_1}(0,p) + \int_{\mathbb{S}^{n-1}} \tilde{\phi_1}(L,p) \\
&= 0 + 0 \\
&= 0,
\end{aligned}
$$

hence we can use it as a test function for $\sigma_1(M, g)$. The same calculations as in (4) show that

$$
\sigma_1(M, g) < \sigma_1^N(A_{1+L/2}). \qquad (5)
$$

Putting Inequality (4) and Inequality (5) together, we get

$$
\begin{aligned}
\sigma_1(M, g) < B_n(L) :&= \min \left\{ \sigma_0^D(A_{1+L/2}),\ \sigma_1^N(A_{1+L/2}) \right\} \\
&= \min \left\{ \frac{(n-2)(1+L/2)^{n-2}}{(1+L/2)^{n-2} - 1}, \frac{(n-1)\left((1+L/2)^n - 1\right)}{(1+L/2)^n + n - 1} \right\}. \qquad (6)
\end{aligned}
$$

We will now prove that the bound $B_n(L)$ is sharp. This means that for each $\varepsilon > 0$, there exists a metric of revolution g_ε on M such that $\sigma_1(M, g_\varepsilon) > B_n(L) - \varepsilon$.

Let $\varepsilon > 0$. Let $M = [0, L] \times \mathbb{S}^{n-1}$ and let $g_\varepsilon(r, p) = dr^2 + h_\varepsilon^2(r) g_0(p)$ be a metric of revolution on M such that:

1. The function h_ε is symmetric: for all $r \in [0, L]$, we have $h_\varepsilon(r) = h_\varepsilon(L - r)$;

2. For all $r \in [0, L/2-\delta]$, we have $h_\varepsilon(r) = (1+r)$, with δ small enough to guarantee that for all $r \in [0, L/2]$, we have

$$\max\{(1+r)^{n-3} - h_\varepsilon(r)^{n-3}, (1+r)^{n-1} - h_\varepsilon(r)^{n-1}\} < \frac{\varepsilon}{B_n(L)} =: \varepsilon^*.$$

   Geometrically, this means that (M, g_ε) looks like two copies of an annulus joined by a smooth curve, see Fig. 3.

Let f_1 be an eigenfunction for $\sigma_1(M, g_\varepsilon)$. Because (M, g_ε) is symmetric, then we can choose f_1 symmetric or anti-symmetric, which means that for all $r \in [0, L]$ and $p \in \mathbb{S}^{n-1}$, we have $|f_1(r, p)| = |f_1(L-r, p)|$.

Moreover, it results from the calculations in (4) that for any symmetric or anti-symmetric function f, we have

$$R_{g_\varepsilon}(f) = \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}\left((\partial_r f)^2 + \frac{1}{h_\varepsilon(r)^2}|\tilde{\nabla} f|^2\right) h_\varepsilon(r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} f^2(0, p) dV_{g_0}}.$$

We will compare

$$R_{g_\varepsilon}(f_1) = \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}\left((\partial_r f_1)^2 + \frac{1}{h_\varepsilon(r)^2}|\tilde{\nabla} f_1|^2\right) h_\varepsilon(r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} (f_1)^2(0, p) dV_{g_0}}$$

with

$$R_{A_{1+L/2}}(f_1) = \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}}\left((\partial_r f_1)^2 + \frac{1}{(1+r)^2}|\tilde{\nabla} f_1|^2\right) (1+r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} (f_1)^2(0, p) dV_{g_0}}.$$

If we call $S := R_{A_{1+L/2}}(f_1) - R_{g_\varepsilon}(f_1)$, we have

$$\begin{aligned}
S &= \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}} (\partial_r f_1)^2\left((1+r)^{n-1} - h_\varepsilon(r)^{n-1}\right) + |\tilde{\nabla} f_1|^2\left((1+r)^{n-3} - h_\varepsilon(r)^{n-3}\right) dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} (f_1)^2(0, p) dV_{g_0}} \\
&< \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}} ((\partial_r f_1)^2 \cdot \varepsilon^* + |\tilde{\nabla} f_1|^2 \cdot \varepsilon^*) dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} (f_1)^2(0, p) dV_{g_0}} \\
&= \varepsilon^* \cdot \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}} ((\partial_r f_1)^2 + |\tilde{\nabla} f_1|^2) dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} (f_1)^2(0, p) dV_{g_0}} \\
&< \varepsilon^* \cdot \frac{\int_0^{L/2}\int_{\mathbb{S}^{n-1}} ((\partial_r f_1)^2 h_\varepsilon(r)^{n-1} + |\tilde{\nabla} f_1|^2 h_\varepsilon(r)^{n-3}) dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} (f_1)^2(0, p) dV_{g_0}} \quad \text{since } h_\varepsilon \geq 1 \\
&= \varepsilon^* \cdot \sigma_1(M, g_\varepsilon) \quad \text{since } f_1 \text{ is an eigenfunction} \\
&< \varepsilon^* \cdot B_n(L) \\
&= \varepsilon.
\end{aligned}$$

Hence, we have

$$R_{A_{1+L/2}}(f_1) < \sigma_1(M, g_\varepsilon) + \varepsilon. \tag{7}$$

We now have two cases:

1. f_1 can be written as $f_1(r, p) = u_0(r)S_0(p)$, where S_0 is a trivial harmonic function of the sphere, i.e S_0 is constant (we can choose $S_0 \equiv 1/\mathrm{Vol}(\mathbb{S}^{n-1})$), and u_0 is smooth. Hence f_1 is constant on $\{0\} \times \mathbb{S}^{n-1}$,
$$\int_{\{0\}\times\mathbb{S}^{n-1}} f_1(r, p) dV_{g_0} = u_0(0) \neq 0.$$
Moreover, since $|f_1(r, p)| = |f_1(L - r, p)|$ for all $r \in [0, L]$ and since
$$\int_\Sigma f_1(r, p) dV_\Sigma = 0,$$
we have
$$f_1\left(\frac{L}{2}, p\right) = 0.$$
Therefore, we can use $f_1|_{[0,L/2]\times\mathbb{S}^{n-1}}$ as a test function for $\sigma_0^D(A_{1+L/2})$, and we can state
$$\sigma_0^D(A_{1+L/2}) \leq R_{A_{1+L/2}}(f_1).$$
2. f_1 can be written as $f_1(r, p) = u_1(r)S_1(p)$, where S_1 is a non constant harmonic function of the sphere associated with the first non zero eigenvalue and u_1 is smooth. Hence
$$\int_{\{0\}\times\mathbb{S}^{n-1}} f_1(r, p) dV_{g_0} = 0.$$
Moreover, we have $u_1(L/2) > 0$.
Added with the fact that $|f_1(r, p)| = |f_1(L - r, p)|$ for all $r \in [0, L]$ and because f_1 is smooth, we can conclude
$$\partial_r f_1\left(\frac{L}{2}, p\right) = 0.$$
Therefore, we can use $f_1|_{[0,L/2]\times\mathbb{S}^{n-1}}$ as a test function for $\sigma_1^N(A_{1+L/2})$ and we can state
$$\sigma_1^N(A_{1+L/2}) \leq R_{A_{1+L/2}}(f_1).$$

But we defined $B_n(L)$ as
$$B_n(L) = \min\{\sigma_0^D(A_{1+L/2}), \sigma_1^N(A_{1+L/2})\}.$$
Hence we have
$$B_n(L) \leq R_{A_{1+L/2}}(f_1) \overset{(7)}{<} \sigma_1(M, g_\varepsilon) + \varepsilon$$
and then
$$\sigma_1(M, g_\varepsilon) > B_n(L) - \varepsilon.$$
□

From this result we can prove Corollary 4.

Proof By Theorem 3, the inequality (6) holds which is
$$\sigma_1(M, g) < \min\left\{\frac{(n-2)\,(1 + L/2)^{n-2}}{(1 + L/2)^{n-2} - 1}, \frac{(n-1)\left((1 + L/2)^n - 1\right)}{(1 + L/2)^n + n - 1}\right\}.$$

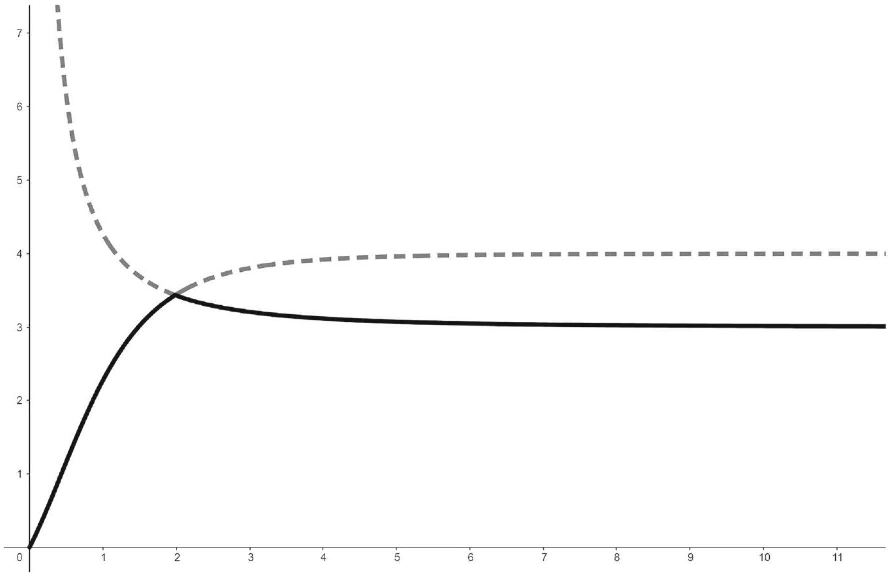

Fig. 4 Representation of the case $n = 5$. The decreasing smooth curve is $L \longmapsto \sigma_0^D(A_{1+L/2})$ while the increasing smooth curve is $L \longmapsto \sigma_1^N(A_{1+L/2})$. The solid curve is the bound $B_5(L)$ given by Theorem 3

We consider the two functions

$$L \longmapsto \frac{(n-2)\,(1+L/2)^{n-2}}{(1+L/2)^{n-2}-1} = \sigma_0^D(A_{1+L/2})$$
$$L \longmapsto \frac{(n-1)\,\big((1+L/2)^n-1\big)}{(1+L/2)^n+n-1} = \sigma_1^N(A_{1+L/2}).$$

We can show that $L \longmapsto \sigma_0^D(A_{1+L/2})$ is strictly decreasing with L (Fig. 4). Indeed, let $L' > L$ and let φ_0 be an eigenfunction for $\sigma_0^D(A_{1+L/2})$. We consider

$$\bar{\varphi}_0 : \left[0, \frac{L'}{2}\right] \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$$

the extension by 0 of φ_0 to the annulus $A_{1+L'/2}$. We get

$$\sigma_0^D(A_{1+L'/2}) < R_{A_{1+L/2}}(\bar{\varphi}_0) = R_{A_{1+L'/2}}(\varphi_0) = \sigma_0^D(A_{1+L/2}),$$

where the strict inequality comes from the fact that $\bar{\varphi}_0$ is not an eigenfunction associated with $\sigma_0^D(A_{1+L'/2})$. Indeed, if we suppose that $\bar{\varphi}_0$ is an eigenfunction for $\sigma_0^D(A_{1+L'/2})$, then it is harmonic in $A_{1+L'/2}$ (since it satisfies the Steklov–Dirichlet problem), and since $\bar{\varphi}_0$ vanishes on the open set $A_{1+L'/2} \backslash A_{1+L/2}$, then by [1] $\bar{\varphi}_0$ is constant, which is a contradiction.

In the same way, we can show that $L \longmapsto \sigma_1^N(A_{1+L/2})$ is strictly increasing with L (Fig. 4). Indeed, let $L' > L$ and let ϕ_1 be an eigenfunction for $\sigma_1^N(A_{1+L'/2})$. We consider

$$\bar{\phi}_1 : \left[0, \frac{L}{2}\right] \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$$

the restriction of ϕ_1 to the annulus $A_{1+L/2}$. We get

$$\sigma_1^N(A_{1+L/2}) \leq R_{A_{1+L/2}}(\bar{\phi}_1) < R_{A_{1+L'/2}}(\phi_1) = \sigma_1^N(A_{1+L'/2}).$$

Hence the bound we gave possesses a maximum depending only on the dimension n, given by

$$\sigma_1(M,g) < B_n := \frac{(n-2)\,(1+L_1/2)^{n-2}}{(1+L_1/2)^{n-2}-1}$$

where L_1 is the unique positive solution of the equation

$$(1+L/2)^{2n-2} - (n-1)\,(1+L/2)^n - (n-1)^2\,(1+L/2))^{n-2} + n - 1 = 0.$$

In order to prove that this bound is sharp, let $\varepsilon > 0$. We define $M_\varepsilon := [0, L_1] \times \mathbb{S}^{n-1}$. Theorem 3 guarantees that there exists a metric of revolution g_ε on M_ε such that $\sigma_1(M_\varepsilon, g_\varepsilon) > B_n(L_1) - \varepsilon = B_n - \varepsilon$, which ends the proof. □

We continue by proving Proposition 5.

Proof We know that there exists a unique positive value of L, that we call $L_1 = L_1(n)$, such that the equality

$$(1+L/2)^{2n-2} - (n-1)\,(1+L/2)^n - (n-1)^2\,(1+L/2))^{n-2} + n - 1 = 0$$

holds. To ease notation, we substitute $(1 + L/2)$ by R and we can state that there is a unique value of $R \in (1,\infty)$ such that the equality

$$R^{2n-2} - (n-1)R^n - (n-1)^2 R^{n-2} + n - 1 = 0$$

holds. This equation is equivalent to

$$R^{n-2}\left(R^n - (n-1)R^2 - (n-1)^2\right) + n - 1 = 0,$$

and we call $R_1 = R_1(n)$ its unique solution in $(1,\infty)$. We prove that $R_1(n) \underset{n\to\infty}{\longrightarrow} 1$.

We call

$$\psi_n(R) := R^n - (n-1)R^2 - (n-1)^2$$

and

$$\Psi_n(R) := R^{n-2}\left(R^n - (n-1)R^2 - (n-1)^2\right) + n - 1.$$

Then, for R_1 to be such that $\Psi_n(R_1) = 0$, it is necessary that $\psi_n(R_1) < 0$.

Thus,

$$\begin{aligned} R_1^n &< (n-1)R_1^2 + (n-1)^2 \\ &< (n-1)^2(R_1^2+1) \\ &< (n-1)^2 \cdot 2R_1^2 && \text{since } R_1 > 1 \\ &< (n-1)^3 R_1^2 && \text{since } n-1 \geq 2. \end{aligned}$$

Therefore,

$$n\ln(R_1) < 3\ln(n-1) + 2\ln(R_1)$$

so

$$\ln(R_1) < \frac{3\ln(n-1)}{n-2}$$

and

$$R_1 < e^{\frac{3\ln(n-1)}{n-2}}.$$

As we substituted $(1 + L/2)$ by R, and we can state that

$$L_1(n) < 2\left(e^{\frac{3\ln(n-1)}{n-2}} - 1\right).$$

Therefore, since $\frac{3\ln(n-1)}{n-2} \underset{n\to\infty}{\longrightarrow} 0$, we have

$$L_1(n) \underset{n\to\infty}{\longrightarrow} 0.$$

Moreover, we have

$$n - 1 > B_n > n - 2 \underset{n\to\infty}{\longrightarrow} \infty.$$

□

5 Stability properties of hypersurfaces of revolution

The goal of this section is to prove Theorems 6 and 7, which show some stability properties of the hypersurfaces we are studying in this paper. For Theorem 6, the key idea is to choose $L \neq L_1$ and compare $\sigma_1(M = [0, L] \times \mathbb{S}^{n-1}, g)$ with the first non trivial eigenvalue of M when endowed with the degenerated maximizing metric, namely $B_n(L)$. For the case of Theorem 7, the strategy consists of showing that among all metrics of revolution that are not close (in a sense properly defined) to the degenerated maximizing metric, none of them induces a first non trivial eigenvalue that is close to $B_n(L)$. We prove these theorems now.

5.1 Proof of Theorem 6

Recall that here we suppose $L \neq L_1$.

Proof Let g be any metric of revolution on $M = [0, L] \times \mathbb{S}^{n-1}$. Then we have

$$\sigma_1(M, g) < B_n(L),$$

where $B_n(L)$ is given by Theorem 3.

We define $C(n, L) := B_n - B_n(L)$, which is strictly positive since we assumed $L \neq L_1$. Then we have

$$B_n - \sigma_1(M, g) \geq B_n - B_n(L) = C(n, L).$$

Let $0 < \delta < \frac{B_n-(n-2)}{2}$, and let us suppose $|B_n - \sigma_1(M, g)| < \delta$. Therefore, we have $|B_n - \sigma_1(M, g^*)| < \delta$, where we wrote g^* the degenerated maximizing metric on M. We consider two cases:

1. We suppose $L_1 < L$. In this case, we have $B_n(L) = \sigma_0^D(A_{1+L/2}) = \frac{(n-2)(1+L/2)^{n-2}}{(1+L/2)^{n-2}-1}$. We write

$$R := 1 + L/2 \quad \text{and} \quad \sigma_1(R) := \frac{(n-2)R^{n-2}}{R^{n-2} - 1}.$$

Hence we have $|B_n - \sigma_1(R)| < \delta \implies R \in [R_1, R_\delta]$, where $R_1 = 1 + L_1/2$ and R_δ is defined by $\sigma_1(R_\delta) = B_n - \delta$. Note that R_δ exists since we assumed $\delta < B_n - (n-2)$. We can calculate that

$$R_\delta = \left(\frac{B_n - \delta}{B_n - (n-2) - \delta}\right)^{\frac{1}{n-2}} \quad \text{and} \quad R_1 = \left(\frac{B_n}{B_n - (n-2)}\right)^{\frac{1}{n-2}}.$$

Thus, we have

$$|R_1 - R| \leq R_\delta - R_1 = \left(\frac{B_n - \delta}{B_n - (n-2) - \delta}\right)^{\frac{1}{n-2}} - \left(\frac{B_n}{B_n - (n-2)}\right)^{\frac{1}{n-2}}.$$

To estimate this expression, we use the identity $x^{n-2} - y^{n-2} = (x-y)(x^{n-3} + x^{n-4}y + \cdots + xy^{n-4} + y^{n-3})$, with $x = R_\delta$ and $y = R_1$. On the one hand, we can compute that

$$R_\delta^{n-2} - R_1^{n-2} = \frac{(n-2)\delta}{(B_n - (n-2) - \delta)(B_n - (n-2))} \leq \frac{2(n-2)\delta}{(B_n - (n-2))^2},$$

where the inequality comes from the assumption $\delta < \frac{B_n - (n-2)}{2}$. On the other hand, we can compute that

$$R_\delta^{n-3} + R_\delta^{n-4} R_1 + \cdots + R_\delta R_1^{n-4} + R_1^{n-3} \geq (n-2) \cdot \left(\frac{B_n}{B_n - (n-2)}\right)^{\frac{n-3}{n-2}}.$$

Therefore,

$$R_\delta - R_1 \leq \frac{2/(B_n - (n-2))^2}{(B_n/(B_n - (n-2)))^{\frac{n-3}{n-2}}} \cdot \delta := C_1(n) \cdot \delta.$$

Since we wrote $R = 1 + L/2$, we can conclude that, for $L_1 < L$ and $0 < \delta < \frac{B_n - (n-2)}{2}$, we have

$$B_n - \sigma_1(M, g) < \delta \implies L - L_1 < 2C_1(n) \cdot \delta.$$

2. Now we suppose $L < L_1$ and we do a similar calculation, this time with $B_n(L) = \sigma_1^N(A_{1+L/2}) = \frac{(n-1)((1+L/2)^n - 1)}{(1+L/2)^n + n - 1}$. We obtain a constant

$$C_2(n) := \frac{(n-2)^2/(n-1-B_n)^2}{n\left(((n-1)B_n + 1)/(n-1-B_n)\right)^{\frac{1}{n}}}$$

such that

$$B_n - \sigma_1(M, g) < \delta \implies |L_1 - L| \leq 2C_2(n) \cdot \delta.$$

Defining $C(n) := 2 \cdot \max\{C_1(n), C_2(n)\}$ concludes the proof. □

5.2 Proof of Theorem 7

Recall that we fixed $m \in [1, 1 + L/2)$ and that we defined $\mathcal{M}_m :=$ {metrics of revolution g induced by a function h such that $\max_{r \in [0,L]}\{h(r)\} \leq m$}.

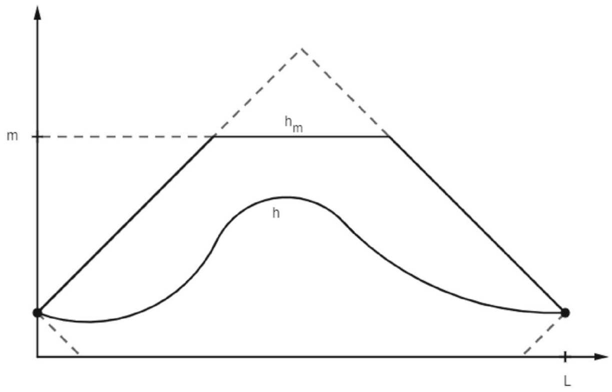

Fig. 5 Since $g \in \mathcal{M}_m$, the function h which induces g satisfies $h \leq h_m$

Proof Let $g \in \mathcal{M}_m$, and let $h : [0, L] \longrightarrow \mathbb{R}_+^*$ be the function which induces g. We define a new function $h_m : [0, L] \longrightarrow \mathbb{R}_+^*$ as follows:

$$h_m(r) = \begin{cases} 1 + r & \text{if } 0 \leq r \leq m - 1 \\ m & \text{if } m - 1 \leq r \leq L - m + 1 \\ 1 + L - r & \text{if } L - m + 1 \leq r \leq L. \end{cases}$$

We call g_m the metric induced by h_m. Notice that g_m is not a metric of revolution in the sense of Definition 1 since h_m is not smooth (Fig. 5).

In the same spirit as in Sect. 3, for any smooth function f on M, we have

$$R_g(f) = \frac{\int_M \left((\partial_r f)^2 + \frac{1}{h^2} |\tilde{\nabla} f|_{g_0}^2 \right) h^{n-1} dV_{g_0} dr}{\int_\Sigma |f|^2 dV_\Sigma}$$

and

$$R_{g_m}(f) = \frac{\int_M \left((\partial_r f)^2 + \frac{1}{h_m^2} |\tilde{\nabla} f|_{g_0}^2 \right) h_m^{n-1} dV_{g_0} dr}{\int_\Sigma |f|^2 dV_\Sigma}.$$

Therefore, since $n \geq 3$ and $h \leq h_m$, we have

$$\sigma_1(M, g) \leq \sigma_1(M, g_m).$$

We can now consider a new function $\tilde{h}_m$, obtained from h_m by smoothing out the two non-smooth points, with $\tilde{h}_m$ satisfying:

1. For all $r \in [0, L]$, we have $h_m(r) \leq \tilde{h}_m(r)$;
2. The metric $\tilde{g}_m$ induced by $\tilde{h}_m$ is a metric of revolution in the sense of Definition 1.

Remark that since $h_m \leq \tilde{h}_m$, we have $\sigma_1(M, g_m) \leq \sigma_1(M, \tilde{g}_m)$.

We define $C(n, L, m) := B_n(L) - \sigma_1(M, \tilde{g}_m)$, which is strictly positive by Theorem 3. Then we have

$$B_n(L) - \sigma_1(M, g) \geq B_n(L) - \sigma_1(M, \tilde{g}_m) = C(n, L, m).$$

$\square$

6 Upper bounds for higher Steklov eigenvalues

In this section, we want to compute some sharp upper bound for higher Steklov eigenvalues of hypersurfaces of revolution. Therefore, we will have to deal with the multiplicity of the eigenvalues. We write $\lambda_{(k)}, \sigma_{(k)}, \sigma^D_{(k)}, \sigma^N_{(k)}$ for the (k)th eigenvalue counted without multiplicity.

Before we can state and prove our results, we first recall some known properties of the multiplicities of the eigenvalues under consideration.

Given a hypersurface of revolution $(M = [0, L] \times \mathbb{S}^{n-1}, g)$, we want to provide information about the multiplicity of the Steklov eigenvalues of (M, g).

For the classical Laplacian problem $\Delta S = \lambda S$ on $(\mathbb{S}^{n-1}, g_0)$, we know [2, pp. 160–162] that the set of eigenvalues is $\{\lambda_{(k)} = k(n + k - 2) : k \geq 0\}$, where the multiplicity m_0 of $\lambda_{(0)} = 0$ is 1 and the multiplicity of $\lambda_{(k)}$ is

$$m_k := \frac{(n+k-3)(n+k-4)\dots n(n-1)}{k!}(n+2k-2). \tag{8}$$

As such, given $k \geq 0$, there exist m_k independent functions $S^1_k, \dots, S^{m_k}_k$ such that $\Delta S^i_k = \lambda_{(k)} S^i_k,\ i = 1, \dots, m_k$.

Given $k \geq 0$, there are m_k independent Steklov–Dirichlet eigenfunctions associated with the eigenvalue $\sigma^D_{(k)}(A_{1+L/2})$, that can be written $\varphi^i_k(r, p) = \alpha_k(r)S^i_k(p),\ i = 1, \dots, m_k$. For the Steklov–Neumann case, the eigenfunctions associated with $\sigma^N_{(k)}(A_{1+L/2})$ can be written $\phi^i_k(r, p) = \beta_k(r)S^i_k(p),\ i = 1, \dots, m_k$. Indeed, for each of these problems, the multiplicity of the (k)th eigenvalue is exactly m_k, see, for example, [5, Prop. 3].

6.1 Upper bound for $\sigma_2(M, g), \dots, \sigma_{m_1}(M, g)$

In this section, we prove the following theorem:

Theorem 18 *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and dimension $n \geq 3$. Let m_1 be the multiplicity of the first non trivial eigenvalue of the classical Laplacian problem on $(\mathbb{S}^{n-1}, g_0)$. Then we have*

$$\sigma_2(M, g) = \dots = \sigma_{m_1}(M, g) < B^2_n(L) = \dots = B^{m_1}_n(L) =: \sigma^N_{(1)}(A_{1+L/2}).$$

Moreover, this bound is sharp: for all $\varepsilon > 0$ there exists a metric of revolution g_ε on M such that

$$\sigma_2(M, g_\varepsilon) = \dots = \sigma_{m_1}(M, g_\varepsilon) > \sigma^N_{(1)}(A_{1+L/2}) - \varepsilon.$$

Proof We consider two cases.

1. Let $M = [0, L] \times \mathbb{S}^{n-1}$, with $L \leq L_1$. We write f^1_1 an eigenfunction associated with $\sigma_1(M, g)$. Since $L \leq L_1$, we have $B_n(L) = \sigma^N_1(A_{1+L/2})$ and therefore $f^1_1(r, p) = u_1(r)S^1_1(p)$.
   We consider now a new function denoted f^2_1 given by $f^2_1(r, p) = u_1(r)S^2_1(p)$. We can check that

$$\int_\Sigma f^2_1(r, p)dV\Sigma = 0 \quad \text{and} \quad \int_\Sigma f^1_1(r, p)f^2_1(r, p)dV\Sigma = 0.$$

Moreover, we have

$$\sigma_1(M,g) = R_g(f_1^1) = R_g(f_1^2).$$

In the same way, we write

$$f_1^i(r,p) = u_1(r)S_1^i(p), \quad i = 1, \dots, m_1$$

and we can conclude

$$\sigma_1(M,g) = \sigma_2(M,g) = \cdots = \sigma_{m_1}(M,g).$$

Therefore, we already have a sharp upper bound for these eigenvalues, which is given by $\sigma_1^N(A_{1+L/2})$.

2. Let $M = [0,L] \times \mathbb{S}^{n-1}$, with $L > L_1$. We call f_1 an eigenfunction associated with $\sigma_1(M,g)$. Since $L > L_1$, we have $B_n(L) = \sigma_0^D(A_{1+L/2})$. Therefore $f_1(r,p) = u_0(r)S_0(p)$.
We write now $f_2^1(r,p) = u_2(r)S_1^1(p)$ an eigenfunction associated with $\sigma_2(M,g)$. As before, we then consider m_1 functions denoted $f_2^i(r,p) = u_2(r)S_1^i(p), i = 1, \dots, m_1$ and we get

$$\sigma_2(m,g) = \cdots = \sigma_{m_1+1}(M,g).$$

We consider a function $\phi_1(r,p) = \beta_1(r)S_1(p)$ associated with $\sigma_{(1)}^N(A_{1+L/2})$. In the same spirit as before, we define a function

$$\begin{aligned} \tilde{\phi_1} : [0,L] \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{R} \\ (r,p) &\longmapsto \begin{cases} \phi_1(r,p) & \text{if } 0 \le r \le L/2 \\ \phi_1(L-r,p) & \text{if } L/2 \le r \le L. \end{cases} \end{aligned}$$

We can check that the function $\tilde{\phi_1}$ is continuous and that $\int_\Sigma \tilde{\phi_1} dV_\Sigma = 0$. Moreover, it is immediate that $\int_\Sigma \tilde{\phi_1} f_1 dV_\Sigma = 0$. Hence we can use $\tilde{\phi_1}$ as a test function for $\sigma_2(M,g)$ and as we did before, we can see that

$$\begin{aligned} \sigma_2(M,g) &\le R_g(\tilde{\phi_1}) \\ &< R_{\tilde{g}}(\tilde{\phi_1}) \text{ where } \tilde{g} \text{ comes from Theorem 2} \\ &= \frac{\int_0^L \int_{\mathbb{S}^{n-1}} \left((\partial_r \tilde{\phi_1})^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \tilde{\phi_1}|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{\int_\Sigma \tilde{\phi_1}^2 (0,p) dV_{g_0}} \\ &= \frac{2 \times \int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \tilde{\phi_1})^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \tilde{\phi_1}|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{2 \times \int_{\mathbb{S}^{n-1}} \tilde{\phi_1}^2 (0,p) dV_{g_0}} \\ &= \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \phi_1)^2 + \frac{1}{\tilde{h}(r)^2} |\tilde{\nabla} \phi_1|^2 \right) \tilde{h}(r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \phi_1^2 (0,p) dV_{g_0}} \\ &< \frac{\int_0^{L/2} \int_{\mathbb{S}^{n-1}} \left((\partial_r \phi_1)^2 + \frac{1}{(1+r)^2} |\tilde{\nabla} \phi_1|^2 \right) (1+r)^{n-1} dV_{g_0} dr}{\int_{\mathbb{S}^{n-1}} \phi_1^2 (0,p) dV_{g_0}} \\ &= \sigma_{(1)}^N(A_{1+L/2}). \end{aligned}$$

Therefore, regardless of the value of $L > 0$, we have

$$\sigma_2(M, g) = \cdots = \sigma_{m_1}(M, g) < B_n^2(L) = \cdots = B_n^{m_1}(L) := \sigma_{(1)}^N(A_{1+L/2}).$$

Moreover, this bound is sharp: for all $\varepsilon > 0$, there exists a metric g_ε on $M = [0, L] \times \mathbb{S}^{n-1}$ such that $\sigma_2(M, g_\varepsilon) = \cdots = \sigma_{m_1}(M, g_\varepsilon) > \sigma_{(1)}^N(A_{1+L/2}) - \varepsilon$. Indeed, as before it is sufficient to choose the metric $g_\varepsilon = dr^2 + h_\varepsilon^2 g_0$, with the function h_ε such that

1. h_ε is symmetric;
2. For all $r \in [0, L/2 - \delta]$, we have $h_\varepsilon(r) = 1 + r$, with δ small enough.

The proof of sharpness goes as in the proof of Theorem 3. □

The upper bound we gave, namely $\sigma_{(1)}^N(A_{1+L/2})$, depends on the dimension of M and the meridian length L of M. It is easy to see that $\sigma_{(1)}^N(A_{1+L/2})$, which is strictly increasing, satisfies

$$\sigma_{(1)}^N(A_{1+L/2}) = \frac{(n-1)\left((1+L/2)^n - 1\right)}{(1+L/2)^n + n - 1} \underset{L\to\infty}{\longrightarrow} n - 1.$$

Therefore, we have got a bound that depends only on the dimension n of M. Given a hypersurface of revolution (M, g) with two boundary components, we have

$$\sigma_2(M, g) = \cdots = \sigma_{m_1}(M, g) < B_n^2 = \cdots = B_n^{m_1} := n - 1.$$

Moreover, this bound is sharp, in the sense that for all $\varepsilon > 0$, there exists a hypersurface of revolution $(M_\varepsilon, g_\varepsilon)$ such that $\sigma_2(M_\varepsilon, g_\varepsilon) = \cdots = \sigma_{m_1}(M_\varepsilon, g_\varepsilon) > n - 1 - \varepsilon$. Indeed, we can choose L_ε large enough for $\sigma_1^N(A_{1+L_\varepsilon/2})$ to be $\frac{\varepsilon}{2}$-close to $n-1$, and then define $M_\varepsilon := [0, L_\varepsilon] \times \mathbb{S}^{n-1}$. Now we can put a metric g_ε on M_ε such that $\sigma_2(M_\varepsilon, g_\varepsilon) = \cdots = \sigma_{m_1}(M_\varepsilon, g_\varepsilon) > \sigma_1^N(A_{1+L_\varepsilon/2}) - \frac{\varepsilon}{2}$, and we are done.

Our calculations showed that the eigenvalues $k = 2, \ldots, m_1$ have a critical length at infinity.

6.2 Upper bound for $\sigma_{m_1+1}(M, g)$

Now we are interested in the next eigenvalue, namely $\sigma_{m_1+1}(M, g)$. For that reason, we define a new special meridian length L_2: it is the unique solution of the equation $\sigma_0^D(A_{1+L/2}) = \sigma_{(2)}^N(A_{1+L/2})$. We remark that we have $L_2 < L_1$. Indeed, for all $L > 0$, we have $\sigma_{(2)}^N(A_{1+L/2}) > \sigma_{(1)}^N(A_{1+L/2})$, and the function $L \longmapsto \sigma_0^D(A_{1+L/2})$ is strictly decreasing. Hence, comparing the intersection of the curves gives $L_2 < L_1$. We prove the following theorem:

Theorem 19 *Let $(M = [0, L] \times \mathbb{S}^{n-1}, g)$ be a hypersurface of revolution in Euclidean space with two boundary components each isometric to $\mathbb{S}^{n-1}$ and dimension $n \geq 3$. Let m_1 be the multiplicity of the first non trivial eigenvalue of the classical Laplacian problem on $(\mathbb{S}^{n-1}, g_0)$. Then we have*

$$\sigma_{m_1+1}(M, g) < B_n^{m_1+1}(L) := \begin{cases} \sigma_{(2)}^N(A_{1+L/2}) & \text{if } L \leq L_2 \\ \sigma_{(0)}^D(A_{1+L/2}) & \text{if } L_2 < L \leq L_1 \\ \sigma_{(1)}^N(A_{1+L/2}) & \text{if } L_1 < L. \end{cases}$$

Moreover, this bound is sharp: for all $\varepsilon > 0$, *there exists a metric of revolution* g_ε *on* M *such that*

$$\sigma_{m_1+1}(M, g_\varepsilon) > B_n^{m_1+1}(L) - \varepsilon.$$

A plot of the function $L \longmapsto B_n^{m_1+1}(L)$ can be useful to visualize the sharp upper bound, see Fig. 6.

Proof Now we have to distinguish three cases.

1. Let $M = [0, L] \times \mathbb{S}^{n-1}$, with $L \leq L_2$. We call $f_1^1(r, p) = u_1(r)S_1^1(p), \dots, f_1^{m_1}(r, p) = u_1(r)S_1^{m_1}(p)$ the Steklov eigenfunctions associated with $\sigma_{(1)}(M, g) = \sigma_1(M, g) = \dots = \sigma_{m_1}(M, g)$.
   There exists an eigenfunction $\phi_2(r, p) = \beta_2(r)S_2(p)$ associated with $\sigma_{(2)}^N(M, g) = \sigma_{m_1+1}^N(M, g)$. We define a new function

$$\tilde{\phi_2} : [0, L] \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$$
$$(r, p) \longmapsto \begin{cases} \phi_2(r, p) & \text{if } 0 \leq r \leq L/2 \\ \phi_2(L - r, p) & \text{if } L/2 \leq r \leq L. \end{cases}$$

   This function is continuous, satisfies $\int_\Sigma \tilde{\phi_2} dV_\Sigma = 0$ and we can check that for all $i = 1, \dots, m_1$,

$$\int_\Sigma \tilde{\phi_2} f_1^i dV_\Sigma = 0.$$

   Hence we can use $\tilde{\phi_2}$ as a test function for $\sigma_{m_1+1}(M, g)$. The same kind of calculations as in Inequality (5) show that we have

$$\sigma_{m_1+1}(M, g) < \sigma_{(2)}^N(A_{1+L/2}),$$

   which is a sharp upper bound.
2. Let $M = [0, L] \times \mathbb{S}^{n-1}$, with $L_2 < L \leq L_1$. We call $f_1^1(r, p) = u_1(r)S_1^1(p), \dots, f_1^{m_1}(r, p) = u_1(r)S_1^{m_1}(p)$ the Steklov eigenfunctions associated with $\sigma_{(1)}(M, g) = \sigma_1(M, g) = \dots = \sigma_{m_1}(M, g)$.
   There exists an eigenfunction $\varphi_0(r, p) = \alpha_0(r)S_0(p)$ associated with $\sigma_0^D(M, g)$. We use the function $\tilde{\varphi_0}$ we defined before, namely

$$\tilde{\varphi_0} : [0, L] \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$$
$$(r, p) \longmapsto \begin{cases} \varphi_0(r, p) & \text{if } 0 \leq r \leq L/2 \\ -\varphi_0(L - r, p) & \text{if } L/2 \leq r \leq L. \end{cases}$$

   We already saw that $\tilde{\varphi_0}$ is continuous, that $\int_\Sigma \tilde{\varphi_0} dV_\Sigma = 0$ and we can check that for all $i = 1, \dots, m_1$,

$$\int_\Sigma \tilde{\varphi_0} f_1^i dV_\Sigma = 0.$$

   Using $\tilde{\varphi_0}$ as a test function, we get

$$\sigma_{m_1+1}(M, g) < \sigma_{(0)}^D(A_{1+L/2}),$$

   which is a sharp upper bound.

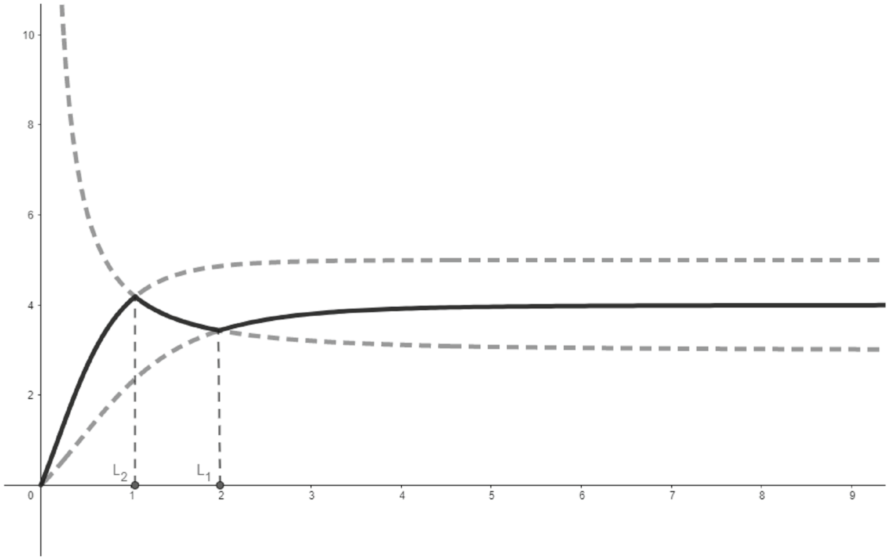

Fig. 6 Representation of the case $n = 5$. The solid curve is the bound given in Theorem 19

3. Let $M = [0, L]\times\mathbb{S}^{n-1}$, with $L_1 \le L$. Then $\sigma_1(M, g) < \sigma_2(M, g) = \cdots = \sigma_{m_1+1}(M, g)$. We already dealt with this case in the proof of Theorem 18 and we saw that

$$\sigma_{m_1+1}(M, g) < \sigma^N_{(1)}(A_{1+L/2}),$$

which is a sharp upper bound.

Therefore, given a hypersurface of revolution $(M = [0, L] \times \mathbb{S}^{n-1}, g)$, we have a sharp upper bound for $\sigma_{m_1+1}(M, g)$, depending on n and L, given by

$$\sigma_{m_1+1}(M, g) < B_n^{m_1+1}(L) := \begin{cases} \sigma^N_{(2)}(A_{1+L/2}) & \text{if } L \le L_2 \\ \sigma^D_{(0)}(A_{1+L/2}) & \text{if } L_2 < L \le L_1 \\ \sigma^N_{(1)}(A_{1+L/2}) & \text{if } L_1 < L. \end{cases}$$

The proof of sharpness goes as in the proof of Theorem 3. □

From this, one can once again look for a sharp upper bound for $\sigma_{m_1+1}(M, g)$ that depends only on the dimension n of M. This bound is given by

$$\sigma_{m_1+1}(M, g) < B_n^{m_1+1} := \max\left\{\sigma_0^D(A_{1+L_2/2}), n-1\right\}$$
$$= \begin{cases} \sigma_0^D(A_{1+L_2/2}) & \text{if } 3 \le n \le 6 \\ n-1 & \text{if } 7 \le n. \end{cases} \quad (9)$$

A proof of (9) is given in Appendix A.

Therefore, the eigenvalue $k = m_1 + 1$ possesses a finite critical length if $3 \le n \le 6$, and it has a critical length at infinity if $7 \le n$.

Remark 20 It is then tempting to search for an expression for $B_n^k := \sup_{L\in\mathbb{R}_+^*}\{B_n^k(L)\}$ for any n and k; but it seems to be hard to give an explicit formula for it. Indeed, as Sects. 6.1

and 6.2 suggest, the function $L \longmapsto B_n^k(L)$ is hard to determine and can be either smooth (as in Sect. 6.1 for instance) or piecewise smooth (as Sect. 6.2 for instance). In the second case, there are possibly many irregular points that we have to consider. Moreover, depending on the value of n and k:

1. Either k has a finite critical length, i.e $B_n^k = B_n^k(L_k)$ for a certain $L_k \in \mathbb{R}_+^*$. That is for instance the case of $\sigma_1(M, g)$ or σ_{m_1+1} if $n = 3, 4, 5$ or 6;
2. Or k has a critical length at infinity, i.e $B_n^k = \lim_{L\to\infty} B_n^k(L)$. That is for instance the case of $\sigma_2(M, g), \ldots, \sigma_{m_1}(M, g)$.

Furthermore, we will prove in Sect. 7 that for all $n \geq 3$, there are infinitely many k that have a finite critical length associated to them. In all these cases, the function $L \longmapsto B_n^k(L)$ is piecewise smooth.

7 Critical lengths of hypersurfaces of revolution

We recall that given $n \geq 3$, we are interested in giving information about the set of finite critical lengths. We want to prove Theorem 9, i.e that there are infinitely many k such that $B_n^k = B_n^k(L_k)$ for a certain finite $L_k \in \mathbb{R}_+^*$, and that the sequence of critical lengths converges to 0.

Proof As before, for $j \geq 0$, we denote by m_j the number given by the formula (8), which is the multiplicity of $\sigma_{(j)}^D(A_R)$ as well as the multiplicity of $\sigma_{(j)}^N(A_R)$. Let $i \geq 2$ be an integer. We claim that for all k such that

$$m_0 + \sum_{j=1}^{i-1} 2m_j + m_i < k \leq m_0 + \sum_{j=1}^{i} 2m_j, \tag{10}$$

we have

$$B_n^k = B_n^k(L_k)$$

for a certain $L_k \in \mathbb{R}_+^*$.

Indeed, let k satisfy (10). Then, because of the asymptotic behaviour of the functions $L \longmapsto \sigma_{(j)}^D(A_{1+L/2})$ and $L \longmapsto \sigma_{(j)}^N(A_{1+L/2})$ as $L \to \infty$, there exists $C > 0$ such that for all $L > C$, we have $B_n^k(L) = \sigma_{(i)}^D(A_{1+L/2})$. But we can compute that

$$\frac{\partial}{\partial L}\sigma_{(i)}^D(A_{1+L/2}) = -\frac{4(L+2)(2i+n-2)^2(1+L/2)^{2i+n}}{\left(4(1+L/2)^{2i+n} - L^2 - 4L - 4\right)^2} < 0,$$

which means that the function $L \longmapsto \sigma_{(i)}^D(A_{1+L/2})$ is strictly decreasing. Hence, for $L > L' > C$, we have $B_n^k(L) < B_n^k(L')$. Therefore, for such a k, we have

$$B_n^k = B_n^k(L_k)$$

with L_k finite, that is L_k is a finite critical length associated to k.

Then, defining $k_1 := 1$ and for each $i \geq 2$, defining $k_i := m_0 + \sum_{j=1}^{i} 2m_j$, we get a sequence $(k_i)_{i=1}^{\infty}$ such that

$$B_n^{k_i} = B_n^{k_i}(L_i)$$

for a certain $L_i \in \mathbb{R}_+^*$ finite.

Now we want to prove that the sequence of finite critical lengths $(L_i)_{i=1}^{\infty}$ converges to 0.

Let $i \geq 1$. We know that L_i has to be a solution of the equation $\sigma^D_{(j_1)}(A_{1+L/2}) = \sigma^N_{(j_2)}(A_{1+L/2})$ for a certain ordered pair $(j_1, j_2) \in \mathbb{N}^2$. As said before, for $L > C$, we have $B_n^{k_i}(L) = \sigma^D_{(i)}(A_{1+L/2})$, hence $L_i \leq L_i^*$, where L_i^* is the unique solution of the equation

$$\sigma^D_{(i)}(A_{1+L/2}) = \sigma^N_{(i+1)}(A_{1+L/2}).$$

Therefore, in order to prove that $L_i \underset{i\to\infty}{\longrightarrow} 0$, we prove that $L_i^* \underset{i\to\infty}{\longrightarrow} 0$.

Using Propositions 13 and 15, making some calculations and substituting $(1 + L/2)$ by R, we can see that solving

$$\sigma^D_{(i)}(A_{1+L/2}) = \sigma^N_{(i+1)}(A_{1+L/2})$$

is equivalent to finding the unique value $R_i \in (1, \infty)$ which solves the equation

$$(i+1)R^{2i+n-2}\left(R^{2i+n} - R^2(2i+n-1) - \frac{(i+n-1)(2i+n-1)}{i+1}\right) + (i+n-1) = 0.$$

We call

$$\underbrace{(i+1)R^{2i+n-2}\left(\underbrace{R^{2i+n} - R^2(2i+n-1) - \frac{(i+n-1)(2i+n-1)}{i+1}}_{=:\psi_i(R)}\right) + (i+n-1)}_{=:\Psi_i(R)}.$$

Because $(i+1)R^{2i+n-2} > 0$ and $(i+n-1) > 0$, then for R_i to be the solution of the equation $\Psi_i(R) = 0$, it is necessary that $\psi_i(R_i) < 0$.

Then we have

$$\begin{aligned} R^{2i+n} &< R^2(2i+n-1) + \frac{(i+n-1)(2i+n-1)}{i+1} \\ &< R^2\left((2i+n-1) + \frac{(i+n-1)(2i+n-1)}{i+1}\right) \\ &= R^2\left(\frac{(2i+n-1)(2i+n)}{i+1}\right) \\ &< R^2(2i+n)^2. \end{aligned}$$

Therefore we have

$$\ln(R) < \frac{2\ln(2i+n)}{2i+n-2}$$

and thus

$$R < e^{\frac{2\ln(2i+n)}{2i+n-2}}$$

Remember that we substituted $(1+L/2)$ by R, and then the unique solution of the equation

$$\sigma^D_{(i)}(A_{1+L/2}) = \sigma^N_{(i+1)}(A_{1+L/2})$$

is a value L_i^* which satisfies

$$0 < L_i^* < 2\left(e^{\frac{2\ln(2i+n)}{2i+n-2}} - 1\right).$$

Therefore, since $\frac{2\ln(2i+n)}{2i+n-2} \underset{i\to\infty}{\longrightarrow} 0$, we have

$$L_i < L_i^* \underset{i\to\infty}{\longrightarrow} 0.$$

In particular, for each $\delta > 0$ there exists $k_0 \in \mathbb{N}$ such that for each $k > k_0$ which has a finite critical length L_k, then $L_k < \delta$. □

Remark 21 The condition given by (10) is sufficient, but is not a necessary one. Indeed, $k = 1$ does not meet condition (10) but we have $B_n^1 = B_n^1(L_1)$, where L_1 is given by Corollary 4. This consideration naturally leads to the following open question:

Question 22 Given $n \geq 3$, are there finitely or infinitely many $k \in \mathbb{N}$ such that k has a critical length at infinity?

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Appendix A. Proof of equality (9)

We know that there exists a unique $L_2 > 0$ such that $\sigma_{(0)}^D(A_{1+L_2/2}) = \sigma_{(2)}^N(A_{1+L_2/2})$. We want to choose, depending on the value of n, if $\sigma_{(0)}^D(A_{1+L_2/2})$ is bigger or smaller than $n-1$. For this purpose, we call L_D the unique positive value such that $\sigma_{(0)}^D(A_{1+L_D/2}) = n-1$, and we call L_N the unique positive value such that $\sigma_{(2)}^N(A_{1+L_N/2}) = n-1$.

Then we have the following fact: if $L_N < L_D$, we have $\sigma_{(0)}(A_{1+L_2/2}) > n-1$. On the contrary, if $L_D < L_N$, we have $\sigma_{(0)}(A_{1+L_2/2}) < n-1$ (Fig. 7).

Hence, we solve the equation $\sigma_{(0)}^D(A_{1+L_D/2}) = n-1$, i.e we find the unique $L_D > 0$ such that

$$\frac{(n-2)(1+L_D/2)^{n-2}}{(1+L_D/2)^{n-2}-1} = n-1.$$

We find

$$L_D = 2(n-1)^{\frac{1}{n-2}} - 2.$$

Similarly, solving the equation

$$\frac{2n((1+L_N/2)^{n+2}-1)}{2(1+L_N/2)^{n+2}+n} = n-1$$

leads to

$$L_N = 2\left(\frac{n(n+1)}{2}\right)^{\frac{1}{n+2}} - 2.$$

We have to find for which values of n we have $L_D < L_N$ and vice versa. This leads to the inequality

$$\left(\frac{n(n+1)}{2}\right)^{\frac{1}{n+2}} > (n-1)^{\frac{1}{n-2}},$$

which is equivalent to

$$\left(\frac{n(n+1)}{2}\right)^{\frac{n-2}{n+2}} > n-1.$$

We suppose $n \geq 9$.

$$\left(\frac{n(n+1)}{2}\right)^{\frac{n-2}{n+2}} > \left(\frac{n^2}{2}\right)^{\frac{n-2}{n+2}} = \frac{1}{2^{\frac{n-2}{n+2}}} n^{\frac{2n-4}{n+2}} > \frac{1}{2} n^{\frac{2n-4}{n+2}} \geq \frac{1}{2} n^{\frac{14}{11}}.$$

We analyze the function $f : [9, \infty) \longrightarrow \mathbb{R}, x \longmapsto \frac{1}{2} x^{\frac{14}{11}}$. We have $f(9) > 8$, and $f'(x) = \frac{14}{22} x^{\frac{3}{11}}$. We can compute that $f'(9) > 1$ and since $f''(x) = \frac{42}{242} x^{\frac{-9}{11}} > 0$ for all $x \in [9, \infty)$, we can conclude $f'(x) > 1$ for all $x \in [9, \infty)$. Hence, $f(x) > x - 1$ for all $x \in [9, \infty)$.

Therefore, for all integers $n \geq 9$, we have

$$\left(\frac{n(n+1)}{2}\right)^{\frac{n-2}{n+2}} > n-1$$

and then $L_D < L_N$. We can compute the cases $n = 3, \ldots, 8$ and we can conclude that

$$\begin{cases} L_N < L_D & \text{if } 3 \leq n \leq 6 \\ L_D < L_N & \text{if } 7 \leq n. \end{cases}$$

Therefore, we have

$$\max\left\{\sigma_0^D(A_{1+L_2/2}), n-1\right\} = \begin{cases} \sigma_0^D(A_{1+L_2/2}) & \text{if } 3 \leq n \leq 6 \\ n-1 & \text{if } 7 \leq n. \end{cases}$$

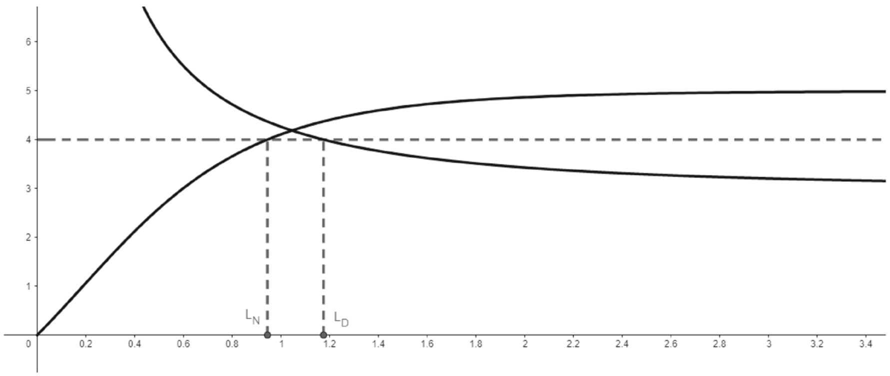

Fig. 7 Representation of the case $n = 5$. Since $L_N < L_D$, then $\sigma_{(0)}(A_{1+L_2/2}) > n - 1$

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