Interpolation of Beilinson–Kato elements and p -adic L -functions

Denis Benois¹ · Kâzım Büyükboduk²

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Abstract

Our objective in this series of two articles, of which the present article is the first, is to give a Perrin-Riou-style construction of p -adic L -functions (of Bellaïche and Stevens) over the eigencurve. As the first ingredient, we interpolate the Beilinson–Kato elements over the eigencurve (including the neighborhoods of θ -critical points). Along the way, we prove étale variants of Bellaïche’s results describing the local properties of the eigencurve. We also develop the local framework to construct and establish the interpolative properties of these p -adic L -functions away from θ -critical points.

Résumé

Ce texte est le premier d’une série de deux articles consacrés à la construction des fonctions L p -adiques de Bellaïche et Stevens sur la courbe de Hecke par la méthode de Perrin-Riou. Nous commençons par interpoler les éléments de Beilinson–Kato le long de la courbe de Hecke, y compris aux voisinages des points θ -critiques. Chemin faisant, nous prouvons une version étale des résultats de Bellaïche sur la structure locale de la courbe. Nous donnons un cadre pour construire les fonctions L p -adiques et établir leurs propriétés d’interpolation en dehors des points θ -critiques.

Keywords Eigencurve · θ -criticality · Triangulations · Beilinson–Kato elements · p -adic L -functions

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To Bernadette Perrin-Riou on the occasion of her 65th birthday, with admiration.

✉ Kâzım Büyükboduk
kazim.buyukboduk@ucd.ie

Denis Benois
denis.benois@math.u-bordeaux.fr

¹ Institut de Mathématiques, Université de Bordeaux, 351, Cours de la Libération, 33405 Talence, France

² UCD School of Mathematics and Statistics, University College, Dublin, Ireland

Contents

1	Introduction	232
1.1	Set up	233
1.2	Main results	234
1.2.1	θ -critical case	235
1.2.2	Layout	236
2	Linear algebra	236
2.1	Tate algebras	236
2.2	Generalized eigenspaces	237
2.3	Specializations	238
2.4	Duality	238
3	Exponential maps and triangulations	239
3.1	Cohomology of (φ, Γ) -modules	239
3.2	The Perrin-Riou exponential map	241
3.3	Triangulations in families	245
4	Two variable p -adic L -functions: the abstract definitions	246
4.1	Duality (bis)	246
4.2	Perrin-Riou-style two variable p -adic L -functions	247
5	Local description of the eigencurve	249
5.1	Modular symbols	249
5.2	Local description of the Coleman–Mazur–Buzzard eigencurve	252
5.3	A variant of Bellaïche’s construction	253
6	Interpolation of Beilinson–Kato elements	258
6.1	Modular curves and Iwasawa sheaves	258
6.2	Big Beilinson–Kato elements	260
6.3	Overconvergent étale sheaves	265
6.4	Big Galois representations	267
6.5	Beilinson–Kato elements over the eigencurve	270
6.6	Normalized Beilinson–Kato elements	273
7	“Étale” construction of p -adic L -functions	277
	Appendix A: Integrality of normalizations	280
	A.1 The normalization factors revisited	281
	A.2 Dependence on c and d	281
	A.3 Integrality of partial normalizations	282
	References	285

1 Introduction

Let $p \geq 5$ be a prime number and let us fix forever an embedding $\iota_\infty : \overline{\mathbb{Q}} \rightarrow \mathbb{C}$ as well as an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$. Let us also put $\iota_p := \iota \circ \iota_\infty$. Let $f = \sum_n a_n(f) q^n \in S_{k+2}(\Gamma_1(N)) \cap \Gamma_0(p, \varepsilon)$ be a cuspidal eigenform (for all Hecke operators $\{T_\ell\}_{\ell \nmid N}$ and $\{U_\ell, \langle \ell \rangle\}_{\ell \mid Np}$) of weight $k+2 \geq 2$ with $p \nmid N$. When $\text{ord}_p(\iota_p(a_p(f))) < k+1$, Amice–Vélu in [2] and Višik in [34] have given a construction of a p -adic L -function $L_p(f, s)$, which is characterized with the property that it interpolates the critical values of Hecke L -functions attached to (twists) of f .

The analogous result in the extreme case when $\text{ord}_p(\iota_p(a_p(f))) = k+1$ (in which case we say that f has critical slope) was established by Pollack–Stevens in [32] and Bellaïche [10]. Note that if $\text{ord}_p(\iota_p(a_p(f))) = k+1$ then f is necessarily p -old unless $k=0$. It is worthwhile to note that the p -adic L -functions of Pollack–Stevens and Bellaïche are not characterized in terms of their interpolation property, but rather via the properties of the f -isotypic Hecke eigensubspace of the space of modular symbols. The work of Pollack–Stevens assumes in addition that f is not in the image of the p -adic θ -operator $\theta^{k+1} :=$

$(q(d/dq))^{k+1}$ on the space of overconvergent modular forms of weight $-k$ (i.e., f is not θ -critical); Bellaïche’s work removes this restriction.

Both constructions in [10, 32] take place in Betti cohomology. Our objective in this series of two articles, of which the present article is the first, is to recover these results in the context of p -adic (étale) cohomology. More precisely, we shall recast the results of Bellaïche and Pollack–Stevens in terms of the Beilinson–Kato elements and the triangulation over the Coleman–Mazur–Buzzard eigencurve. Along the way, we interpolate the Beilinson–Kato elements over a neighborhood on the eigencurve (including the neighborhoods of θ -critical points).

1.1 Set up

We put $\Gamma_p = \Gamma_1(N) \cap \Gamma_0(p)$ and let $f_0^{\alpha_0} \in S_{k_0+2}(\Gamma_p)$ be a p -stabilized cuspidal eigenform, where $U_p f_0^{\alpha_0} = \alpha_0 f_0^{\alpha_0}$ and N is coprime to p . We let f_0 be the newform associated to $f_0^{\alpha_0}$ (note that it may happen that $f_0 = f_0^{\alpha_0}$, in which case $f_0^{\alpha_0}$ is not of critical slope). We fix a real number $\nu \geq v_p(\alpha_0)$, where $v_p(\cdot)$ is the p -adic valuation on $\overline{\mathbb{Q}}_p$ normalized so that $v_p(p) = 1$.

We shall call $\text{Hom}_{\text{cts}}(\mathbb{Z}_p^\times, \mathbb{G}_m)$ the weight space, which we think of as a rigid analytic space over $\mathbb{Q}_p$. Let $\mathcal{W}$ be a nice affinoid neighborhood (in the sense of [10], Definition 3.5) about k_0 of the weight space, which is adapted to slope ν (in the sense of [10], §3.2.4). We will adjust our choice of $\mathcal{W}$ on shrinking it as necessary for our arguments.

Let $\mathcal{C}$ be the Coleman–Mazur–Buzzard eigencurve and let $x_0 \in \mathcal{C}$ be the point that corresponds to $f_0^{\alpha_0}$. We let $\mathcal{C}_{\mathcal{W}, \nu} \subset \mathcal{C}$ denote an open affinoid subspace of the eigencurve that lies over $\mathcal{W}$ and U_p acts by slope at most ν . By shrinking $\mathcal{W}$ as necessary, there is a unique connected component $\mathcal{X} \subset \mathcal{C}_{\mathcal{W}, \nu}$ that contains x_0 , which is an affinoid neighborhood of x_0 .

For any E -valued point $x \in \mathcal{X}(E)$ (where E is a sufficiently large extension of $\mathbb{Q}_p$, which contains the image of the Hecke field of $f_0^{\alpha_0}$ under the fixed isomorphism ι) of classical weight $w(x) \in \mathbb{Z}_{\geq 0}$ in the irreducible component $\mathcal{X} \subset \mathcal{C}$, we let $f_x \in S_{w(x)+2}(\Gamma)$ denote the corresponding p -stabilized eigenform. We let V'_{f_x} denote Deligne’s representation attached to f_x ; see §6.4.1 for its precise description. There is a natural free $\mathcal{O}_{\mathcal{X}}$ -module $V'_{\mathcal{X}}$ of rank 2, which is equipped with an $\mathcal{O}_{\mathcal{X}}$ -linear continuous $G_{\mathbb{Q}}$ -action such that $V'_{\mathcal{X}} \otimes_x E \xrightarrow{\sim} V'_{f_x}$ (c.f. (51) and (55) below).

As in [22, §5], let ξ denote either the symbol $a(B)$ with $a, B \in \mathbb{Z}$ and $B \geq 1$ or an element of $\text{SL}_2(\mathbb{Z})$. For each positive integer m coprime to p , we set $S := \begin{cases} \text{primes}(mBp), & \text{if } \xi = a(B) \\ \text{primes}(mNp), & \text{if } \xi \in \text{SL}_2(\mathbb{Z}) \end{cases}$, where $\text{primes}(M)$ stands for the set of prime divisors of M . Let (c, d) be a pair of positive integers satisfying the following conditions: $(cd, 6) = 1 = (d, N)$ and $\text{prime}(cd) \cap S = \emptyset$. For each integer r and natural number n , we let

$$c, d \text{BK}_{N, mp^n}(f_x, j, r, \xi) \in H^1(\mathbb{Q}(\zeta_{mp^n}), V'_{f_x}(2-r))$$

denote the Beilinson–Kato element associated to the eigenform f_x , given as in Theorem 6.8(iv) (see also Sect. 6.2 for details).

For any abelian group G , let us denote by $X(G)$ its character group. We also put $\Lambda(G) := \mathbb{Z}_p[[G]]$ for its completed group ring. We shall denote by $\Lambda(G)^t$ for the free $\Lambda(G)$ -module of rank one on which G acts via the character $g \mapsto g^{-1} \in \Lambda(G)^\times$.

For each positive integer m , let us put $\Gamma_{\mathbb{Q}(\zeta_m)} := \text{Gal}(\mathbb{Q}(\zeta_{mp^\infty})/\mathbb{Q}(\zeta_m))$ and set $\Gamma := \Gamma_{\mathbb{Q}}$. We denote by χ the p -adic cyclotomic character. We put $\mathcal{H}_r(\Gamma)$ for the image of tempered

distributions of order ν under the Amice transform and define $\mathcal{H}(\Gamma) := \varinjlim \mathcal{H}_r(\Gamma)$. We also set $\mathcal{H}_r(\Gamma) := \mathcal{H}(\Gamma) \widehat{\otimes} (?)$, where $(?) = E, \mathcal{O}_{\mathcal{W}}, \mathcal{O}_{\mathcal{X}}$.

Fix a generator ε of $\mathbb{Z}_p(1)$.

1.2 Main results

We will briefly overview of the results in this article. In a nutshell, our work has two threads: The first concerns the interpolation of Beilinson–Kato elements along the eigencurve $\mathcal{C}$, the second concerns p -local aspects, such as the (properties of the) triangulation over $\mathcal{C}$ and the formalism of Perrin–Riou exponential maps. The first part of Theorem A below belongs to the first thread and corresponds to Theorem 6.8 below. The second part corresponds to Theorem 7.3 in the main body of our article and dwells on the second thread, granted the first.

Theorem A

(i) For each $m \geq 1$ and coprime to p , there exists a cohomology class

$${}_{c,d}\mathbb{BK}_{N,m}^{[\mathcal{X}]}(j, \xi) \in H^1(\mathbb{Q}(\zeta_m), V'_{\mathcal{X}} \widehat{\otimes} \Lambda(\mathbb{Q}(\zeta_m))^t(1))$$

which interpolates the Beilinson–Kato classes ${}_{c,d}\mathbf{BK}_{N,mp^n}(f_x, j, r, \xi)$ as $x \in \mathcal{X}^{\text{cl}}(E)$ and integers r, n vary.

ii) Suppose that $\mathcal{X}$ is étale over $\mathcal{W}$. We let $\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) \in H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}} \widehat{\otimes} \Lambda(\Gamma)^t(1))$ denote the partially normalized Beilinson–Kato element, given as in Definition 6.14 and Proposition 6.15 below. Let $L_p^{\pm}(\mathcal{X}; j, \xi) \in \mathcal{H}_{\mathcal{O}_{\mathcal{X}}}(\Gamma)$ denote the images of the $\pm$ -parts of the class $\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi)$ under the Perrin–Riou dual exponential map (see Definition 7.1). On shrinking the neighborhood $\mathcal{X}$ of x_0 as necessary, the following hold true.

- (a) There exist (j_+, ξ_+) and (j_-, ξ_-) such that $L_p^+(\mathcal{X}; j_+, \xi_+, x_0)$ and $L_p^-(\mathcal{X}; j_-, \xi_-, x_0)$ are nonzero elements of $\mathcal{H}_E(\Gamma)$.
- (b) Assume that $(j_{\pm}, \xi_{\pm})$ satisfy the conditions in (a). Then for each $x \in \mathcal{X}^{\text{cl}}(E)$ such that $v_p(\alpha(x)) < w(x) + 1$, the p -adic L -functions $L_p^{\pm}(\mathcal{X}; j_{\pm}, \xi_{\pm}, x)$ agree with the Manin–Višik p -adic L -functions attached to f_x , up to multiplication by $D^{\pm} \mathcal{E}_N(x)$ where $D^{\pm} \in E^{\times}$ and $\mathcal{E}_N(x)$ is the product of bad Euler factors, given as in (64).
- (c) Assume that $v_p(\alpha_0) = k_0 + 1$, but f is not θ -critical. Then $L_p^{\pm}(\mathcal{X}; j_{\pm}, \xi_{\pm}, x_0)$ agree with the one-variable p -adic L -functions of Pollack–Stevens [32] up to multiplication by $D^{\pm} \mathcal{E}_N(x)$, where $D^{\pm} \in E^{\times}$.
- (d) Let $L_p(\mathcal{X}, \Phi^{\pm})$ denote the two-variable p -adic L -functions of Bellaïche and Stevens associated to modular symbols $\Phi^{\pm} \in \text{Symb}_{\Gamma_p}^{\pm}(\mathcal{X})$ (c.f. [10], Theorem 3). Then there exist functions $u^{\pm}(x) \in O_{\mathcal{X}}^{\times}$ such that

$$L_p^{\pm}(\mathcal{X}; j_{\pm}, \xi_{\pm}) = u^{\pm}(x) \mathcal{E}_N(x) L_p(\mathcal{X}, \Phi^{\pm}).$$

See Definition 6.7(iii) where the class ${}_{c,d}\mathbb{BK}_{N,m}^{[\mathcal{X}]}(j, \xi)$ is introduced and see Definition 6.14 where we introduce its partial normalization (as well as Appendix A where we establish the regularity properties of this normalization). We call the p -adic L -function $L_p^{\pm}(\mathcal{X}; j, \xi) \in \mathcal{H}_{\mathcal{O}_{\mathcal{X}}}(\Gamma)$ given as in Definition 7.1(i) (which is denoted by $L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi)$ in the main body of our text) the arithmetic p -adic L -function.

Remark 1.1 Results similar to Theorem A are proved in the recent paper of Wang [35] via different methods. Some parts of this result has been also announced in the independent

preprints of Hansen [17] and Ochiai [29]. Our approach to establish Theorem A(i) (which applies even when $\mathcal{X}$ is not étale over $\mathcal{W}$; this is crucial for our results in the θ -critical case) is a synthesis of the techniques of [17, 26], relying on the overconvergent étale cohomology of Andreatta–Iovita–Stevens [3] and Kings’ theory of Λ -adic sheaves developed in [23]. Along the way, we verify (relying on the results of Ash–Stevens and Pollack–Stevens) that the construction of the local pieces (including neighborhoods of θ -critical points, where $\mathcal{X}$ is no longer étale over $\mathcal{W}$) of the cuspidal eigencurve using different types of distribution spaces over the weight space, or compactly vs non-compactly supported cohomology produce the same end result. See Sect. 5.3, §5.1.5 and Sect. 6.3 for further details (most particularly, Lemma 5.2, Proposition 5.8(ii) and Proposition 6.5) concerning this technical point, which we believe is of independent interest.

Remark 1.2 We note that in the particular case when $f_0 = f_0^{\alpha_0}$ is a newform of level $\Gamma_0(Np)$ and weight $k_0 + 2$ with $\alpha_0 = a_p(f_0) = p^{k_0/2}$, the conclusions of Theorem 7.3 play a crucial role in [6].

1.2.1 θ -critical case

We conclude Sect. 1.2 with a brief summary of our results in our companion article [5], where we focus on the θ -critical scenario (i.e. in the situation when $\mathcal{X}$ is *not* étale over $\mathcal{W}$) and give an étale construction of Bellaïche’s p -adic L -function.

The first key ingredient in [5] is Theorem A(i), namely the construction of a big Beilinson–Kato class about a θ -critical point on the eigencurve. The local aspects turn out to be significantly more challenging when $\mathcal{X}$ fails to be étale over $\mathcal{W}$. To prove that the Perrin–Riou style p -adic L -functions $L_p^\pm(\mathcal{X}; j, \xi)$, defined in an analogous way, has the required properties, we introduce in op. cit. a new local argument (called the “eigenspace transition via differentiation”) and prove the following results (which we state in vague form to avoid digression and refer readers to [5] for details):

Theorem B *Suppose that the ramification index of $\mathcal{X}$ over $\mathcal{W}$ at x_0 equals¹ to 2. As before, we let $L_p^\pm(\mathcal{X}; j, \xi) \in \mathcal{H}_{\mathcal{O}_{\mathcal{X}}}(\Gamma)$ denote the images of the $\pm$ -parts of the class $\mathbb{B}\mathbb{K}_N^{l_{\mathcal{X}}}(j, \xi)$ under the Perrin–Riou dual exponential map. There exists two pairs (j_+, ξ_+) and (j_-, ξ_-) with the following properties:*

- (i) *We have $L_p^\pm(x_0; j_\pm, \xi_\pm, \rho\chi^r) = 0$ for all integers $1 \leq r \leq k_0 + 1$ and characters $\rho \in X(\Gamma)$ of finite order.*
- (ii) *We define the improved arithmetic p -adic L -functions at the critical point x_0 on setting*

$$L_p^{[1], \pm}(x_0; j_\pm, \xi_\pm) := \frac{\partial}{\partial X} L_p^\pm(X; j_\pm, \xi_\pm) \Big|_{X=0} \in \mathcal{H}_E(\Gamma)$$

Here, as in [10, §4.4], we denote by X a fixed choice of a uniformizer of $\mathcal{X}$ about x_0 and consider the p -adic L -functions $L_p^\pm(\mathcal{X}; j, \xi)$ in the neighborhood $\mathcal{X}$ of x_0 as the functions $L_p^\pm(X; j, \xi)$ with X in a neighborhood of 0. Then the improved arithmetic p -adic L -functions verify the same interpolation property that Bellaïche’s improved p -adic L -functions do, up to the bad Euler factors $\mathcal{E}_N(x_0)$ and constants that depend only on the choices of Shimura periods.

¹ In an unpublished note (see [9] Proposition 1), Bellaïche explains that a conjecture of Jannsen combined with Greenberg’s conjecture (that “locally split implies CM”) and the important result of Breuil–Emerton (“ θ -critical implies locally split”) would yield $e = 2$. We are grateful to R. Pollack for bringing Bellaïche’s note to our attention.

This theorem supplies us with a new construction of Bellaïche's p -adic L -functions (with Euler factors at primes dividing the tame conductor removed). One of the consequences of the étale construction of the p -adic L -functions is the leading term formulae for these p -adic L -functions. These will be explored in the sequels to the present article.

1.2.2 Layout

We close our introduction reviewing the layout of our article.

After a very general preparation in Sect. 2 (where we axiomatise various constructions in [10], §4.3), we give a general overview of triangulations in Sect. 3. In Sect. 3.2, we also define the Perrin-Riou exponential map, which is one of the crucial inputs defining the “arithmetic” p -adic L -functions.

We then introduce Perrin-Riou-style (abstract) two-variable p -adic L -functions in Sect. 4 and study their interpolative properties. These results are later applied in Sect. 7 in the context of the Coleman–Mazur–Buzzard eigencurve and with the Beilinson–Kato element ${}_{c,d}\mathrm{Bk}_{N,m}^{[\mathcal{X}]}(j, \xi)$ of Theorem A.

In Sect. 5, we review Bellaïche's results in [10] on the local description of the eigencurve and prove variants (in Sect. 5.3) of these results involving slightly different local systems as coefficients and non-compactly supported cohomology (at a level of generality that covers also the neighborhoods of θ -critical point). We utilize these in Sect. 6 (together with the work of Andreatta–Iovita–Stevens [3]) to deduce the required properties for the Galois representation $V'_{\mathcal{X}}$, where the interpolated Beilinson–Kato elements take coefficients in. The variants of Bellaïche's construction that we discuss in Sect. 5.3 allow us (through Proposition 6.5) to establish the properties of $V_{\mathcal{X}}$ as an $\mathcal{O}_{\mathcal{X}}$ -module, where the images of Perrin-Riou exponential maps take coefficients in.

In Sect. 6.5, we introduce the interpolated Beilinson–Kato elements (which are denoted by ${}_{c,d}\mathrm{Bk}_{N,m}^{[\mathcal{X}]}(j, \xi)$ in Theorem A(i)) as part of Definition 6.7(iii) (see also Sect. 6.6 where we introduce their normalized versions). Our construction builds primarily on the ideas in [23, 24, 26]. As we have noted in Remark 1.1, our approach in this portions is, in some sense, a synthesis of the techniques of [26] and [17] (which we crucially enhance to apply also about θ -critical points). In Sect. 7, we apply the general results in Sect. 4 to define the “arithmetic” p -adic L -functions and study their interpolation properties, proving our Theorem A(ii).

2 Linear algebra

In Sect. 2, we fix some notation and conventions from linear algebra, which will be used in the remainder of the paper (as well as in the companion paper [5]). We also axiomatize various constructions in [10, §4.3].

2.1 Tate algebras

Let E be a finite extension of $\mathbb{Q}_p$. Fix an integer $e \geq 1$ and denote by R the Tate algebra $R = E\langle Y/p^{re} \rangle$, where $r \geq 0$ is some fixed integer. Let $A = R[X]/(X^e - Y)$. Then $A = E\langle X/p^r \rangle$.

Set $\mathcal{W} = \mathrm{Spm}(R)$ and $\mathcal{X} = \mathrm{Spm}(A)$. We will consider $\mathcal{W}$ as a weight space in the following sense. Fix an integer $k_0 \geq 2$ and denote by $D(k_0, p^{re-1}) = k_0 + p^{re-1}\mathbb{Z}_p$ the closed disk with center k_0 and radius $1/p^{re-1}$. We identify each $y \in D(k_0, p^{re-1})$ with the

point of $\mathcal{W}$ that corresponds to the maximal ideal

$$\mathfrak{m}_y = \left((1+Y) - (1+p)^{y-k_0} \right).$$

We have a canonical morphism

$$w : \mathcal{X} \rightarrow \mathcal{W}, \quad (1)$$

which we will call the weight map. If $x \in \mathcal{X}$, we say that $y = w(x)$ is the weight of x . Let $x_0 \in \mathcal{X}$ denote the unique point such that $w(x_0) = k_0$. Set

$$\mathcal{X}^{\text{cl}} = \{x \in \mathcal{X} \mid w(x) \in \mathbb{Z}, w(x) \geq 2\}.$$

2.2 Generalized eigenspaces

2.2.1. Let M be an A -module. For any $x \in \mathcal{X}(E)$ and $y \in \mathcal{W}(E)$ define

$$\begin{aligned} M_x &= M \otimes_A A/\mathfrak{m}_x, \\ M_y &= M \otimes_R R/\mathfrak{m}_y. \end{aligned}$$

We denote by

$$\pi_x : M_{w(x)} \longrightarrow M_x$$

the canonical projection.

Consider X as a function on $\mathcal{X}$ and denote by $X(x)$ the value of X at $x \in \mathcal{X}(E)$. For any endomorphism f of M , set $M[f] = \ker(f)$.

Definition 2.1 We denote by $M[x] = M_{w(x)}[X - X(x)]$, the submodule of $M_{w(x)}$ annihilated by $X - X(x) \in A$. We also put $M[[x]] := \bigcup_{n \geq 1} M_{w(x)}[(X - X(x))^n]$ and call it the generalized eigenspace associated to x .

2.2.2. Suppose that $M \cong A^d$ is a free A -module of rank d , so that we also have $M \cong (R[X]/(X^e - Y))^d$. Let us put $P_x(X) := \frac{X^e - X(x)^e}{X - X(x)} \in E[X]$. Then,

$$\begin{aligned} M_{w(x)} &= (E[X]/(X^e - X(x)^e)E[X])^d, \\ M[x] &= (P_x(X)E[X]/(X^e - X(x)^e)E[X])^d, \\ M_x &= (E[X]/(X - X(x)))^d, \end{aligned}$$

and the multiplication by $P_x(X)$ gives an isomorphism

$$M_x \xrightarrow[\times P_x(X)]{\sim} M[x].$$

We consider separately the following cases:

- (a) When $x \neq x_0$ (so $w(x) \neq k_0$), then the polynomial $X^e - X(x)^e$ is separable. In this scenario, the A -module $M_{w(x)}$ is semi-simple and $M[x] = P_x(X)M_{w(x)} \subset M_{w(x)}$ is an A -direct summand. The restriction of the natural surjection $\pi_x : M_{w(x)} \longrightarrow M_x$ to $M[x]$ is an isomorphism, which we shall denote with the same symbol unless there is a chance of confusion. Also, $M[[x]] = M[x]$.

- (b) When $x = x_0$, we have $X(x_0) = 0$ and $P_{x_0}(X) = X^{e-1}$. In this case, $M_{k_0} = (E[X]/X^e E[X])^d$ as an A -module and

$$\begin{aligned} M_{k_0} &= M[[x_0]] \xrightarrow{\sim} (E[X]/X^e E[X])^d, \\ M[x_0] &= M_{k_0}[X] = X^{e-1} M_{k_0}, \\ M_{x_0} &\xrightarrow{\sim} E^d. \end{aligned}$$

If $e \geq 2$, the restriction $\pi_{x_0}|_{M[x_0]} : M[x_0] \rightarrow M_{x_0}$ of the surjection π_{x_0} is evidently the zero map.

Observe also that in either of the cases (a) or (b) above, $M[x]$ can be identified with an A -equivariant image of $M_{w(x)}$ under the multiplication-by- $P_x(X)$ map.

2.3 Specializations

Let M be an A -module, and let $x \in \mathcal{X}(E)$. For any $m \in M$, we denote by $m_x \in M_x$ the image of m under the specialization map $M \rightarrow M_x$. On the other hand, $M_A = M \otimes_R A$ has the A -module structure via the A -action on the second factor. For any $x \in \mathcal{X}(E)$, the specialization of M_A at x is

$$(M_A)_x := (M \otimes_R A) \otimes_{A,x} E = M \otimes_{R,w(x)} E = M_{w(x)}.$$

Definition 2.2 For an A -module M and $x \in \mathcal{X}(E)$, we let sp_x denote the specialization map $M_A \rightarrow (M_A)_x$.

We need the following lemma.

Lemma 2.3 *In the notation of Definition 2.2, suppose that M is a free A -module of rank one. Let m denote a generator of M and put $\Phi := \sum_{i=0}^{e-1} X^i m \otimes X^{e-1-i} \in M \otimes_R A$. Then*

- (i) *We have $(X \otimes 1)\Phi = (1 \otimes X)\Phi$.*
- (ii) *The element $\text{sp}_x(\Phi) \in M_{w(x)}$ generates the E -vector space $M[x]$.*
- (iii) *We have $\pi_x \circ \text{sp}_x(\Phi) = X(x)^{e-1} m_x$.*

Proof The first part is the abstract version of [10, Lemma 4.13] and the second part is that of [10, Proposition 4.14]. Final assertion follows from a direct calculation. $\square$

Remark 2.4 Suppose in this remark that $e = 1$, so that $A = R$. In this scenario, we have $M_A = M$ and $M_{w(x)} \xrightarrow[\pi_x]{\sim} M_x$. Moreover, the specialization map sp_x is simply the canonical projection $M \rightarrow M_x$.

2.4 Duality

Let M and M' be two free A -modules of finite rank equipped with an R -linear pairing

$$(\cdot, \cdot) : M' \otimes_R M \rightarrow R.$$

Assume that this pairing satisfies the following property:

Adj) For every $m' \in M'$ and $m \in M$, we have $(Xm', m) = (m', Xm)$.

For any $x \in \mathcal{X}(E)$, we denote by

$$(\cdot, \cdot)_{w(x)} : M'_{w(x)} \otimes_E M_{w(x)} \rightarrow E$$

the specialization of $(\cdot, \cdot)$ at $w(x)$.

Lemma 2.5 *There exists a unique pairing*

$$(\cdot, \cdot)_x : M'_x \otimes_E M[x] \rightarrow E$$

such that the restriction of $(\cdot, \cdot)_{w(x)}$ to the subspace $M'_{w(x)} \otimes_E M[x]$ factors as

$$\begin{array}{ccc} M'_{w(x)} \otimes_E M[x] & \xrightarrow{(\cdot, \cdot)_{w(x)}} & E \\ \pi_x \otimes \text{id} \searrow & & \nearrow (\cdot, \cdot)_x \\ & M'_x \otimes_E M[x] & \end{array} \quad (2)$$

Proof If $x \neq x_0$, the vector space M'_x is a direct summand of $M'_{w(x)}$, and in this case, $(\cdot, \cdot)_x$ coincides with the restriction of $(\cdot, \cdot)_{w(x)}$ on $M'_x \otimes_E M[x]$.

If $x = x_0$, then $M'_{x_0} = M'/XM'$ and $M[x_0] = X^{e-1}M_{k_0}$. The property **Adj** shows that $(\cdot, \cdot)_{k_0}$ is trivial on $XM'_{k_0} \otimes M[x_0]$, and therefore it factorizes uniquely through $M'_{x_0} \otimes_E M[x_0]$. $\square$

3 Exponential maps and triangulations

3.1 Cohomology of (φ, Γ) -modules

3.1.1. We set $K_n = \mathbb{Q}_p(\zeta_{p^n})$, $K_\infty = \bigcup_{n=0}^\infty K_n$ and $\Gamma = \text{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$. For any $n \geq 1$ we set $\Gamma_n = \text{Gal}(K_\infty/K_n)$, and put $G_n = \text{Gal}(K_n/\mathbb{Q}_p)$. Let us fix topological generators $\gamma_n \in \Gamma_n$ such that $\gamma_n^p = \gamma_{n+1}$ for all $n \geq 1$ and $\gamma_0^{p-1} = \gamma_1$. For any group G and left G -module M we denote by M^t the right G -module whose underlying group is M and on which G acts by $m \cdot g = g^{-1}m$.

If G is a finite abelian group, we denote by $X(G)$ its group of characters. If E is a fixed field such that $\rho \in X(E)$ takes values in E , we denote by

$$e_\rho = \frac{1}{|G|} \sum_{g \in G} \rho^{-1}(g)g$$

the corresponding idempotent of $E[G]$. For any $E[G]$ -module M we denote by $M^{(\rho)} = e_\rho M$ its ρ -isotypic component. For any map $f : N \rightarrow M$ we denote by $f^{(\rho)}$ the compositum

$$f^{(\rho)} : N \xrightarrow{f} M \xrightarrow{[e_\rho]} M^{(\rho)}.$$

3.1.2. In this section, we review the construction of the Bloch–Kato exponential map for crystalline (φ, Γ) -modules. Let E be a finite extension of $\mathbb{Q}_p$. For each $n \geq 0$, we denote by $\mathcal{R}_E$ the Robba ring of formal power series $f(\pi) = \sum_{m \in \mathbb{Z}} a_m \pi^m$ converging on some annulus of the form $r(f) \leq |\pi|_p < 1$. Recall that $\mathcal{R}_E^+ := \mathcal{R}_E \cap E[[\pi]]$ is the ring of formal power

series converging on the open unit disk. We equip $\mathcal{R}_E$ with the usual E -linear actions of the Frobenius operator φ and the cyclotomic Galois group Γ given by

$$\begin{aligned}\varphi(\pi) &= (1 + \pi)^p - 1, \\ \gamma(\pi) &= (1 + \pi)^{\chi(\gamma)} - 1, \quad \gamma \in \Gamma.\end{aligned}$$

Let ψ denote the left inverse of φ defined by

$$\psi(f(\pi)) = \frac{1}{p} \varphi^{-1} \left(\sum_{\zeta^p=1} f(\zeta(1 + \pi) - 1) \right).$$

Then $(\mathcal{R}_E^+)^{\psi=0}$ is a $\mathcal{H}(\Gamma)$ -module of rank one, generated by $1 + \pi$. The differential operator $\partial = (1 + \pi) \frac{d}{d\pi}$ is a bijection from $(\mathcal{R}^+)^{\psi=0}$ to itself. Furthermore, we have

$$\partial \circ \gamma = \chi(\gamma) \gamma \circ \partial.$$

Let $t = \log(1 + \pi)$ denote the “additive generator of $\mathbb{Z}_p(1)$ ”. Recall that

$$\varphi(t) = pt, \quad \gamma(t) = \chi(\gamma)t, \quad \gamma \in \Gamma.$$

Let $A = E \langle X/p^r \rangle$ be a Tate algebra over E as above. We denote by $\mathcal{R}_A = A \widehat{\otimes}_E \mathcal{R}_E$ the Robba ring with coefficients in A . The action of Γ , φ , ψ and ∂ can be extended to $\mathcal{R}_A$ by linearity.

3.1.3. Recall that a (φ, Γ) -module over $\mathcal{R}_A$ is a finitely generated projective module over $\mathcal{R}_A$ equipped with commuting semilinear actions of φ and Γ and satisfying some additional technical properties which we shall not record here (see [12, 25] for details). The cohomology $H^i(K_n, \mathbf{D})$ of $\mathbf{D}$ over K_n is defined as the cohomology of the Fontaine–Herr complex

$$C_{\varphi, \gamma_n}^\bullet(\mathbf{D}) : 0 \longrightarrow \mathbf{D} \xrightarrow{d_0} \mathbf{D} \oplus \mathbf{D} \xrightarrow{d_1} \mathbf{D} \longrightarrow 0,$$

where $d_0(x) := ((\varphi - 1)x, (\gamma_n - 1)x)$ and $d_1(y, z) := (\gamma_n - 1)y - (\varphi - 1)z$.

3.1.4. For any (φ, Γ) -module $\mathbf{D}$ over $\mathcal{R}_A$ we set $\mathcal{D}_{\text{cris}}(\mathbf{D}) = (\mathbf{D}[1/t])^\Gamma$. Recall that $\mathcal{D}_{\text{cris}}(\mathbf{D})$ is a finitely generated free A -module equipped with an A -linear Frobenius φ and a decreasing filtration $(\text{Fil}^i \mathcal{D}_{\text{cris}}(\mathbf{D}))_{i \in \mathbb{Z}}$. In [27], Nakamura defined A -linear maps

$$\exp_{\mathbf{D}, K_n} : \mathcal{D}_{\text{cris}}(\mathbf{D}) \otimes_{\mathbb{Q}_p} K_n \longrightarrow H^1(K_n, \mathbf{D}), \quad n \geq 0$$

which extends the definition of Bloch–Kato exponential maps to (φ, Γ) -modules.

3.1.5. Let V be a p -adic representation of $G_{\mathbb{Q}_p}$ with coefficients in A . The theory of (φ, Γ) -modules associates to V a (φ, Γ) -module $\mathbf{D}_{\text{rig}, A}^\dagger(V)$ over $\mathcal{R}_A$. The functor $\mathbf{D}_{\text{rig}, A}^\dagger$ is fully faithful and we have functorial isomorphisms

$$H^i(K_n, V) \simeq H^i(K_n, \mathbf{D}_{\text{rig}, A}^\dagger(V)).$$

If V is crystalline in the sense of [12], we have a functorial isomorphism between the “classical” filtered Dieudonné module $\mathbf{D}_{\text{cris}}(V)$ associated to V and $\mathcal{D}_{\text{cris}}(\mathbf{D}_{\text{rig}, A}^\dagger(V))$. Moreover, the diagram

$$\begin{array}{ccc} \mathbf{D}_{\text{cris}}(V) \otimes_{\mathbb{Q}_p} K_n & \xrightarrow{\exp_{V, K_n}} & H^1(K_n, V) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{D}_{\text{cris}}(\mathbf{D}_{\text{rig}, A}^\dagger(V)) \otimes_{\mathbb{Q}_p} K_n & \xrightarrow{\exp_{\mathbf{D}_{\text{rig}, A}^\dagger(V), K_n}} & H^1(K_n, \mathbf{D}_{\text{rig}, A}^\dagger(V)), \end{array}$$

where the upper row is the Bloch–Kato exponential map, commutes.

3.1.6. We refer the reader to [25, Section 4.4] and [30] for the proofs of the results reviewed in this subsection and for further details. Let $\mathcal{H}_E$ denote the algebra of formal power series $f(z) = \sum_{j=0}^{\infty} a_j z^j$ with coefficients in E that converges on the open unit disk. We put

$$\begin{aligned}\mathcal{H}_E(\Gamma_1) &:= \{f(\gamma_1 - 1) \mid f \in \mathcal{H}_E\}, \quad \mathcal{H}_E(\Gamma) := E[\Delta] \otimes_E \mathcal{H}_E(\Gamma_1), \\ \mathcal{H}_A(\Gamma) &:= A \widehat{\otimes}_E \mathcal{H}_E(\Gamma),\end{aligned}$$

where $\Delta = \text{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$. Note that $\mathcal{H}_A(\Gamma)$ contains the Iwasawa algebra $\Lambda_A := A \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]]$.

The Iwasawa cohomology $H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D})$ of a (φ, Γ) -module $\mathbf{D}$ over $\mathcal{H}_A$ is defined as the cohomology of the complex

$$\mathbf{D} \xrightarrow{\psi-1} \mathbf{D}$$

concentrated in degrees 1 and 2. In particular, $H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}) = \mathbf{D}^{\psi=1}$. We have canonical projections

$$\text{pr}_n : H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}) \rightarrow H^1(K_n, \mathbf{D}), \quad n \geq 0.$$

If $\mathbf{D} = \mathbf{D}_{\text{rig}, A}^{\dagger}(V)$ is the (φ, Γ) -module associated to a p -adic representation V over A , we then have a functorial isomorphism

$$\mathcal{H}_A(\Gamma) \otimes_{\Lambda_A} H_{\text{Iw}}^1(\mathbb{Q}_p, V) \simeq H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}_{\text{rig}, A}^{\dagger}(V)), \quad (3)$$

where $H_{\text{Iw}}^1(\mathbb{Q}_p, V)$ denotes the classical Iwasawa cohomology with coefficients in V . The isomorphism (3) composed with the projections pr_n coincide with the natural morphisms

$$\mathcal{H}_A(\Gamma) \otimes_{\Lambda_A} H_{\text{Iw}}^1(\mathbb{Q}_p, V) \longrightarrow H^1(K_n, V)$$

induced by the Iwasawa theoretic descent maps $H_{\text{Iw}}^1(\mathbb{Q}_p, V) \rightarrow H^1(K_n, V)$.

For each integer m , the cyclotomic twist

$$\text{Tw}_m : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_p, V(m)), \quad \text{Tw}_m(x) = x \otimes \varepsilon^{\otimes m}$$

can be extended to a map

$$\begin{aligned}\text{Tw}_m : \mathcal{H}_A(\Gamma) \otimes_{\Lambda_A} H_{\text{Iw}}^1(\mathbb{Q}_p, V) &\longrightarrow \mathcal{H}_A(\Gamma) \otimes_{\Lambda_A} H_{\text{Iw}}^1(\mathbb{Q}_p, V(m)) \\ f(\gamma - 1) \otimes x &\longmapsto f(\chi^{-m}(\gamma)\gamma - 1) \otimes \text{Tw}_m(x).\end{aligned}$$

3.2 The Perrin-Riou exponential map

3.2.1. In this subsection, we shall review fragments of Perrin-Riou's theory of large exponential maps. Define the operators

$$\begin{aligned}\nabla &= \frac{\log(\gamma_1)}{\log \chi(\gamma_1)}, \quad \gamma_1 \in \Gamma_1, \\ \ell_j &= j - \nabla, \quad j \in \mathbb{Z}.\end{aligned}$$

Note that ∇ does not depend on the choice of $\gamma_1 \in \Gamma_1$. It is easy to check by induction that

$$\prod_{j=0}^{h-1} \ell_j = (-1)^h t^h \partial^h \quad (\text{on } \mathcal{H}_E).$$

Let V be a p -adic crystalline representation of $G_{\mathbb{Q}_p}$ with coefficients in E satisfying the following condition:

LE) $\mathbf{D}_{\text{cris}}(V)^{\varphi=p^i} = 0$ for all $i \in \mathbb{Z}$.

Note that this assumption can be relaxed (see, for example, [8, 31]), but it simplifies the presentation. In particular, it implies that $H^0(K_\infty, V) = 0$ and therefore also that $H_{\text{Iw}}^1(\mathbb{Q}_p, V)$ is torsion free over the Iwasawa algebra (c.f. [31], Lemme 3.4.3 and Proposition 3.2.1).

Let us set

$$\mathfrak{D}(V) = (\mathcal{R}_E^+)^{\psi=0} \otimes_E \mathbf{D}_{\text{cris}}(V).$$

For any $\alpha \in \mathfrak{D}(V)$, the equation

$$(1 - \varphi)(F) = \alpha$$

has a unique solution $F \in \mathcal{R}_E^+ \otimes_E \mathbf{D}_{\text{cris}}(V)$. We define the maps

$$\Xi_{V,n} : \mathfrak{D}(V) \rightarrow \mathbf{D}_{\text{cris}}(V) \otimes_{\mathbb{Q}_p} K_n, \quad n \geq 0$$

on setting

$$\Xi_{V,n}(\alpha) = \begin{cases} p^{-n}(\text{id} \otimes \varphi)^{-n}(F)(\zeta_{p^n} - 1) & \text{if } n \geq 1, \\ \left(\frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} \right) \alpha(0) & \text{if } n = 0. \end{cases}$$

The following is the main result of [31].

Theorem 3.1 (Perrin-Riou) *Let V be a crystalline representation which satisfies the condition **LE)** above. Then for integers $h \geq 1$ such that $\text{Fil}^{-h} \mathbf{D}_{\text{cris}}(V) = \mathbf{D}_{\text{cris}}(V)$, there exists a $\mathcal{H}_E(\Gamma)$ -homomorphism*

$$\text{Exp}_{V,h} : \mathfrak{D}(V) \longrightarrow \mathcal{H}_E(\Gamma) \otimes_{\Lambda_E} H_{\text{Iw}}^1(\mathbb{Q}_p, V)$$

satisfying the following properties:

(i) For all $n \geq 0$ the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{D}(V) & \xrightarrow{\text{Exp}_{V,h}} & \mathcal{H}_E(\Gamma) \otimes_{\Lambda_E} H_{\text{Iw}}^1(\mathbb{Q}_p, V) \\ \downarrow \Xi_{V,n} & & \downarrow \text{pr}_n \\ \mathbf{D}_{\text{cris}}(V) \otimes K_n & \xrightarrow{(h-1)! \text{exp}_{V,K_n}} & H^1(K_n, V). \end{array}$$

(ii) Let us denote by $e_{-1} := \varepsilon^{-1} \otimes t$ the canonical generator of $\mathbf{D}_{\text{cris}}(\mathbb{Q}_p(-1))$. Then

$$\text{Exp}_{V(1),h+1} = -\text{Tw}_1 \circ \text{Exp}_{V,h} \circ (\partial \otimes e_{-1}).$$

(iii) We have

$$\text{Exp}_{V,h+1} = \ell_h \text{Exp}_{V,h}.$$

Let us put

$$\text{Exp}_{V,h,n} = \text{pr}_n \circ \text{Exp}_{V,h} : \mathfrak{D}(V) \longrightarrow H^1(K_n, V). \quad (4)$$

3.2.2. Set $G_n = \text{Gal}(K_n/K)$. Without loss of generality, we can assume that the characters of G_n take values in E . Recall that Shapiro's lemma gives an isomorphism of $E[G_n]$ -modules

$$H^1(K_n, V) \xrightarrow{\sim} H^1(\mathbb{Q}_p, V \otimes_E E[G_n]{}^c).$$

On taking the ρ -isotypic components, we obtain isomorphisms

$$H^1(K_n, V)^{(\rho)} \xrightarrow{\sim} H^1(\mathbb{Q}_p, V(\rho^{-1})), \quad \rho \in X(G_n).$$

We shall make use of the following elementary lemma giving the ρ -isotypic component of the map $\Xi_{V,n}$.

Lemma 3.2 *Assume that $\rho \in X(G_n)$ is a primitive character. Then for any $\alpha = f(\pi) \otimes d \in \mathfrak{D}(V)$*

$$\Xi_{V,n}^{(\rho)}(\alpha) = \begin{cases} e_\rho(f(\zeta_{p^n} - 1)) \otimes p^{-n} \varphi^{-n}(d) & \text{if } n \geq 1, \\ (1 - p^{-1} \varphi^{-1})(1 - \varphi)^{-1}(d) & \text{if } n = 0. \end{cases}$$

Proof See [13, Lemma 4.10]. $\square$

3.2.3. We recall the explicit construction of the Perrin–Riou exponential that was discovered by Berger in [12]. Let, as before, $F \in \mathcal{R}_E^+ \otimes \mathbf{D}_{\text{cris}}(V)$ denote the solution of the equation $(1 - \varphi)F = \alpha$. Define

$$\Omega_{V,h}(\alpha) := -\frac{\log \chi(\gamma_1)}{p} \ell_{h-1} \ell_{h-2} \cdots \ell_0(F(\pi)).$$

It is not difficult to see that $\Omega_{V,h}(\alpha) \in \mathbf{D}_{\text{rig},E}^+(V)^{\psi=1}$ and an explicit computation shows that $\Omega_{V,h}$ satisfies properties i–iii in Theorem 3.1.

See also [27] for a generalization of this approach to de Rham representations.

3.2.4. In the remainder of Sect. 3.2, we shall review the construction of the Perrin–Riou exponential maps for families of (φ, Γ) -modules of rank one.

For any continuous character $\delta : \mathbb{Q}_p^\times \rightarrow A^\times$ we denote by $\mathbf{D}_\delta = \mathcal{R}_A e_\delta$ the (φ, Γ) -module of rank one over the relative Robba ring $\mathcal{R}_A$, generated by e_δ and such that

$$\varphi(e_\delta) = \delta(p)e_\delta, \quad \gamma(e_\delta) = \delta(\chi(\gamma))e_\delta \quad \gamma \in \Gamma.$$

Set $\mathbf{D}_\delta^+ := \mathcal{R}_A^+ e_\delta$. We say that $\mathbf{D}_\delta$ is of Hodge–Tate weight $-m \in \mathbb{Z}$ (sic!) if

$$\delta(u) = u^m, \quad \forall u \in \mathbb{Z}_p^\times.$$

If $\mathbf{D}_\delta$ is of Hodge–Tate weight $-m$, then $\mathcal{D}_{\text{cris}}(\mathbf{D}_\delta)$ is the free A -module of rank one generated by $d_\delta = t^{-m} e_\delta$. It has the unique filtration break at $-m$, and φ acts on $\mathcal{D}_{\text{cris}}(\mathbf{D}_\delta)$ as the multiplication by $p^{-m} \delta(p)$ map.

3.2.5. Let $\mathbf{D}_\delta$ be a (φ, Γ) -module of rank one and Hodge–Tate weight $-m \in \mathbb{Z}$. The direct analogue of the condition **LE**) for families of (φ, Γ) -modules is the following condition:

LE*) For any $i \in \mathbb{Z}$, $1 - p^i \delta(p) \in A$ does not vanish on $\mathcal{X} = \text{Spm}(A)$.

For any point $x \in \text{Spm}(A)$, let $\mathbf{D}_{\delta,x}$ denote the specialization of $\mathbf{D}_\delta$ at x . To facilitate a comparison with **LE**), we remark that the condition **LE*)** implies that

$$\mathcal{D}_{\text{cris}}(\mathbf{D}_{\delta,x})^{\varphi=p^i} = 0, \quad \forall x \in \text{Spm}(A), \quad \forall i \in \mathbb{Z}.$$

Let us put $\mathfrak{D}(\mathbf{D}_\delta) = \mathcal{R}_A^{\psi=0} \otimes_A \mathcal{D}_{\text{cris}}(\mathbf{D}_\delta)$. We shall explain the construction of a family of maps (which we will call Perrin–Riou exponential maps)

$$\text{Exp}_{\mathbf{D}_\delta,h} : \mathfrak{D}(\mathbf{D}_\delta) \longrightarrow H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}_\delta), \quad h \geq m$$

modelled on the discussion in §3.2.3. Let us set $\alpha(\pi) := f(\pi) \otimes d_\delta \in \mathfrak{D}(\mathbf{D}_\delta)$. The equation

$$(1 - \delta(p)\varphi)F_m(\pi) = \partial^m f(\pi)$$

has a unique solution in $\mathcal{R}_A^+$. It is easy to see that

$$F_{-m}(\pi) \otimes e_\delta \in \mathbf{D}_\delta^{\psi=1} = H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}_\delta).$$

We define

$$\text{Exp}_{\mathbf{D}_\delta, m}(\alpha) := (-1)^{m-1} \frac{\log \chi(\gamma_1)}{p} F_m(\pi) \otimes e_\delta,$$

and set

$$\text{Exp}_{\mathbf{D}_\delta, h}(\alpha) = \left(\prod_{j=m}^{h-1} \ell_j \right) \circ \text{Exp}_{\mathbf{D}_\delta, m}(\alpha), \quad h \geq m+1.$$

The following result can be extracted from [28, Section 4] or proved directly using Berger's arguments.

Proposition 3.3 (i) *For any $h \geq m$ we have*

$$\text{Exp}_{\mathbf{D}_\delta, h+1} = \ell_h \text{Exp}_{\mathbf{D}_\delta, h}.$$

(ii) *For any $h \geq m$ the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{D}(\mathbf{D}_{\delta\chi}) & \xrightarrow{\text{Exp}_{\mathbf{D}_{\delta\chi}, h+1}} & H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}_{\delta\chi}) \\ e_1 \otimes \partial \downarrow & & \uparrow -\text{Tw}_1 \\ \mathfrak{D}(\mathbf{D}_\delta) & \xrightarrow{\text{Exp}_{\mathbf{D}_\delta, h}} & H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}_\delta) \end{array}$$

(iii) *Assume that $h \geq m \geq 1$. For any $\alpha(\pi) = f(\pi) \otimes d_\delta \in \mathcal{R}_A^{\psi=0} \otimes_A \mathcal{D}_{\text{cris}}(\mathbf{D}_\delta)$, the equation*

$$(1 - \varphi)(F) = \alpha(\pi)$$

has a unique solution $F \in \mathcal{D}_{\text{cris}}(\mathbf{D}_\delta) \otimes \mathcal{R}_A^+$ and it verifies

$$\text{Exp}_{\mathbf{D}_\delta, h}(\alpha) = -\frac{\log \chi(\gamma_1)}{p} \prod_{j=0}^{h-1} \ell_j(F).$$

Moreover,

$$\prod_{j=0}^{h-1} \ell_j(F) = (-1)^{h-1} t^h \delta^h(F) \in \left(t^{h-m} \mathbf{D}_\delta^+ \right)^{\psi=1}.$$

(iv) *Under the conditions and notation of iii), let us put*

$$\Xi_{\mathbf{D}_\delta, n}(\alpha) := \begin{cases} \left(\frac{1-p^{-1}\varphi^{-1}}{1-\varphi} \right) \alpha(0) & \text{if } n = 0, \\ p^{(m-1)n} \delta(p)^{-n} F(\zeta_{p^n} - 1) & \text{if } n \geq 1. \end{cases}$$

Then the map $\Xi_{\mathbf{D}_\delta, n} : \mathfrak{D}(\mathbf{D}_\delta) \rightarrow \mathcal{D}_{\text{cris}}(\mathbf{D}_\delta) \otimes_{\mathbb{Q}_p} K_n$ is surjective, and the diagram

$$\begin{array}{ccc} \mathfrak{D}(\mathbf{D}_\delta) & \xrightarrow{\text{Exp}_{\mathbf{D}_\delta, h}} & H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}_\delta) \\ \downarrow \Xi_{\mathbf{D}_\delta, n} & & \downarrow \text{pr}_n \\ \mathcal{D}_{\text{cris}}(\mathbf{D}_\delta) \otimes_{\mathbb{Q}_p} K_n & \xrightarrow{(h-1)! \text{exp}_{\mathbf{D}_\delta, K_n}} & H^1(K_n, \mathbf{D}_\delta) \end{array}$$

commutes.

3.3 Triangulations in families

3.3.1. In Sect. 3.3, we shall work in the setting of Sect. 2 and introduce various objects with which we shall apply the general constructions in Sect. 2. To that end, let us fix an integer $e \geq 1$ and put $A = R[X]/(X^e - Y)$ as before, where $R = E \langle Y/p^{r^e} \rangle$ is a Tate algebra. Set $\mathcal{W} = \text{Spm}(R)$, $\mathcal{X} = \text{Spm}(A)$ and denote by $w : \mathcal{X} \rightarrow \mathcal{W}$ the weight map. We fix an integer $k_0 \geq 2$ and identify $\mathcal{W}$ with the closed disk $D(k_0, p^{r^e-1})$ as in Sect. 2.1. Let $x_0 \in \mathcal{X}$ denote the unique point such that $w(x_0) = k_0$. We put

$$\mathcal{X}^{\text{cl}} = \{x \in \mathcal{X} \mid w(x) \in \mathbb{Z}, w(x) \geq 0\}.$$

We let V denote a free A -module of rank 2 which is endowed with a continuous action of the Galois group $G_{\mathbb{Q},S}$. In accordance with the notation of Sect. 2, for any $y \in \mathcal{W}(E)$ and $x \in \mathcal{X}$, we set $V_y = V \otimes_{R,w} E$, $V_x = V/\mathfrak{m}_x V$ and denote by $\pi_x : V_{w(x)} \rightarrow V_x$ the canonical projection. Note that $V_{k_0} = V/X^e V$, $V_{x_0} = V/XV$ and $V[x_0] = X^{e-1}V/X^e V$.

3.3.2. We shall assume that V verifies following conditions:

- C1)** For each $x \in \mathcal{X}^{\text{cl}}$ of integer weight $w(x) \geq 0$ the restriction of V_x on the decomposition group at p is semistable of Hodge–Tate weights $(0, w(x) + 1)$.
- C2)** There exists $\alpha \in A$ such that for all $x \in \mathcal{X}^{\text{cl}}$ the eigenspace $\mathbf{D}_{\text{st}}(V_x)^{\varphi=\alpha(x)}$ is one dimensional.
- C3)** For each $x \in \mathcal{X}^{\text{cl}} - \{x_0\}$

$$\mathbf{D}_{\text{st}}(V_x)^{\varphi=\alpha(x)} \cap \text{Fil}^{w(x)+1} \mathbf{D}_{\text{st}}(V_x) = 0.$$

On shrinking $\mathcal{X}$ as necessary, it follows from [25] that one can construct a unique (φ, Γ) -submodule $\mathbf{D} \subseteq \mathbf{D}_{\text{rig},A}^{\dagger}(V)$ of rank one with the following properties:

- $\varphi\Gamma_1) \mathbf{D} = \mathbf{D}_{\delta}$ with $\delta : \mathbb{Q}_p^{\times} \rightarrow A^{\times}$ such that $\delta(p) = \alpha(x)$ and $\delta|_{\mathbb{Z}_p^{\times}} = 1$.
- $\varphi\Gamma_2) \mathcal{D}_{\text{cris}}(\mathbf{D}_x) = \mathbf{D}_{\text{st}}(V_x)^{\varphi=\alpha(x)}$ for each $x \in \mathcal{X}^{\text{cl}}$, where $\mathbf{D}_x := \mathbf{D} \otimes_{A,x} E$.
- $\varphi\Gamma_3)$ For each $x \in \mathcal{X}^{\text{cl}} - \{x_0\}$, the (φ, Γ) -module $\mathbf{D}_x$ is saturated in $\mathbf{D}_{\text{rig},E}^{\dagger}(V_x)$.

Let $\mathbf{D}_{x_0}^{\text{sat}}$ denote the saturation of the specialization $\mathbf{D}_{x_0}$ of the (φ, Γ) -module $\mathbf{D}$ at x_0 . Then, as explained in the final section of [25], we have $\mathbf{D}_{x_0} = t^m \mathbf{D}_{x_0}^{\text{sat}}$ for some $m \geq 0$. We will consider the following two scenarios:²

$$\neg \Theta) \mathbf{D}_{\text{st}}(V_{x_0})^{\varphi=\alpha(x_0)} \cap \text{Fil}^{k_0+1} \mathbf{D}_{\text{st}}(V_{x_0}) = 0.$$

Since $\mathcal{D}_{\text{cris}}(\mathbf{D}_{x_0}^{\text{sat}}) = \mathbf{D}_{\text{st}}(V_{x_0})^{\varphi=\alpha(x_0)}$, the condition $\neg \Theta$) implies that the Hodge–Tate weight of $\mathbf{D}_{\text{st}}(V_{x_0})^{\varphi=\alpha(x_0)}$ is 0. Since the Hodge–Tate weight of $\mathbf{D}_{x_0}$ is also 0, we deduce that in this case $m = 0$ and $\mathbf{D}_{x_0}$ is saturated in $\mathbf{D}_{\text{rig},E}^{\dagger}(V_x)$.

$$\Theta) \mathbf{D}_{\text{st}}(V_{x_0})^{\varphi=\alpha(x_0)} = \text{Fil}^{k_0+1} \mathbf{D}_{\text{st}}(V_{x_0}).$$

Suppose that Θ) holds. Then $\mathcal{D}_{\text{cris}}(\mathbf{D}_{x_0}^{\text{sat}}) = \text{Fil}^{k_0+1} \mathbf{D}_{\text{st}}(V_{x_0})$. Therefore $\mathbf{D}_{x_0}^{\text{sat}}$ is of Hodge–Tate weight $k_0 + 1$ and $m = k_0 + 1$. Let $\beta(x_0)$ denote the other eigenvalue of φ on $\mathbf{D}_{\text{st}}(V_{x_0})$. By the weak admissibility of $\mathbf{D}_{\text{st}}(V_{x_0})$, we infer that $v_p(\alpha(x_0)) \geq k_0 + 1$, $v_p(\beta(x_0)) \geq 0$ and that $v_p(\alpha(x_0)) + v_p(\beta(x_0)) = k_0 + 1$. Thence, $v_p(\alpha(x_0)) = k_0 + 1$ and $v_p(\beta(x_0)) = 0$, and the eigenspaces $\mathbf{D}_{\text{st}}(V_{x_0})^{\varphi=\alpha(x_0)}$ and $\mathbf{D}_{\text{st}}(V_{x_0})^{\varphi=\beta(x_0)}$ are weakly admissible. They are

² In the context of elliptic modular forms, the condition $\neg \Theta$) (resp., Θ)) translates into the requirement that the corresponding eigenform is non- θ critical (resp., is θ -critical) in the sense of Coleman.

therefore admissible and the restriction of V_{x_0} to the decomposition group at p decomposes into a direct sum

$$V_{x_0} = V_{x_0}^{(\alpha)} \oplus V_{x_0}^{(\beta)} \quad (5)$$

of two one-dimensional $G_{\mathbb{Q}_p}$ -representations. Let us $\mathbf{D}^{(\alpha)} = \mathbf{D}_{\text{rig}, E}^{\dagger}(V_{x_0}^{(\alpha)})$ and $\mathbf{D}^{(\beta)} = \mathbf{D}_{\text{rig}, E}^{\dagger}(V_{x_0}^{(\beta)})$. Then

$$\mathbf{D}^{(\alpha)} = \mathcal{R}_E e^{(\alpha)} \quad , \quad \mathbf{D}^{(\beta)} = \mathcal{R}_E e^{(\beta)} ,$$

where

$$\begin{aligned} \gamma(e^{(\alpha)}) &= \chi^{-1-k_0}(\gamma)e^{(\alpha)} , & \varphi(e^{(\alpha)}) &= \alpha(x_0)p^{-1-k_0}e^{(\alpha)} , \\ \gamma(e^{(\beta)}) &= e^{(\beta)} , & \varphi(e^{(\beta)}) &= \beta(x_0)e^{(\beta)} , \end{aligned} \quad \gamma \in \Gamma.$$

We have

$$\begin{aligned} \mathbf{D}_{\text{cris}}(V_{x_0}^{(\alpha)}) &= \mathbf{D}_{\text{st}}(V_{x_0})^{\varphi=\alpha(x_0)} , \\ \mathbf{D}_{\text{cris}}(V_{x_0}^{(\beta)}) &= \mathbf{D}_{\text{st}}(V_{x_0})^{\varphi=\beta(x_0)} , \\ \mathbf{D}^{(\alpha)} &= \mathbf{D}_{x_0}^{\text{sat}} = t^{-(k_0+1)}\mathbf{D}_{x_0} . \end{aligned} \quad (6)$$

We remark that $\mathcal{D}_{\text{cris}}(\mathbf{D}_{x_0})$ and $\mathbf{D}_{\text{cris}}(V_{x_0}^{(\alpha)})$ are isomorphic as φ -modules but not as filtered modules: they have Hodge–Tate weights 0 and $k_0 + 1$, respectively.

4 Two variable p -adic L -functions: the abstract definitions

Suppose that we are given another free A -module V' of rank two which is equipped with a continuous $G_{\mathbb{Q}, S}$ -action, together with a Galois equivariant R -linear pairing

$$(\ , \ ) : V' \otimes_R V \longrightarrow R \quad (7)$$

satisfying the condition **Adj**) of Sect. 2:

$$\text{For every } v' \in V' \text{ and } v \in V, \text{ we have } (Xv', v) = (v', Xv).$$

4.1 Duality (bis)

We have a canonical $\mathcal{H}_R(\Gamma)$ -bilinear pairing

$$\langle \ , \ \rangle : (\mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V'(1))) \otimes (\mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V)^t) \longrightarrow \mathcal{H}_R(\Gamma), \quad (8)$$

which has the following explicit description. Let

$$(\ , \ )_n : H^1(K_n, V'(1)) \times H^1(K_n, V) \longrightarrow R, \quad n \geq 0$$

denote the cup-product pairings induced by (7). Recall from §3.1.6. the projection maps

$$\begin{aligned} \text{pr}_n &: (\mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V)) \longrightarrow H^1(K_n, V), \\ \text{pr}'_n &: (\mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V'(1))) \longrightarrow H^1(K_n, V'(1)), \quad n \geq 0. \end{aligned}$$

We then have

$$\langle x, y \rangle \equiv \sum_{\tau \in \Gamma/\Gamma_n} (\tau^{-1} \text{pr}'_n(x), \text{pr}_n(y))_n \tau \pmod{(\gamma_n - 1)} \quad (9)$$

(see [31], Section 3.6). In particular, for any finite character $\rho \in X(\Gamma)$ of conductor p^n , we have

$$\rho((x, y^t)) = \sum_{\tau \in \Gamma/\Gamma_n} (\tau^{-1} \text{pr}'_n(x), \text{pr}_n(y))_n \rho(\tau) = (e_\rho(x), e_\rho(y^t))_{\rho,0},$$

where $(,)_{\rho,0}$ stands for the cup-product pairing

$$H^1(\mathbb{Q}_p, V'(\chi\rho^{-1})) \otimes_R H^1(\mathbb{Q}_p, V(\rho)) \xrightarrow{(\cdot, \cdot)_{\rho,0}} R.$$

Note that the pairing (8) can be recast entirely in terms of the Iwasawa cohomology of associated (φ, Γ) -modules; c.f. [25, Section 4.2].

4.1.1. We apply the formalism of Sect. 2.4 to our situation. Equip the tensor product

$$A \otimes_{\Lambda_R} (\mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V)) \simeq \mathcal{H}_A(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V)$$

with the action of A through the first factor. We extend the pairing (8) by linearity to the pairing

$$\langle \cdot, \cdot \rangle_A : (\mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V'(1))) \otimes_{\mathcal{H}_R(\Gamma)} (\mathcal{H}_A(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V)^t) \longrightarrow \mathcal{H}_A(\Gamma), \quad (10)$$

which is $\mathcal{H}_R(\Gamma)$ -linear (respectively, $\mathcal{H}_A(\Gamma)$ -linear) with respect to the first (respectively second) factor.

For any $x \in \mathcal{X}(E)$, the functoriality of the cup-products gives rise to the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V'(1))) & \otimes_{\mathcal{H}_R(\Gamma)} & (\mathcal{H}_A(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V)^t) \xrightarrow{\langle \cdot, \cdot \rangle_A} \mathcal{H}_A(\Gamma) \\ \downarrow w(x) & & \downarrow \text{sp}_x \quad \quad \quad \downarrow x \\ \mathcal{H}_E(\Gamma) \otimes_{\Lambda_E} H_{\text{Iw}}^1(\mathbb{Q}_p, V'_{w(x)}(1)) & \otimes_{\mathcal{H}_E(\Gamma)} & (\mathcal{H}_E(\Gamma) \otimes_{\Lambda_E} H_{\text{Iw}}^1(\mathbb{Q}_p, V_{w(x)})^t) \xrightarrow{\langle \cdot, \cdot \rangle_{w(x)}} \mathcal{H}_E(\Gamma). \end{array} \quad (11)$$

where the left and the right vertical maps are the specializations at $w(x)$ and x respectively, and the middle vertical map is the specialization defined in 40314031 Sect. 2.4.

By Lemma 2.5, for any $x \in \mathcal{X}(E)$ the pairing $(,)$ induces a pairing

$$(,)_x : V_x \otimes_E V[x] \longrightarrow E.$$

The pairing in the bottom row of (11) therefore factors as

$$\begin{array}{ccc} \mathcal{H}_E(\Gamma) \otimes_{\Lambda_E} H_{\text{Iw}}^1(\mathbb{Q}_p, V'_{w(x)}(1)) & \otimes & (\mathcal{H}_E(\Gamma) \otimes_{\Lambda_E} H_{\text{Iw}}^1(\mathbb{Q}_p, V[x])^t) \xrightarrow{\langle \cdot, \cdot \rangle_{w(x)}} \mathcal{H}_E(\Gamma) \\ \downarrow x & & \parallel \\ \mathcal{H}_E(\Gamma) \otimes_{\Lambda_E} H_{\text{Iw}}^1(\mathbb{Q}_p, V'_x(1)) & \otimes & (\mathcal{H}_E(\Gamma) \otimes_{\Lambda_E} H_{\text{Iw}}^1(\mathbb{Q}_p, V[x])^t) \xrightarrow{\langle \cdot, \cdot \rangle_x} \mathcal{H}_E(\Gamma). \end{array} \quad (12)$$

4.2 Perrin-Riou-style two variable p -adic L -functions

Recall from §3.2.5 the Perrin-Riou exponential

$$\text{Exp}_{\mathbf{D},h} : \mathfrak{D}(V) \longrightarrow H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}).$$

Note that we have a canonical injection

$$H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}) \longrightarrow \mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V) = \mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V).$$

Let us fix an R -module generator $\eta \in \mathcal{D}_{\text{cris}}(\mathbf{D})$.

Let us set $\mathfrak{D}(V)_A = \mathfrak{D}(V) \otimes_R A$. We have a canonical isomorphism

$$\mathfrak{D}(V)_A \simeq \mathcal{D}_{\text{cris}}(\mathbf{D}) \otimes_R (\mathcal{R}_A^+)^{\psi=0}$$

The Perrin-Riou exponential map $\text{Exp}_{\mathbf{D},h}$ can be extended by linearity to an A -linear map

$$\text{Exp}_{\mathbf{D},h}^A : \mathfrak{D}(V)_A \xrightarrow{\text{Exp}_{\mathbf{D},h} \otimes \text{id}_A} H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}) \otimes_R A.$$

Note that we have a canonical injection

$$H_{\text{Iw}}^1(\mathbb{Q}_p, \mathbf{D}) \otimes_R A \longrightarrow A \otimes_R (\mathcal{H}_R(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V)) = \mathcal{H}_A(\Gamma) \otimes_{\Lambda_R} H_{\text{Iw}}^1(\mathbb{Q}_p, V).$$

Fix an A -module generator $\eta \in \mathcal{D}_{\text{cris}}(\mathbf{D})$ and set

$$\begin{aligned} \mathfrak{v} &:= \sum_{i=0}^{e-1} X^i \eta \otimes X^{e-1-i} \in \mathcal{D}_{\text{cris}}(\mathbf{D}) \otimes_R A, \\ \widetilde{\mathfrak{v}} &:= \mathfrak{v} \otimes (1 + \pi) \in \mathfrak{D}(V)_A. \end{aligned}$$

We remark that for each $x \in \mathcal{X}(E)$, the specialization $\mathfrak{v}_x := \text{sp}_x(\mathfrak{v})$ is a generator of $\mathcal{D}_{\text{cris}}(\mathbf{D})[x]$ by Lemma 2.3. Let us put $\widetilde{\mathfrak{v}}_x := \mathfrak{v}_x \otimes (1 + \pi)$.

Definition 4.1

(i) For each $h \geq 0$ and cohomology class $\mathbf{z} \in H_{\text{Iw}}^1(\mathbb{Q}_p, V'(1))$, we define

$$\begin{aligned} \mathfrak{L}_{\mathbf{D}, \mathfrak{v}, 1-h}^A &: H_{\text{Iw}}^1(\mathbb{Q}_p, V'(1)) \longrightarrow \mathcal{H}_A(\Gamma), \\ \mathfrak{L}_{\mathbf{D}, \mathfrak{v}, 1-h}^A(\mathbf{z}) &:= \left\langle \mathbf{z}, c \circ \text{Exp}_{\mathbf{D},h}^A(\widetilde{\mathfrak{v}}) \right\rangle_A. \end{aligned}$$

(ii) We define the two-variable p -adic L -function associated to $\mathbf{z}$ on setting

$$L_{p, \mathfrak{v}}(\mathbf{z}) := \mathfrak{L}_{\mathbf{D}, \mathfrak{v}, 1}^A(\mathbf{z}) \in \mathcal{H}_A(\Gamma).$$

For each $x \in \mathcal{X}^{\text{cl}}(E)$, we similarly define the one-variable p -adic L -function

$$L_{p, \mathfrak{v}_x}(\mathbf{z}_x) := \left\langle \mathbf{z}_x, c \circ \text{Exp}_{\mathbf{D}[x], 0}(\widetilde{\mathfrak{v}}_x) \right\rangle_x \in \mathcal{H}_E(\Gamma).$$

(iii) For any $x \in \mathcal{X}(E)$, finite character $\rho \in X(\Gamma)$, we denote by

$$L_{p, \mathfrak{v}}(\mathbf{z}, x, \rho) := \rho \circ x \circ L_{p, \mathfrak{v}}(\mathbf{z}, X).$$

the value of $L_{p, \mathfrak{v}}(\mathbf{z})$ at (x, ρ) .

Proposition 4.2 Suppose $e = 1$. For any $x \in \mathcal{X}^{\text{cl}}(E)$ we have

$$L_{p, \mathfrak{v}}(\mathbf{z})(x) = L_{p, \mathfrak{v}_x}(\mathbf{z}_x).$$

Proof This is an immediate consequence of the diagram (11), combined with the discussion in Remark 2.4. $\square$

5 Local description of the eigencurve

In Sect. 5, we review Bellaïche's results in [10] on the local description of the eigencurve (most particularly, about a θ -critical eigenform). We also follow his exposition very closely here, particularly of Sects. 2.1.1, 2.1.3, 3.1, 3.2 and 3.4 in op. cit.; and rely also on the notation set therein for the most part (e.g., unless we declare otherwise).

5.1 Modular symbols

5.1.1. Let $\mathcal{WS} = \text{Hom}(\mathbb{Z}_p^\times, \mathbb{G}_m)$ denote the weight space which we consider as a rigid analytic space over a finite extension E of $\mathbb{Q}_p$. If $y \in \mathcal{WS}(E)$, we denote by $\kappa_y : \mathbb{Z}_p^\times \rightarrow E^\times$ the associated character. We consider $\mathbb{Z}$ as a subset of $\mathcal{WS}(E)$ identifying $k \in \mathbb{Z}$ with the character $u \mapsto u^k$. Set

$$\mathcal{W}^* = \left\{ y \in \mathcal{WS} \mid v_p(\kappa_y(a)^{p-1} - 1) > \frac{1}{p-1} \quad \forall a \in \mathbb{Z}_p^\times \right\}.$$

Note that $\mathbb{Z} \subset \mathcal{W}^*$. If $U \subset \mathcal{W}^*$ is either an affinoid disk or a wide open disk we will write O_U° for the ring of analytic functions on U that are bounded by 1 and set $O_U = O_U^\circ[1/p]$.

We denote by $\kappa_U : \mathbb{Z}_p^\times \rightarrow O_U^{\circ,\times}$ the universal weight character characterized by the property

$$\kappa_y = y \circ \kappa, \quad \forall y \in U(E).$$

We remark that our assumption that $U \subset \mathcal{W}^*$ is to guarantee that κ_U lands in $O_U^{\circ,\times}$.

5.1.2. Let $\mathbb{A}(U)^\circ$ (resp., $\mathbb{A}'(U)^\circ$) denote the space of functions $f : \mathbb{Z}_p^\times \times \mathbb{Z}_p \rightarrow O_U^\circ$ (resp., $f : p\mathbb{Z}_p \times \mathbb{Z}_p^\times \rightarrow O_U^\circ$) satisfying the following properties:

- f is homogeneous of weight κ_U , namely

$$f(ax, ay) = \kappa_U(a)f(x, y), \quad \forall a \in \mathbb{Z}_p^\times;$$

- The one variable function $f(1, z)$ (resp., $f(pz, 1)$) is analytic, namely it can be written in the form

$$\sum_{m=0}^{\infty} c_m z^m, \quad c_m \in O_V^\circ$$

where c_m goes to zero when $m \rightarrow \infty$. Note that in the scenario when U is wide open, we work with the m_U -adic topology.

Set $\mathbb{D}(U)^\circ := \text{Hom}_{O_U^{\circ,\text{cont}}}(\mathbb{A}(U)^\circ, O_U^\circ)$, $\mathbb{A}(U) = \mathbb{A}(U)^\circ[1/p]$ and $\mathbb{D}(U) := \mathbb{D}(U)^\circ[1/p]$. Similarly, we set $\mathbb{D}'(U)^\circ := \text{Hom}_{O_U^{\circ,\text{cont}}}(\mathbb{A}'(U)^\circ, O_U^\circ)$, $\mathbb{A}'(U) = \mathbb{A}'(U)^\circ[1/p]$ and $\mathbb{D}'(U) := \mathbb{D}'(U)^\circ[1/p]$.

Let

$$\Sigma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{Z}_p) : p \nmid a, p \mid c, ad - bc \neq 0 \right\}$$

and

$$\Sigma'_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{Z}_p) : p \nmid d, p \mid c, ad - bc \neq 0 \right\}.$$

The monoid $\Sigma_0(p)$ acts naturally on $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ and therefore on $\mathbb{A}(U)$ and $\mathbb{D}(U)$. Similarly, the monoid $\Sigma'_0(p)$ acts on $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ and therefore on $\mathbb{A}'(U)$ and $\mathbb{D}'(U)$.

If $\mathcal{W} = \text{Spm}(R)$ is an affinoid, our $\mathbb{D}(\mathcal{W})$ corresponds to the space of distributions $\mathbf{D}(R)[0]$ in [10]. If U is an open wide disk, our $\mathbb{D}(U)$ and $\mathbb{D}'(U)$ correspond to $D_{U,0}(T_0)$ and $D_{U,0}(T'_0)$ of [26]. If U reduces to one point $y \in \mathcal{WS}(E)$, we write $\mathbb{D}_y(E)$ (resp., $\mathbb{D}'_y(E)$) instead of $\mathbb{D}(U)$ (resp., $\mathbb{D}'(U)$) and $\mathbb{A}_y(E)$ (resp., $\mathbb{A}'_y(E)$) instead of $\mathbb{A}(U)$.

5.1.3. If $\mathcal{W} \subset \mathcal{WS}$ is an affinoid disk we will also work with the space of overconvergent modular symbols $\mathbb{D}^\dagger(\mathcal{W})$ which corresponds to the space $\mathbf{D}^\dagger[0](R)$ in [10]. To be more precise, we let $\mathbb{A}[r](\mathcal{W})$ denote the space of functions $f : \mathbb{Z}_p \rightarrow \mathcal{O}_{\mathcal{W}}$ that are analytic on the closed disk of radius r and let $\mathbb{D}[r](\mathcal{W}) = \text{Hom}_{\mathcal{O}_{\mathcal{W}}, \text{cpt}}(\mathbb{A}[r](\mathcal{W}), \mathcal{O}_{\mathcal{W}})$ denote the space of compact continuous linear maps $\mathbb{A}[r](\mathcal{W}) \rightarrow \mathcal{O}_{\mathcal{W}}$. In particular, $\mathbb{A}[1](\mathcal{W}) = \mathbb{A}(\mathcal{W})$ and $\mathbb{D}[1](\mathcal{W}) = \mathbb{D}(\mathcal{W})$. For any $r' > r$ the natural map $\mathbb{A}[r'](\mathcal{W}) \rightarrow \mathbb{A}[r](\mathcal{W})$ induces a map $\mathbb{D}[r](\mathcal{W}) \rightarrow \mathbb{D}[r'](\mathcal{W})$, and one defines

$$\mathbb{D}^\dagger(\mathcal{W}) = \varprojlim_{r>0} \mathbb{D}[r](\mathcal{W}).$$

If $\mathcal{W} \subset U$, where U is a wide open, then we have the restriction map $\mathbb{A}_U \rightarrow \mathbb{A}(\mathcal{W})$, which induces an isomorphism of $\mathcal{O}_{\mathcal{W}}$ -modules

$$\mathbb{D}(\mathcal{W}) \simeq \mathbb{D}(U)_{\mathcal{W}}, \quad \mathbb{D}(U)_{\mathcal{W}} := \mathbb{D}(U) \widehat{\otimes}_{\mathcal{O}_U} \mathcal{O}_{\mathcal{W}}$$

(c.f. [3], Lemma 3.8). In particular, we have an injection

$$\mathbb{D}^\dagger(\mathcal{W}) \longrightarrow \mathbb{D}(U)_{\mathcal{W}}. \quad (13)$$

For any $y \in \mathcal{W}(E)$, the natural map

$$\mathbb{D}^\dagger(\mathcal{W}) \otimes_{\mathcal{O}_{\mathcal{W}}, y} E \longrightarrow \mathbb{D}^\dagger_y(E) \quad (14)$$

is a $\Sigma_0(p)$ -equivariant isomorphism of E -Banach spaces (c.f. [10], Lemma 3.6).

We refer the reader to [3, 10] and [26, Section 4] for further details pertaining to these objects.

5.1.4. Put $\Gamma_p = \Gamma_1(N) \cap \Gamma_0(p) \subset \text{SL}_2(\mathbb{Z})$; note that this congruence subgroup is denoted by Γ in [10]. We denote by $H^*(\Gamma_p, -)$ (resp., $H^*_c(\Gamma_p, -)$) the cohomology of Γ_p (resp., the cohomology with compact support). Note that $H^0_c(\Gamma_p, -)$ and $H^2(\Gamma_p, -)$ vanish since Γ_p is the fundamental group of an open curve.

Lemma 5.1 *Let $\mathcal{W} \subset \mathcal{W}^*$ be an affinoid disk. Then:*

(i) *One has*

$$H^2_c(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W})) \cong \begin{cases} \{0\} & \text{if } 0 \notin \mathcal{W}(E), \\ E & \text{if } 0 \in \mathcal{W}(E). \end{cases}$$

(ii) *For any $y \in \mathcal{W}(E)$, the specialization at y gives rise to the short exact sequence*

$$0 \longrightarrow \mathbb{D}^\dagger(\mathcal{W}) \longrightarrow \mathbb{D}^\dagger(\mathcal{W}) \xrightarrow{y} \mathbb{D}^\dagger_y(E) \longrightarrow 0 \quad (15)$$

(iii) *The exact sequence (15) induces an isomorphism*

$$H^1(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W})) \otimes_{\mathcal{O}_{\mathcal{W}}, y} E \simeq H^1(\Gamma_p, \mathbb{D}^\dagger_y(E)) \quad (16)$$

and an injection

$$H^1_c(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W})) \otimes_{\mathcal{O}_{\mathcal{W}}, y} E \hookrightarrow H^1_c(\Gamma_p, \mathbb{D}^\dagger_y(E)) \quad (17)$$

- (iv) If $y \neq 0$, the map (17) is an isomorphism. If $y = 0$, then $\text{coker}(17)$ is isomorphic to E .
 (v) The statements i-iv) remain true if $\mathbb{D}^\dagger(\mathcal{W})$ is replaced by $\mathbb{D}(U)$, where $U \subset \mathcal{W}^*$ is a wide open.

Proof The part (i) is proved in [10, Lemma 3.9] and the same arguments compute H_c^2 with coefficients in $\mathbb{D}(U)$.

The exactness of (15) in part (ii) is proved in [10, Lemma 3.6], whereas its analogue for $\mathbb{D}(U)$ in [3, Proposition 3.11].

The isomorphism (16) in part (iii) follows from the long exact sequence associated to the specialization exact sequence and the vanishing of $H^2(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W}))$. The inclusion (17) for the cohomology with compact support and the point (iv) follow formally from (i) (c.f. [10, Theorem 3.10]), and the proofs of these statements for cohomology with coefficients in $\mathbb{D}(U)$ are the same. $\square$

5.1.5. Consider the spaces of modular symbols

$$\text{Symb}_{\Gamma_p}(\mathbb{D}) := \text{Hom}_{\Gamma_p}(\text{Div}^0(\mathbb{P}^1(\mathbb{Q})), \mathbb{D}), \quad \mathbb{D} = \mathbb{D}(U), \mathbb{D}_y^\dagger, \mathbb{D}^\dagger(\mathcal{W}),$$

where $\text{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ is the group of divisors of degree 0 on $\mathbb{P}^1(\mathbb{Q})$. Recall the functorial Hecke equivariant isomorphism of Ash and Stevens [4, Proposition 4.2]

$$\text{Symb}_{\Gamma_p}(\mathbb{D}) \xrightarrow{\sim} H_c^1(\Gamma_p, \mathbb{D}). \quad (18)$$

For a positive real number ν , we denote by $\text{Symb}_{\Gamma_p}(\mathbb{D})^{\leq \nu}$ the space of modular symbols of slope bounded by ν and by

$$\text{Symb}_{\Gamma_p}^\pm(\mathbb{D})^{\leq \nu} \subset \text{Symb}_{\Gamma_p}(\mathbb{D})^{\leq \nu}$$

the ± 1 -eigenspace for the involution ι given by the action of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (see for example, [10, Section 3.2.4]).

Lemma 5.2 *Let U be a wide open and $\mathcal{W} \subset U$ an affinoid disk. Then we have canonical isomorphisms*

$$\begin{aligned} \text{Symb}_{\Gamma_p}(\mathbb{D}(U)_{\mathcal{W}}) &\simeq \text{Symb}_{\Gamma_p}(\mathbb{D}(U))_{\mathcal{W}} \\ H^1(\Gamma_p, \mathbb{D}(U)_{\mathcal{W}}) &\simeq H^1(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}} \end{aligned}$$

where $\mathbb{D}(U)_{\mathcal{W}} = \mathbb{D}(U) \widehat{\otimes}_{O_U} O_{\mathcal{W}}$ and $\text{Symb}_{\Gamma_p}(\mathbb{D}(U))_{\mathcal{W}} = \text{Symb}_{\Gamma_p}(\mathbb{D}(U)) \widehat{\otimes}_{O_U} O_{\mathcal{W}}$.

Proof Only in this proof, A shall denote a general ring. For any complex $C^\bullet$ of A -modules over a ring A and a morphism of rings $A \rightarrow B$, one has the spectral sequence

$$E_2^{ij} = \text{Tor}_{-i}^A(H^j(C^\bullet), B) \implies H^{i+j}(C^\bullet \otimes_A B).$$

Let us choose $C^\bullet$ as the complex computing the cohomology with compact support with coefficients in $\mathbb{D}(U)^\circ$, $A = O_U^\circ$ and $B = O_{\mathcal{W}}^\circ/p^n$. Then our spectral sequence induces an exact sequence

$$\begin{aligned} 0 \longrightarrow H_c^1(\Gamma_p, \mathbb{D}(U)^\circ) \otimes_{O_U^\circ} O_{\mathcal{W}}^\circ/p^n &\longrightarrow H_c^1(\Gamma_p, \mathbb{D}(U)^\circ \otimes_{O_U^\circ} O_{\mathcal{W}}^\circ/p^n) \\ &\longrightarrow \text{Tor}_1^{O_U^\circ}(H_c^2(\Gamma_p, \mathbb{D}(U)^\circ), O_{\mathcal{W}}^\circ/p^n). \end{aligned} \quad (19)$$

It follows from Lemma 5.1(i) (combined with (v) of the same lemma) that $H_c^2(\Gamma_p, \mathbb{D}(U)^\circ)$ is a finitely generated O_E -module and therefore, there exists a natural number N such that for any $n \geq 1$, the O_E -module

$$\mathrm{Tor}_1^{O_U^\circ}(H_c^2(\Gamma_p, \mathbb{D}(U)^\circ), O_{\mathcal{W}}^\circ/p^n)$$

is annihilated by p^N . On passing to projective limits in (19) and using [3, Lemma 3.13], we infer that

$$0 \longrightarrow H_c^1(\Gamma_p, \mathbb{D}(U)^\circ) \widehat{\otimes}_{O_U^\circ} O_{\mathcal{W}}^\circ \longrightarrow H_c^1(\Gamma_p, \mathbb{D}(U)^\circ) \widehat{\otimes}_{O_U^\circ} O_{\mathcal{W}}^\circ \longrightarrow C$$

where C is a finitely generated module that is annihilated by p^N . This concludes the proof of the first isomorphism. The proof of the second isomorphism is analogous (an even simpler because $H^2(\Gamma_p, \mathbb{D}(U)^\circ)$ vanishes). $\square$

5.1.6. For a non-negative integer k , we let $\mathcal{P}_k(E) \subset E[Z]$ denote the space of polynomials of degree less or equal to k , which is equipped with a left $\mathrm{GL}_2(\mathbb{Z}_p)$ -action; see [10, 3.2.5] for a description of this action. We let $\mathcal{V}_k(E)$ denote the E -linear dual of $\mathcal{P}_k(E)$, which also carries an induced $\mathrm{GL}_2(\mathbb{Z}_p)$ -action.

Regarding elements $\mathcal{P}_k(E)$ as analytic functions, we have an induced map

$$\rho_k^* : \mathbb{D}_k^\dagger(E) \longrightarrow \mathcal{V}_k(E)$$

which is a $\Sigma_0(p)$ -equivariant surjection. By Stevens' control theorem (c.f. [32], Theorem 5.4) it induces a Hecke equivariant isomorphism of E -vector spaces

$$\rho_k^* : \mathrm{Symb}_{\Gamma_p}(\mathbb{D}_k^\dagger(E))^{<k+1} \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_p}(\mathcal{V}_k(E))^{<k+1} \quad (20)$$

We also have the non-compact version of this isomorphism, proved by Ash and Stevens [1, Theorem 1] :

$$H^1(\Gamma_p, \mathbb{D}_k^\dagger(E))^{<k+1} \xrightarrow{\sim} H^1(\Gamma_p, \mathcal{V}_k(E))^{<k+1}. \quad (21)$$

5.2 Local description of the Coleman–Mazur–Buzzard eigencurve

Our main objective in this subsection is to record Proposition 5.3 and Theorem 5.4 (which is due to Bellaïche; see the second paragraph in [10, §1.5]).

5.2.1. We now briefly recall the Coleman–Mazur–Buzzard construction of the eigencurve. We retain the notation from Sect. 5.1 and continue following the exposition in [10, §2.1]. Let $\mathcal{H}$ denotes the Hecke algebra generated over $\mathbb{Z}$ by the Hecke operators $\{T_\ell\}_{\ell \nmid Np}$, the Atkin–Lehner operator U_p and diamond operators $\{\langle d \rangle\}_{d \in (\mathbb{Z}/N\mathbb{Z})^\times}$. For any affinoid $\mathcal{W}$ we set $\mathcal{H}_{\mathcal{W}} = \mathcal{H} \otimes_{\mathbb{Z}} O_{\mathcal{W}}$.

We fix a nice affinoid disk $\mathcal{W} = \mathrm{Spm}(O_{\mathcal{W}})$ (in the sense of [10, Definition 3.5]) of the weight space $\mathcal{WS}$ adapted to slope v in the sense of [10], §3.2.4. Let us denote by $M^\dagger(\Gamma_p, \mathcal{W})$ Coleman's space of overconvergent modular forms of level Γ_p and weight in $\mathcal{W}$; and let $M^\dagger(\Gamma_p, \mathcal{W})^{\leq v}$ denote its $O_{\mathcal{W}}$ -submodule on which U_p acts with slope at most v . We similarly let $S^\dagger(\Gamma_p, \mathcal{W})$ denote the space of cuspidal overconvergent modular forms of level Γ_p and weight in the affinoid disk $\mathcal{W}$; and $S^\dagger(\Gamma_p, \mathcal{W})^{\leq v} \subset S^\dagger(\Gamma_p, \mathcal{W})$.

Let $\mathbb{T}_{\mathcal{W},v}$ (resp., $\mathbb{T}_{\mathcal{W},v}^{\mathrm{cusp}}$) denote the image of $\mathcal{H}_{\mathcal{W}}$ in $\mathrm{End}_{O_{\mathcal{W}}}(M^\dagger(\Gamma_p, \mathcal{W}))$ (resp., $\mathrm{End}_{O_{\mathcal{W}}}(S^\dagger(\Gamma_p, \mathcal{W}))$). Then $\mathcal{C}_{\mathcal{W},v} := \mathrm{Spm}(\mathbb{T}_{\mathcal{W},v})$ (resp., $\mathcal{C}_{\mathcal{W},v}^{\mathrm{cusp}} := \mathrm{Spm}(\mathbb{T}_{\mathcal{W},v}^{\mathrm{cusp}})$) is an open affinoid subspace of the Coleman–Mazur–Buzzard eigencurve $\mathcal{C}$ (resp., cuspidal eigencurve $\mathcal{C}^{\mathrm{cusp}}$) that lies over $\mathcal{W}$.

We let $\mathbb{T}_{\mathcal{W},v}^{\pm}$ denote the image of $\mathcal{H}_{\mathcal{W}}$ in $\text{End}_{O_{\mathcal{W}}}(\text{Symb}_{\Gamma_p}^{\pm}(\mathbb{D}(\mathcal{W}))^{\leq v})$. We then define the open affinoids $\mathcal{C}_{\mathcal{W},v}^{\pm} := \text{Spm}(\mathbb{T}_{\mathcal{W},v}^{\pm})$. By [10, Theorem 3.30], there exist canonical closed immersions

$$\mathcal{C}_{\mathcal{W},v}^{\text{cusp}} \hookrightarrow \mathcal{C}_{\mathcal{W},v}^{\pm} \hookrightarrow \mathcal{C}_{\mathcal{W},v}, \quad \mathcal{C}_{\mathcal{W},v} = \mathcal{C}_{\mathcal{W},v}^{+} \cup \mathcal{C}_{\mathcal{W},v}^{-}. \quad (22)$$

The open affinoids $\mathcal{C}_{\mathcal{W},v}^{\pm}$ admissibly cover the Coleman–Mazur–Buzzard eigencurve $\mathcal{C}$ as $(\mathcal{W}, v)$ varies (see [10], §3).

5.2.2. In the remainder of this subsection, we fix a p -stabilization of a newform f_0 and denote by x_0 the corresponding point of $\mathcal{C}$. The following proposition will allow us to apply the formalism of Sects. 3–4 in the context of eigencurves. We denote by $\mathcal{W}$ an affinoid neighborhood of $k_0 = w(x_0)$ in the weight space, where $O_{\mathcal{W}}$ is of the form $O_{\mathcal{W}} = E \langle Y/p^m \rangle$.

Proposition 5.3 (Bellaïche) *Up to shrinking $\mathcal{W}$ and enlarging E , there exists an affinoid neighborhood $\mathcal{X}$ of $x_0 \in \mathcal{C}_{\mathcal{W},v}^{\text{cusp}}$ such that*

- (a) $\mathcal{X}$ is a connected component of $\mathcal{C}_{\mathcal{W},v}$;
- (b) *There exist integers $r, e \geq 1$ and an element $a \in O_{\mathcal{X}}$ such that $O_{\mathcal{W}} = E \langle Y/p^{re} \rangle$ and the map $X \mapsto a$ induces an isomorphism of R -algebras*

$$O_{\mathcal{W}}[X]/(X^e - Y) \simeq O_{\mathcal{X}}.$$

Proof By [10, Proposition 4.11], there exists an affinoid neighborhood $\mathcal{X}$ of $\mathcal{C}_{\mathcal{W},v}^{\pm}$ which satisfies the condition (b). By [10, Corollary 2.17], the eigencurves $\mathcal{C}$ and $\mathcal{C}^{\text{cusp}}$ are locally isomorphic at x_0 . Together with (22), this concludes the proof of (a). $\square$

5.2.3. We set

$$\text{Symb}_{\Gamma_p}^{\pm}(\mathcal{X})^{\leq v} := \text{Symb}_{\Gamma_p}^{\pm}(\mathbb{D}^{\dagger}(\mathcal{W}))^{\leq v} \otimes_{\mathbb{T}_{\mathcal{W},v}^{\pm}} O_{\mathcal{X}},$$

where $\mathcal{X}$ is as in Proposition 5.3.

Theorem 5.4 (Bellaïche)

- (i) *The $O_{\mathcal{X}}$ -module $\text{Symb}_{\Gamma_p}^{\pm}(\mathcal{X})^{\leq v}$ is free of rank one.*
- (ii) *The $O_{\mathcal{X}}$ -module $\text{Symb}_{\Gamma_p}(\mathcal{X})^{\leq v}$ is free of rank 2.*

Proof The first assertion follows as a consequence of the discussion in Section 4.2.1 in op. cit.; see also the first paragraph following the proof of [10, Proposition 4.11]. The second assertion follows from the first one and the decomposition

$$\text{Symb}_{\Gamma_p}(\mathcal{X})^{\leq v} = \text{Symb}_{\Gamma_p}^{+}(\mathcal{X})^{\leq v} \oplus \text{Symb}_{\Gamma_p}^{-}(\mathcal{X})^{\leq v}.$$

$\square$

5.3 A variant of Bellaïche's construction

5.3.1. Fix an affinoid disk $\mathcal{W}$ and a wide open affinoid U such that

$$\mathcal{W} \subset U \subset \mathcal{W}^*$$

where $\mathcal{W}^*$ is given as in §5.1.1. See [3] as well as §5.1.1 for an explanation of this assumption. Note that this condition translates to the requirement that U is 0-accessible in the sense of [26, Definition 4.1.1]. We then have a natural (restriction) map $O_U^{\circ} \rightarrow R$. Recall that

$\kappa_U : \mathbb{Z}_p^\times \rightarrow O_U^{\circ, \times}$ denotes the restriction of the universal weight character to U . In what follows, we will allow ourselves to shrink both U and $\mathcal{W}$ as necessary.

5.3.2. Recall that $\mathcal{H}_{\mathcal{W}}$ denotes the Hecke algebra over $O_{\mathcal{W}}$. In this subsection, we work with the spaces $\mathbb{D}(U)$ and $\mathcal{A}'(U)$. Set

$$\begin{aligned} H_c^1(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}}^{\pm, \leq \nu} &:= H_c^1(\Gamma_p, \mathbb{D}(U))^{\pm, \leq \nu} \otimes_{O_U} O_{\mathcal{W}}, \\ H_c^1(\Gamma_p, \mathcal{A}'(U))_{\mathcal{W}}^{\pm, \leq \nu} &:= H_c^1(\Gamma_p, \mathcal{A}'(U))^{\pm, \leq \nu} \otimes_{O_U} O_{\mathcal{W}} \end{aligned}$$

and let

$$r_1^{\pm} : \mathcal{H}_{\mathcal{W}} \rightarrow \text{End}_{O_{\mathcal{W}}}(H_c^1(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}}^{\pm, \leq \nu}), \quad r_2^{\pm} : \mathcal{H}_{\mathcal{W}} \rightarrow \text{End}_{O_{\mathcal{W}}}(H_c^1(\Gamma_p, \mathcal{A}'(U))_{\mathcal{W}}^{\pm, \leq \nu})$$

denote the canonical representations of $\mathcal{H}_{\mathcal{W}}$. We define the ideals $I_1^{\pm} = \ker(r_1^{\pm})$ and $I_2^{\pm} = \ker(r_2^{\pm})$.

Lemma 5.5 *Let $k_0 \geq 0$ be an integer such that $k_0 \in \mathcal{W} \subset U$. For sufficiently small $\mathcal{W}$ and U the map (13) together with Lemma 5.2 induce natural Hecke equivariant isomorphisms*

$$\begin{aligned} H_c^1(\Gamma_p, \mathbb{D}^{\dagger}(\mathcal{W}))^{\pm, \leq \nu} &\simeq H_c^1(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}}^{\pm, \leq \nu}, \\ H^1(\Gamma_p, \mathbb{D}^{\dagger}(\mathcal{W}))^{\pm, \leq \nu} &\simeq H^1(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}}^{\pm, \leq \nu}. \end{aligned} \quad (23)$$

In particular, the morphism

$$\mathbb{T}_{\mathcal{W}, v}^{\pm} \longrightarrow \text{im}(r_1^{\pm})$$

induced from (23) is an isomorphism as well.

Proof Let us put $M_1 := H_c^1(\Gamma_p, \mathbb{D}^{\dagger}(\mathcal{W}))^{\pm, \leq \nu}$ and $M_2 := H_c^1(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}}^{\pm, \leq \nu}$ and denote by $g : M_1 \rightarrow M_2$ the map induced from (13) together with Lemma 5.2. It follows from (18) and general results about the slope decomposition that M_1 and M_2 are finitely generated free $O_{\mathcal{W}}$ -modules. By Lemma 5.1, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1/\mathfrak{m}_{k_0}M_1 & \longrightarrow & H_c^1(\Gamma_p, \mathbb{D}_{k_0}^{\dagger}(E))^{\pm, \leq \nu} & \longrightarrow & H_c^2(\Gamma_p, \mathbb{D}^{\dagger}(\mathcal{W}))^{\pm, \leq \nu}[k_0] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_2/\mathfrak{m}_{k_0}M_2 & \longrightarrow & H_c^1(\Gamma_p, \mathbb{D}_{k_0}(E))^{\pm, \leq \nu} & \longrightarrow & H_c^2(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}}^{\pm, \leq \nu}[k_0] \longrightarrow 0 \end{array}$$

where $\mathfrak{m}_{k_0} \subset O_{\mathcal{W}}$ is the maximal ideal determined by the weight k_0 . By [32, Lemma 5.3], the middle vertical map of the diagram is an isomorphism. According to Lemma 5.1, both $H_c^2(\Gamma_p, \mathbb{D}^{\dagger}(\mathcal{W}))^{\pm, \leq \nu}[k_0]$ and $H_c^2(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}}^{\pm, \leq \nu}[k_0]$ are both either isomorphic to E or else vanish (depending on whether $k_0 = 0$ or not). Moreover, since the middle vertical arrow is an isomorphism, the right vertical arrow is surjective and therefore an isomorphism as well. We conclude that the left vertical map induced from the vertical isomorphism in the middle is an isomorphism:

$$M_1/\mathfrak{m}_{k_0}M_1 \xrightarrow{\sim} M_2/\mathfrak{m}_{k_0}M_2. \quad (24)$$

In particular, M_1 and M_2 are free $O_{\mathcal{W}}$ -modules with the same rank. Let $G \in O_{\mathcal{W}}$ denote the determinant of g in some bases of M_1 and M_2 . The isomorphism (24) shows that $G(k_0) \neq 0$ and hence, on shrinking the neighborhood $\mathcal{W}$ as necessary, G is non-vanishing on $\mathcal{W}$. This concludes the proof that g is an isomorphism for sufficiently small $\mathcal{W}$ and U , as we have asserted in the statement of our lemma.

The second isomorphism in (23) can be established by the same argument, using the fact that the map $H^1(\Gamma_p, \mathbb{D}_{k_0}^{\dagger}(E))^{\leq \nu} = H^1(\Gamma_p, \mathbb{D}_{k_0}(E))^{\leq \nu}$ by [1, Theorem 5.5.3]. $\square$

5.3.3. Let us set

$$\mathbb{T}'_{\mathcal{W},v}{}^{\pm} = \text{im}(r_2^{\pm}), \quad \mathcal{C}'_{\mathcal{W},v}{}^{\pm} := \text{Spm}(\mathbb{T}'_{\mathcal{W},v}{}^{\pm}).$$

Note then that we have canonical isomorphisms $\mathbb{T}_{\mathcal{W},v}^{\pm} \simeq \mathcal{H}_{\mathcal{W}}/I_1^{\pm}$ and $\mathbb{T}'_{\mathcal{W},v}{}^{\pm} \simeq \mathcal{H}_{\mathcal{W}}/I_2^{\pm}$.

Theorem 5.6

- (i) We have $I_1^{\pm} = I_2^{\pm}$. In particular, there exist canonical isomorphisms $\mathbb{T}_{\mathcal{W},v}^{\pm} \simeq \mathbb{T}'_{\mathcal{W},v}{}^{\pm}$ and $\mathcal{C}_{\mathcal{W},v}^{\pm} \simeq \mathcal{C}'_{\mathcal{W},v}{}^{\pm}$.
- (ii) Suppose $x_0 \in \text{Spm}(\mathcal{H}_{\mathcal{W}}/I^{\pm})$, where $I^{\pm} := I_1^{\pm} = I_2^{\pm}$. For a sufficiently small wide open neighborhood U of $k_0 = w(x_0)$ and an affinoid $\mathcal{W} \subset U$ containing k_0 the following two assertions hold true.
  - (a) x_0 belongs to a unique connected component $\mathcal{X}^{\pm}$ of $\text{Spm}(\mathcal{H}_{\mathcal{W}}/I^{\pm})$.
  - (b) Both $\mathcal{O}_{\mathcal{X}^{\pm}}$ -modules $H_c^1(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}}^{\pm, \leq \nu} \otimes_{\mathcal{H}_{\mathcal{W}}} \mathcal{O}_{\mathcal{X}^{\pm}}$ and $H_c^1(\Gamma_p, \mathbb{A}'(U))_{\mathcal{W}}^{\pm, \leq \nu} \otimes_{\mathcal{H}_{\mathcal{W}}} \mathcal{O}_{\mathcal{X}^{\pm}}$ are free of rank one.
- (iii) Fix a wide open U as well as an affinoid neighborhood $\mathcal{W} \subset U$ of x_0 chosen so as to ensure that the conclusions of Part ii) are verified. If x_0 is cuspidal, then $x_0 \in \mathcal{C}_{\mathcal{W},v}^{+, \leq \nu} \cap \mathcal{C}_{\mathcal{W},v}^{-, \leq \nu}$ and $\mathcal{X}^{+} = \mathcal{X}^{-}$.

Proof

- (i) We first show that for every integer $k \in U$ with $k \geq \nu$ there exists a Hecke equivariant³ isomorphism

$$H_c^1(\Gamma_p, \mathbb{D}_k(E))^{\pm, \leq \nu} \xrightarrow{\sim} H_c^1(\Gamma_p, \mathbb{A}'_k(E))^{\pm, \leq \nu}. \quad (25)$$

For such k , the natural injection

$$H_c^1(\Gamma_p, \mathcal{P}_k(E))^{\leq \nu} \longrightarrow H_c^1(\Gamma_p, \mathbb{A}'_k(E))^{\leq \nu}$$

as well as the natural surjection

$$H_c^1(\Gamma_p, \mathbb{D}_k(E))^{\leq \nu} \longrightarrow H_c^1(\Gamma_p, \mathcal{V}_k(E))^{\leq \nu}$$

of Hecke modules are isomorphisms thanks to the control theorem of Ash and Stevens (c.f. [14], Proposition 4.2.2).

We are therefore reduced to proving that we have an Hecke equivariant isomorphism

$$H_c^1(\Gamma_p, \mathcal{V}_k(E)) \xrightarrow{\sim} H_c^1(\Gamma_p, \mathcal{P}_k(E)).$$

This follows from the isomorphism

$$\mathcal{V}_k(E) \otimes \det^k \xrightarrow[(\det^*)^{-1}]{\sim} \text{Hom}(\mathcal{V}_k(E), E) =: \mathcal{P}_k(E)$$

of $\text{GL}_2(\mathbb{Z}_p)$ -modules (where $\det$ stands for the determinant character of $\text{GL}_2(\mathbb{Z}_p)$), together with the fact that $\det|_{\Gamma_p} = 1$. Here, $\det^*$ is the isomorphism induced from the perfect pairing

$$\begin{aligned} \mathcal{V}_k(E) \otimes \mathcal{V}_k(E) &\longrightarrow E \otimes \det^{-k} \\ v_1 \otimes v_2 &\longmapsto \sum_{j=0}^k \binom{k}{j} (-1)^j v_1(Z^j) v_2(Z^{k-j}); \end{aligned}$$

³ Note here and below that we are not specifying the covariance or the contravariance of the Hecke action that ensures Hecke compatibilities.

see the paragraph following [14, Proposition 3.2] as well as Remark 3.3.2 in op. cit. This proves (25).

In view of [14], Proposition 4.2.1, it follows from the isomorphisms (25) that the representations r_1 and r_2 verify the conditions of [15, Proposition 3.7]. This implies that $I_1^\pm = I_2^\pm$ and the remaining assertions are immediate from this fact and Lemma 5.5.

- (ii) We will only prove the assertion when the coefficients are $\mathbb{A}'(U)$, since the proof in the case of $\mathbb{D}(U)$ is similar. We remark that one may alternatively deduce our claim that the $O_{\mathcal{X}^\pm}$ -module $H_c^1(\Gamma_p, \mathbb{D}(U))^{\pm, \leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}^\pm}$ is free of rank one from [10, Proposition 4.5], using the isomorphism (18) of Ash–Stevens and Proposition 5.5. We will prove that for U sufficiently small the $O_{\mathcal{X}^\pm}$ -module $H_c^1(\Gamma_p, \mathbb{A}'(U))^{\pm, \leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}^\pm}$ is free of rank one. It is clear that one may choose U so as to ensure that the condition a) holds. Let us fix such U . According to [14, Proposition 4.2.1], the $\mathbb{T}_{U, v}^{\pm, \leq v}$ -module $H_c^1(\Gamma_p, \mathbb{A}'(U))_{O_{\mathcal{W}}}^{\pm, \leq v}$ is of finite rank as an $O_{\mathcal{W}}$ -module (therefore also as a $\mathbb{T}_{O_{\mathcal{W}}, v}^{\pm, \leq v}$ -module). For any $y \in U$, the long exact sequence Γ_p -cohomology induced from the short exact sequence of specialization at y

$$0 \longrightarrow \mathbb{A}'(U) \longrightarrow \mathbb{A}'(U) \xrightarrow{y} \mathbb{A}'_y(E) \longrightarrow 0$$

shows that

$$H_c^1(\Gamma_p, \mathbb{A}'(U))[\mathfrak{m}_y] = \text{im}(H_c^0(\Gamma_p, \mathbb{A}'_y(E)) \longrightarrow H_c^1(\Gamma_p, \mathbb{A}'(U))).$$

Since $H_c^0(\Gamma_p, \mathbb{A}'_y(E)) = 0$, this proves that $H_c^1(\Gamma_p, \mathbb{A}'(U))$ is torsion-free over O_U and therefore, the $O_{\mathcal{W}}$ -module $H_c^1(\Gamma_p, \mathbb{A}'(U)) \otimes_{O_U} O_{\mathcal{W}}$ is torsion-free as well. We have proved that the $O_{\mathcal{W}}$ -modules $H_c^1(\Gamma_p, \mathbb{A}'(U))_{\mathcal{W}}^{\pm, \leq v}$ are finitely generated and torsion-free (therefore free, since $O_{\mathcal{W}}$ is a PID).

Set $N^\pm = H_c^1(\Gamma_p, \mathbb{A}'(U))^{\pm, \leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}^\pm}$. Now the arguments of Bellaïche apply verbatim. More precisely, since $O_{\mathcal{X}^\pm}$ are PIDs, [10, Lemma 4.1] tells us that for any $x \in \mathcal{X}^\pm$, the localization $N_{(x)}^\pm$ is a free $O_{\mathcal{X}^\pm, x}$ -module of finite rank. Moreover, for any classical point $x \neq x_0$ of weight $k \geq v$, the isomorphism (25) together with [10, Lemma 2.8] show that $N_{(x)}^\pm$ is of rank one over $O_{\mathcal{X}^\pm, x}$. By the local constancy of the rank, the same also holds true for $N_{(x_0)}^\pm$. We conclude that $N^\pm$ is a free $O_{\mathcal{X}^\pm}$ -module of rank one, when U and $\mathcal{W}$ sufficiently small.

- (iii) This portion follows directly from [10, Theorem 3.30].

□

Corollary 5.7 *Let us fix $\mathcal{W}$ that ensures the validity of the conclusions of Theorem 5.6(ii) and suppose x_0 is cuspidal, so that $\mathcal{X}^+ = \mathcal{X}^-$ according to Theorem 5.6(iii). Put $\mathcal{X} = \mathcal{X}^\pm$. Then both $O_{\mathcal{X}}$ -modules $H_c^1(\Gamma_p, \mathbb{D}(U))_{\mathcal{W}}^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}}$ and $H_c^1(\Gamma_p, \mathbb{A}'(U))_{\mathcal{W}}^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}}$ are free of rank 2.*

Proof Clear, thanks to Theorem 5.6. □

5.3.4. We record the following proposition which will shall use in the proof of Proposition 6.5. It is well known to experts, but we prove it for reader's convenience.

Proposition 5.8 *Let x_0 be a classical cuspidal point on the eigencurve $\mathcal{C}$ and $k_0 = w(x_0)$. Then the following hold true.*

(i) *The natural map*

$$H_c^1(\Gamma_p, \mathcal{V}_{k_0}(E)) \otimes_{\mathcal{H}, x_0} E \longrightarrow H^1(\Gamma_p, \mathcal{V}_{k_0}(E)) \otimes_{\mathcal{H}, x_0} E \quad (26)$$

is an isomorphism of E -vector spaces of dimension 2.

(ii) *There exists an affine neighborhood $\mathcal{W}$ of k_0 and $\nu > 0$ such that for the connected component $\mathcal{X} \subset \mathcal{C}_{\mathcal{W}, \nu}$ of $x_0 \in \mathcal{C}_{\mathcal{W}, \nu}$ the map*

$$H_c^1(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W}))^{\leq \nu} \otimes_{\mathcal{H}_{\mathcal{W}}} \mathcal{O}_{\mathcal{X}} \xrightarrow{j \otimes \text{id}} H^1(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W}))^{\leq \nu} \otimes_{\mathcal{H}_{\mathcal{W}}} \mathcal{O}_{\mathcal{X}} \quad (27)$$

induced from the natural $\mathcal{H}_{\mathcal{W}}$ -equivariant map $H_c^1(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W}))^{\leq \nu} \xrightarrow{j} H^1(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W}))^{\leq \nu}$, is an isomorphism.

Proof

(i) Let us denote by $\tilde{Y}$ the Borel–Serre compactification of the modular curve $Y = \Gamma_p \backslash \mathbf{H}$ (and $\partial \tilde{Y}$ its boundary). The sheaf $\mathcal{V}_{k_0}$ on Y extends to a sheaf on $\tilde{Y}$. It follows from [4, Proposition 4.2] that

$$\text{coker}(26) \hookrightarrow \text{im} \left(H^1(Y, \mathcal{V}_{k_0}) \xrightarrow{\partial_Y^1} H^1(\partial \tilde{Y}, \mathcal{V}_{k_0}) \right) [\mathfrak{m}_{x_0}]$$

and the surjectivity of (26) follows once we verify that

$$\text{im} \left(H^1(Y, \mathcal{V}_{k_0}) \xrightarrow{\partial_Y^1} H^1(\partial \tilde{Y}, \mathcal{V}_{k_0}) \right) [\mathfrak{m}_{x_0}] = 0.$$

This follows from the properties⁴ of the Eisenstein cohomology, since x_0 is cuspidal.

The proof that the map (26) is injective is similar, where one instead relies on the vanishing of the x_0 -isotypic subspace of $\text{im}(H^0(\partial \tilde{Y}, \mathcal{V}_{k_0}) \rightarrow H_c^1(Y, \mathcal{V}_{k_0}))$.

Our assertion on the dimension of these vector spaces is standard, c.f. [10, Proposition 3.18].

(ii) Fix $\nu \geq k_0 + 1$. Let $\mathcal{W}$ be an affinoid neighborhood of k_0 such that the conclusion of Theorem 5.4 hold for $\mathcal{W}$. To simplify notation, set $M_c = H_c^1(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W}))^{\leq \nu} \otimes_{\mathcal{H}_{\mathcal{W}}} \mathcal{O}_{\mathcal{X}}$ and $M = H^1(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W}))^{\leq \nu} \otimes_{\mathcal{H}_{\mathcal{W}}} \mathcal{O}_{\mathcal{X}}$. Then the $\mathcal{O}_{\mathcal{X}}$ -module M_c is free of rank 2. The same argument (that Bellaïche utilizes to prove M_c is free of rank 2) shows that M is also free of rank 2 over $\mathcal{O}_{\mathcal{X}}$. Namely, on shrinking $\mathcal{W}$ if necessarily, we can assume that M is a finitely generated free $\mathcal{O}_{\mathcal{X}}$ -module. For any point x of weight $w(x)$ we have

$$M/\mathfrak{m}_x M \simeq (H^1(\Gamma_p, \mathbb{D}^\dagger(\mathcal{W}))^{\leq \nu} \otimes_{\mathcal{O}_{\mathcal{W}, w(x)}} E) \otimes_{\mathcal{H}, x} E, \quad (28)$$

Together with Lemma 5.1 this gives

$$M/\mathfrak{m}_x M \simeq H^1(\Gamma_p, \mathbb{D}_{w(x)}^\dagger(E))^{\leq \nu} \otimes_{\mathcal{H}, x} E. \quad (29)$$

The control theorem of Ash–Stevens (21) reads

$$H^1(\Gamma_p, \mathbb{D}_k^\dagger(E))^{\leq \nu} \xrightarrow{\sim} H^1(\Gamma_p, \mathcal{V}_k(E))^{\leq \nu}$$

⁴ The required vanishing statement follows from [18, Corollary 4.7] (see also [19], Theorem 1) combined with the strong multiplicity one for GL₂, which follows as a consequence of the classification result due to Jacquet–Shalika [20, 21] (c.f. the proof of [16], Proposition 4.1). In brief terms, Corollary 4.7 of [18] tells us that the systems of Hecke eigenvalues occurring in the image of the map ∂_Y^1 are those of Eisenstein series. By the classification result of Jacquet–Shalika, the set of systems of Hecke eigenvalues of Eisenstein series is disjoint from those of cusp forms. (We are grateful to Fabian Januszewski for indicating these references.)

for each positive integer $k > \nu - 1$. Choosing $x \in \mathcal{X}$ such that $w(x) > \nu - 1$, we deduce that

$$M/\mathfrak{m}_x M \simeq H^1(\Gamma_p, \mathcal{V}_k(E))^{\leq \nu} \otimes_{\mathcal{H},x} E.$$

Now it follows from the classical Eichler–Shimura isomorphism and multiplicity-one that $M/\mathfrak{m}_x M$ has dimension 2 over E , and the $\mathcal{O}_{\mathcal{X}}$ -module M is free of rank 2 as well.

Let $\det_j \in \mathcal{O}_{\mathcal{X}}$ denote the determinant of $j \otimes \text{id} : M_c \rightarrow M$ with respect to fixed bases of M_c and M . If we knew that $\det_j(x_0) \neq 0$, we could shrink $\mathcal{W}$ and the neighborhood $\mathcal{X}$ of x_0 to ensure that $\det_j$ is non-vanishing on $\mathcal{X}$ and thereby conclude that $j \otimes \text{id}$ is an isomorphism, as required. We have therefore reduced to proving that $\det_j(x_0) \neq 0$.

Recall the isomorphism (29) for $x = x_0$:

$$M/\mathfrak{m}_{x_0} M \simeq H^1(\Gamma_p, \mathbb{D}_{k_0}^{\dagger}(E))^{\leq \nu} \otimes_{\mathcal{H},x_0} E. \quad (30)$$

The analogous isomorphism (28) for M_c together with Lemma 5.1 shows that we have an injection

$$M_c/\mathfrak{m}_{x_0} M_c \hookrightarrow H_c^1(\Gamma_p, \mathbb{D}_{k_0}^{\dagger}(E))^{\leq \nu} \otimes_{\mathcal{H},x_0} E,$$

which is an isomorphism if $k_0 \neq 0$. As a matter of fact, it follows from [10, Proposition 3.14] that this map is an isomorphism since x_0 is cuspidal. We therefore have the following commutative diagram:

$$\begin{array}{ccc} M_c/\mathfrak{m}_{x_0} M_c & \xrightarrow{\sim} & H_c^1(\Gamma_p, \mathbb{D}_{k_0}^{\dagger}(E))^{\leq \nu} \otimes_{\mathcal{H},x_0} E \\ \downarrow & & \downarrow \\ M/\mathfrak{m}_{x_0} M & \xrightarrow{\sim} & H^1(\Gamma_p, \mathbb{D}_{k_0}^{\dagger}(E))^{\leq \nu} \otimes_{\mathcal{H},x_0} E. \end{array}$$

To conclude with the proof of the asserted isomorphism in our proposition with sufficiently small $\mathcal{W}$ and $\mathcal{X}$, it suffices to prove that the right vertical map is an isomorphism. Observe that the following diagram commutes:

$$\begin{array}{ccc} H_c^1(\Gamma_p, \mathbb{D}_{k_0}^{\dagger}(E))^{\leq \nu} \otimes_{\mathcal{H},x_0} E & \longrightarrow & H^1(\Gamma_p, \mathbb{D}_{k_0}^{\dagger}(E))^{\leq \nu} \otimes_{\mathcal{H},x_0} E \\ \rho_{k_0}^* \downarrow \sim & & \sim \downarrow \rho_{k_0}^* \\ H_c^1(\Gamma_p, \mathcal{V}_{k_0}(E))^{\leq \nu} \otimes_{\mathcal{H},x_0} E & \xrightarrow[\text{(i)}]{\sim} & H^1(\Gamma_p, \mathcal{V}_{k_0}(E))^{\leq \nu} \otimes_{\mathcal{H},x_0} E \end{array} \quad (31)$$

All objects of this diagram have dimension 2 over E . The surjectivity of the vertical arrow in (31) on the left follows from [32, Lemma 5.1], and we infer that the vertical arrow on the left is an isomorphism. The bottom map is an isomorphism by part i). This implies that all the maps of the diagram are isomorphisms. The proof of our proposition is now complete. $\square$

6 Interpolation of Beilinson–Kato elements

6.1 Modular curves and Iwasawa sheaves

6.1.1. For a pair of positive integers M_1 and M_2 , we recall the modular curves $Y(M_1, M_2)$ from [22, §2]. We then have $Y(N, N) = Y(N)$ and $Y(1, N) = Y_1(N)$, modular curves of level $\Gamma(N)$ and $\Gamma_1(N)$, respectively. Recall from §2.8 of op. cit. also the modular curves denoted

by $Y(M_1, M_2(B))$ and $Y(M_1(B), M_2)$, where B is also a positive integer. In Sections 6.3–6.5, we will be working over the modular curve $Y(1, N(p))$ of $\Gamma_1(N) \cap \Gamma_0(p)$ -level. We note that the modular curve $Y(1, N(p))$ is denoted by $Y(N, p)$ in [3].

If Y denotes any one of the modular curves above, we denote by $\lambda : \mathcal{E} \rightarrow Y$ the universal elliptic curve with the appropriate level structure (which depends on Y , but we suppress this dependence from our notation). We let $\mathcal{T} := R^1\lambda_*\mathbb{Z}_p(1)$ denote the pro-system $(\mathcal{T}_n)_{n \geq 1}$ of étale lisse sheaves $\mathcal{T}_n := R^1\lambda_*\mathbb{Z}_{p^n}$ on the open modular curves $Y[1/p]$ given as in [22, §§1–2]. The sheaf $\mathcal{T}$ has rank 2 and the Poincaré duality identifies it with the p -adic Tate module $T_p(\mathcal{E})$ of $\mathcal{E}$. We write $\mathcal{T}_{\mathbb{Q}_p}$ for the associated sheaf of $\mathbb{Q}_p$ -vector spaces.

For each non-negative integer k and a locally free sheaf $\mathcal{F}$ over $Y[1/p]$, we let $\mathrm{TSym}^k \mathcal{F}$ denote the locally free sheaf on Y of symmetric k -tensors, given as in [24, §2.2].

6.1.2. Let $i_D : D \hookrightarrow \mathcal{E}$ be a subscheme. Assume that D is étale over Y and consider the diagram

$$\begin{array}{ccc} \mathcal{E}[p^n]\langle D \rangle & \longrightarrow & \mathcal{E} \\ \downarrow p_{D,n} & & \downarrow [p^n] \\ D & \xrightarrow{i_D} & \mathcal{E} \\ \downarrow \lambda_D & \nearrow \lambda & \\ Y & & \end{array}$$

where $\mathcal{E}[p^n]\langle D \rangle$ is the fiber product of $\mathcal{E}$ and D over $\mathcal{E}$. We define the pro-étale sheaf $\underline{\Delta}(\mathcal{T}\langle D \rangle) = (\underline{\Delta}_n(\mathcal{T}\langle D \rangle))_{n \geq 1}$ on setting

$$\underline{\Delta}_n(\mathcal{T}\langle D \rangle) = \lambda_{D,*} p_{D,n,*}(\mathbb{Z}/p^n\mathbb{Z}).$$

We refer the reader to [23] where Kings develops these notions in a general framework.

If $D = Y$ and $i_D = s : Y \rightarrow \mathcal{E}$ is a section of λ , we will write $\underline{\Delta}(\mathcal{T}\langle s \rangle)$ instead $\underline{\Delta}(\mathcal{T}\langle D \rangle)$. In particular, we denote by $\underline{\Delta}(\mathcal{T}) = \underline{\Delta}(\mathcal{T}\langle 0 \rangle)$ the sheaf associated to the identity section 0. For any section s , the sheaf $\underline{\Delta}(\mathcal{T}\langle s \rangle)$ is a sheaf of modules of rank one over the sheaf of Iwasawa algebras $\underline{\Delta}(\mathcal{T})$ (c.f. [23], §2.4).

6.1.3. In the remainder of §6.1, we recall a number of notation and constructions from [24, 26].

Fix a positive integer N . We denote by $\lambda_N : \mathcal{E}_N \rightarrow Y_1(N)$ the universal elliptic curve and by $\mathcal{T}_N$ the associated étale sheaf. Recall that $Y_1(N)$ is the moduli space of pairs (E, β_N) , where E is an elliptic curve and $\beta_N : \mathbb{Z}/N\mathbb{Z} \rightarrow E[N]$ is an injection. Then the map $(E, \beta_N) \mapsto \beta_N(1)$ defines a section $s_N : Y_1(N) \rightarrow \mathcal{E}_N$ of λ_N , and we denote by $\underline{\Delta}(\mathcal{T}_N\langle s_N \rangle)$ the associated sheaf.

Let $g_N : Y_1(Np^n) \rightarrow Y_1(N)$ denote the canonical projection. We define the pro-étale sheaf $\underline{\Delta}_N$ on $Y_1(N)_{\text{ét}}$ on setting

$$\underline{\Delta}_N = (\underline{\Delta}_{N,n})_{n \geq 1}, \quad \underline{\Delta}_{N,n} = g_{n,*}(\mathbb{Z}/p^n\mathbb{Z}).$$

If $p \mid N$, the moduli description of $Y_1(Np^n)$ shows that there exists a canonical isomorphism $Y_1(Np^n) \simeq \mathcal{E}[p^n]\langle s_N \rangle$, and therefore we have

$$\underline{\Delta}(\mathcal{T}_N\langle s_N \rangle) \simeq \underline{\Delta}_N. \quad (32)$$

(c.f. the proof of [24, Theorem 4.5.1].) In particular, we can apply the formalism developed in [23] with the sheaf $\underline{\Delta}_N$.

6.1.4. Assume that N is coprime to p and consider the modular curve $Y(1, N(p))$ equipped with the universal elliptic curve $\mathcal{E}_{N(p)}$ and the associated sheaf $\mathcal{T}_{N(p)}$. Recall that $Y(1, N(p))$ is the moduli space for the triples (E, β_N, C) , where E is an elliptic curve, $\beta_N : \mathbb{Z}/N\mathbb{Z} \rightarrow E$ is an injection and $C \subset E$ is a (cyclic) subgroup of order Np that contains $\beta_N(1 + N\mathbb{Z})$. Denote by $\mathcal{C} \subset \mathcal{E}_{N(p)}[p]$ the canonical subgroup of order p and set $D := \mathcal{E}_{N(p)}[p] - \mathcal{C}$ and $D' := \mathcal{C} - \{0\}$. Note that both D and D' are finite étale over $Y(1, N(p))$, of degrees $p^2 - p$ and $p - 1$, respectively, and we denote by $\underline{\Delta}(\mathcal{T}_{N(p)}\langle D \rangle)$ and $\underline{\Delta}(\mathcal{T}_{N(p)}\langle D' \rangle)$ the associated sheaves.

Since both D and D' are contained in $\mathcal{E}_{N(p)}[p]$, the “multiplication-by- p ” morphism induces the trace map

$$\underline{\Delta}(\mathcal{T}_{N(p)}\langle D^? \rangle) \xrightarrow{[p]_*} \underline{\Delta}(\mathcal{T}_{N(p)}), \quad D^? \in \{D, D'\} \quad (33)$$

of sheaves on $Y(1, N(p))$.

The rule $(E, \beta_{Np}) \mapsto (E, \beta_{Np}(p), \text{im}(\beta_{Np}))$ defines a morphism $\text{pr}' : Y(1, Np) \rightarrow Y(1, N(p))$, which in turn induces the cartesian square

$$\begin{array}{ccc} \mathcal{E}_{Np} & \longrightarrow & Y_1(Np) \\ \downarrow & & \downarrow \text{pr}' \\ \mathcal{E}_{N(p)} & \longrightarrow & Y(1, N(p)) \end{array}$$

For each positive integer r , we have a natural morphism

$$\mathcal{E}_{Np}[p^r]\langle s_{Np} \rangle \xrightarrow{[N]_* \circ \text{pr}'} \mathcal{E}_{N(p)}[p^r]\langle D' \rangle. \quad (34)$$

where the schemes $\mathcal{E}_?[p^r]\langle \cdot \rangle$ are given as in [24, Definition 4.1.4]. The composition of this map with (32) gives a map

$$H_{\text{ét}}^*(Y_1(Np), \underline{\Delta}_{Np}) \longrightarrow H_{\text{ét}}^*(Y(1, N(p)), \underline{\Delta}(\mathcal{T}_{N(p)}\langle D' \rangle)), \quad (35)$$

(see also [7, §4.2.5] for further details).

6.2 Big Beilinson–Kato elements

6.2.1. Let us fix positive integers M and N such that $M + N \geq 5$. We consider the following objects:

- The modular curves $Y(M, N)$ and $Y_1(N)$, which come equipped with the universal elliptic curves $\lambda_{M,N} : \mathcal{E}_{M,N} \rightarrow Y(M, N)$ and $\lambda_N : \mathcal{E}_N \rightarrow Y_1(N)$, as well as the sheaves $\mathcal{T}_{M,N} = R^1\lambda_{M,N,*}\mathbb{Z}_p(1)$ and $\mathcal{T}_N = R^1\lambda_{N,*}\mathbb{Z}_p(1)$ on $Y(M, N)_{\text{ét}}$ and $Y_1(N)_{\text{ét}}$, respectively.
- The morphisms $Y(M, N) \rightarrow Y_1(N) \times \mu_M^\circ \rightarrow Y_1(N)$, where $\mu_M^\circ = \text{Spec}(\mathbb{Z}[\zeta_M])$. These morphisms induce a canonical map $\mathcal{T}_{M,N} \rightarrow \mathcal{T}|_{Y(M,N)}$.
- The canonical projections

$$\begin{aligned} f_n : Y(Mp^n, Np^n) &\rightarrow Y(M, N), & h_n : \mu_{Mp^n}^\circ &\rightarrow \mu_M^\circ \\ g_n : Y_1(Np^n) &\rightarrow Y_1(N), & g_n^{\text{cyc}} : Y_1(Np^n) \times \mu_{Mp^n}^\circ &\rightarrow Y_1(N) \times \mu_M^\circ \end{aligned}$$

which give rise to the inverse systems of sheaves

$$\begin{aligned}\underline{\Delta}_{M,N} &= (\underline{\Delta}_{M,N,n})_{n \geq 1}, & \underline{\Delta}_{M,N,n} &= f_{n,*}(\mathbb{Z}/p^n \mathbb{Z}), \\ \underline{\Delta}_{\mu_M^\circ}^{\text{cyc}} &= (\underline{\Delta}_{\mu_M^\circ,n}^{\text{cyc}})_{n \geq 1}, & \underline{\Delta}_{\mu_M^\circ,n}^{\text{cyc}} &= h_{n,*}(\mathbb{Z}/p^n \mathbb{Z}), \\ \underline{\Delta}_N &= (\underline{\Delta}_{N,n})_{n \geq 1}, & \underline{\Delta}_{N,n} &= g_{n,*}(\mathbb{Z}/p^n \mathbb{Z}), \\ \underline{\Delta}_{N,\mu_M^\circ}^{\text{cyc}} &= (\underline{\Delta}_{N,\mu_M^\circ,n}^{\text{cyc}})_{n \geq 1}, & \underline{\Delta}_{N,\mu_M^\circ,n}^{\text{cyc}} &= g_{n,*}^{\text{cyc}}(\mathbb{Z}/p^n \mathbb{Z}).\end{aligned}$$

By the flatness of our sheaves, it follows that

$$\underline{\Delta}_{N,\mu_M^\circ}^{\text{cyc}} \simeq \underline{\Delta}_N \boxtimes \underline{\Delta}_{\mu_M^\circ}^{\text{cyc}}. \quad (36)$$

For any integer L , let us denote by $\text{prime}(L)$ the set of primes dividing L . For any integer $m \geq 1$, set $\Gamma_{\mathbb{Q}(\zeta_m)} = \text{Gal}(\mathbb{Q}(\zeta_{mp^\infty})/\mathbb{Q}(\zeta_m))$ and $\Lambda(\Gamma_{\mathbb{Q}(\zeta_m)}) = \mathbb{Z}_p[[\Gamma_{\mathbb{Q}(\zeta_m)}]]$.

6.2.2. Poincaré duality gives a canonical isomorphism between $\mathcal{T}_{M,N}$ and the Tate module $\underline{T}_p(\mathcal{E}_{M,N})$ of the universal elliptic curve $\mathcal{E}_{M,N}$. The interpretation of $Y(Mp^n, Np^n)$ as the moduli space of pairs (E, α) , where E is an elliptic curve and $\alpha : (\mathbb{Z}/Np^n \mathbb{Z})^2 \simeq E[Np^n]$ is an isomorphism, shows that the restriction of $\underline{\mathcal{E}}_{M,N}[p^n] \simeq \underline{T}_p(\mathcal{E}_{M,N})/p^n$ on $Y(Mp^n, Np^n)$ has a canonical basis $\{e_{1,n}, e_{2,n}\}$ given by $e_{1,n} = \alpha(M, 0)$ and $e_{2,n} = \alpha(0, N)$. Analogously, the interpretation of $Y(Mp^n, Np^n)$ as the moduli space of pairs (E, β) , where E is an elliptic curve and $\beta : \mathbb{Z}/Np^n \mathbb{Z} \rightarrow E[Np^n]$ is an injection, shows that the restriction of the sheaf $\mathcal{T}_N$ on $Y_1(Np^n)$ has the canonical section $\beta(N)$ which we also denote by $e_{2,n}$ to simplify notation.

6.2.3. One has an isomorphism of continuous Galois modules

$$H_{\text{ét}}^1((Y_1(N) \times \mu_M^\circ)_{\overline{\mathbb{Q}}}, \underline{\Delta}_{N,\mu_M^\circ}^{\text{cyc}}) \simeq H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \underline{\Delta}_{N,\mu_M^\circ}^{\text{cyc}}) \otimes \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})]].$$

Since $\underline{\Delta}_{\mu_M^\circ}^{\text{cyc}}$ is a constant sheaf on $Y_1(N)_{\overline{\mathbb{Q}}}$, one also has

$$H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \underline{\Delta}_{N,\mu_M^\circ}^{\text{cyc}}) \simeq H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \underline{\Delta}_N) \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_M)}).$$

The commutative diagrams (for each n)

$$\begin{array}{ccc} Y(Mp^n, Np^n) & \longrightarrow & Y_1(Np^n) \times \mu_{Mp^n}^\circ \\ \downarrow f_n & & \downarrow g_n^{\text{cyc}} \\ Y(M, N) & \longrightarrow & Y_1(N) \times \mu_M^\circ \\ & & \downarrow \\ & & Y_1(N) \times \mu_m^\circ \end{array}$$

show that we have the trace map

$$\begin{aligned} H_{\text{ét}}^1(Y_1(M, N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{M,N}) \otimes \underline{\Delta}_{M,N}) &\longrightarrow H_{\text{ét}}^1((Y_1(N) \times \mu_M^\circ)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_M^\circ}^{\text{cyc}}) \\ &\longrightarrow H_{\text{ét}}^1((Y_1(N) \times \mu_m^\circ)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_M^\circ}^{\text{cyc}}) \\ &\xrightarrow{\sim} H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_M^\circ}^{\text{cyc}}) \otimes \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]]. \end{aligned} \quad (37)$$

for each m dividing M .

6.2.4. More generally, let (a, B, m) be a triple of integers such that $mB \mid M$. Let us choose $L \geq 1$ such that $M \mid L$, $N \mid L$ and

$$\text{prime}(L) = \text{prime}(MN).$$

The construction of [22, §5.2] provides us with a map

$$\begin{aligned} t_{m,a(B)} : H_{\text{ét}}^1(Y_1(M, L)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{M,L}) \otimes \underline{\Delta}_{M,L}) \\ \longrightarrow H_{\text{ét}}^1\left((Y_1(N) \times \mu_m^{\circ})_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_m^{\circ}}^{\text{cyc}}\right) \\ \simeq H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_m^{\circ}}^{\text{cyc}}\right) \otimes \mathbb{Z}_p[\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]. \end{aligned} \quad (38)$$

Note that this map coincides with (37) if $m = M$, $a = 0$, $B = 1$ and $L = N$.

6.2.5. We write

$$\partial_n : \mathcal{O}(Y(Mp^n, Np^n))^{\times} \longrightarrow \varprojlim_r H_{\text{ét}}^1(Y(Mp^n, Np^n), \mathbb{P}_{p^r}) = H_{\text{ét}}^1(Y(Mp^n, Np^n), \mathbb{Z}_p(1))$$

for the Kummer map. For any integer $j \geq 0$, consider the chain of maps

$$\begin{aligned} \text{Ch}_{M,N,j} : \varprojlim_n K_2(Y(Mp^n, Np^n)) \\ \longrightarrow \varprojlim_n H_{\text{ét}}^2(Y(Mp^n, Np^n), \mathbb{Z}_p(2)) \\ \xrightarrow{\cup e_{1,n}^{\otimes j}} \varprojlim_n H_{\text{ét}}^2(Y(Mp^n, Np^n), f_n^*(\text{TSym}^j(\mathcal{T}_{M,N,n}) \otimes \mathbb{P}_{p^n}^{\otimes 2})) \\ \xrightarrow{\sim} H_{\text{ét}}^2(Y(M, N), \text{TSym}^j(\mathcal{T}_{M,N}) \otimes \underline{\Delta}_{M,N}(2)) \\ \longrightarrow H^1\left(\mathbb{Z}[1/MNp], H_{\text{ét}}^1(Y(M, N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{M,N}) \otimes \underline{\Delta}_{M,N}(2))\right), \end{aligned} \quad (39)$$

where the first map is induced by the composition of ∂_n with the cup product in étale cohomology, and the very last map is deduced from the Hochschild–Serre spectral sequence.

Let ${}_c g_{1/Mp^n,0} \in \mathcal{O}(Y(Mp^n, 1))^{\times}$ and ${}_d g_{0,1/Np^n} \in \mathcal{O}(Y(1, Np^n))^{\times}$ denote Kato's Siegel units (see [22, §§1–2] for their definition). As in op. cit., we put

$${}_c, d \mathbb{Z} Mp^n, Np^n := {}_c g_{1/Mp^n,0} \cup {}_d g_{0,1/Np^n} \in K_2(Y(Mp^n, Np^n)).$$

We recall that

$$({}_c, d \mathbb{Z} Mp^n, Np^n)_{n \geq 1} \in \varprojlim_n K_2(Y(Mp^n, Np^n))$$

belongs to the source of the map $\text{Ch}_{M,N,j}$ (c.f. [22], Proposition 2.3).

6.2.6. Let us fix a positive integer N . As in [22, §5], let ξ denote either the symbol $a(B)$ with $a, B \in \mathbb{Z}$ and $B \geq 1$ or an element of $\text{SL}_2(\mathbb{Z})$. For each integer $m \geq 1$, we denote by S the following set of primes:

$$S = \begin{cases} \text{primes}(mBp), & \text{if } \xi = a(B) \\ \text{primes}(mNp), & \text{if } \xi \in \text{SL}_2(\mathbb{Z}). \end{cases}$$

Let (c, d) be a pair of positive integers satisfying the following conditions:

$$(cd, 6) = 1, \quad (d, N) = 1, \quad \text{prime}(cd) \cap S = \emptyset.$$

If $\xi = a(B)$, we choose M and $L \geq 1$ such that

$$\begin{aligned} mB \mid M, \quad M \mid L, \quad N \mid L, \\ \text{prime}(M) = S, \quad \text{prime}(L) = S \cup \text{prime}(N) \end{aligned}$$

and denote by

$$\begin{aligned} \text{Ch}_{M,L,j,\xi} &: \varprojlim_n K_2(Y(Mp^n, Lp^n)) \\ &\longrightarrow H^1\left(\mathbb{Z}[1/S], H_{\text{ét}}^1(Y_1(M, L)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{M,L}) \otimes \underline{\Delta}_{M,L}(2))\right) \\ &\longrightarrow H^1\left(\mathbb{Z}[1/S], H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_m^{\circ}}^{\text{cyc}}(2)\right) \otimes \mathbb{Z}_p[\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]\right) \\ &\xrightarrow{\sim} H^1\left(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_m^{\circ}}^{\text{cyc}}(2)\right)\right) \\ &\xrightarrow{\sim} H^1\left(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_N(2)\right) \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t\right) \end{aligned} \quad (40)$$

the composition of the maps (39) and (38). Here, the symbol ι on the final line means that the target cohomology group is equipped with a continuous action of $\Gamma_{\mathbb{Q}(\zeta_m)} = \text{Gal}(\mathbb{Q}(\zeta_{mp^\infty})/\mathbb{Q}(\zeta_m))$ induced by the right action of this group on $\Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})$ via the canonical involution $\iota(g) = g^{-1}$ (c.f. §3.1.1).

If $\xi \in \text{SL}_2(\mathbb{Z})$, we fix $L \geq 3$ such that

$$m \mid L, \quad N \mid L, \quad \text{prime}(L) = S.$$

The element ξ induces an automorphism of $Y(Lp^n) = Y(Lp^n, Lp^n)$. We consider the map

$$\begin{aligned} \text{Ch}_{L,j,\xi} &: \varprojlim_n K_2(Y(Lp^n)) \\ &\longrightarrow H^1\left(\mathbb{Z}[1/S], H_{\text{ét}}^1(Y(L)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{L,L}) \otimes \underline{\Delta}_{L,L}(2))\right) \\ &\longrightarrow H^1\left(\mathbb{Z}[1/S], H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_m^{\circ}}^{\text{cyc}}(2)\right) \otimes \mathbb{Z}_p[\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]\right) \\ &\longrightarrow H^1\left(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_m^{\circ}}^{\text{cyc}}(2)\right)\right) \\ &\xrightarrow{\sim} H^1\left(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_N(2)\right) \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t\right). \end{aligned} \quad (41)$$

Definition 6.1

(i) If $\xi = a(B)$ we set

$${}_{c,d}\mathbb{BK}_{N,m}(j, \xi) = \text{Ch}_{M,L,j,\xi} \left(({}_{c,d}\mathbb{Z}Mp^n, Lp^n)_{n \geq 1} \right),$$

where $\text{Ch}_{M,L,j,\xi}$ is the map (40).

(ii) If $\xi \in \text{SL}_2(\mathbb{Z})$, we set

$${}_{c,d}\mathbb{BK}_{N,m}(j, \xi) = \text{Ch}_{L,j,\xi} \left((\xi^*({}_{c,d}\mathbb{Z}Lp^n, Lp^n))_{n \geq 1} \right),$$

where $\text{Ch}_{L,j,\xi}$ is the map (41).

Lemma 6.2 *The elements ${}_{c,d}\mathbb{BK}_{N,m}(j, \xi)$ do not depend on the choice of M and L .*

Proof The proof is similar to that of [22, Proposition 8.7]. Let (M', L') be another pair of integers that satisfy (40) and such that $M \mid M', N \mid N'$. Since the trace map commutes with cup products, we have the following commutative diagram:

$$\begin{array}{ccc} \varprojlim_n K_2(Y(M'p^n, L'p^n)) & \longrightarrow & H^1\left(\mathbb{Z}[1/S], H_{\text{ét}}^1(Y_1(M', L')_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{M', L'}) \otimes \underline{\Delta}_{M', L'}(2))\right) \\ \downarrow & & \downarrow \\ \varprojlim_n K_2(Y(Mp^n, Lp^n)) & \longrightarrow & H^1\left(\mathbb{Z}[1/S], H_{\text{ét}}^1(Y_1(M, L)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{M, L}) \otimes \underline{\Delta}_{M, L}(2))\right) \end{array}$$

On the other hand, the left vertical map sends $(c, d\mathcal{Z}_{M'p^n, L'p^n})_{n \geq 1}$ to $(c, d\mathcal{Z}_{Mp^n, Lp^n})_{n \geq 1}$ by [22, Proposition 2.3]. This proves the lemma in the case $\xi = a(B)$; the case $\xi \in \text{SL}_2(\mathbb{Z})$ is analogous. $\square$

6.2.7. The isomorphism (32) allows us to consider the moment maps

$$\text{mom}_N^{[k]} : \underline{\Delta}_N \rightarrow \text{TSym}^k(\mathcal{T}_N) \quad (42)$$

given as in [23, Section 2.5] (see also [24], §4.4). Let $\text{mom}_{N,n}^{[k]}$ denote the map $\text{mom}_N^{[k]}$ modulo p^n . Then the map $\text{mom}_{N,n}^{[k]}$ can be described as the composition

$$\begin{aligned} g_{n,*}(\mathbb{Z}/p^n\mathbb{Z}) &\xrightarrow{\cup e_{2,n}^{\otimes k}} g_{n,*}\left(\mathbb{Z}/p^n\mathbb{Z} \otimes g_n^*(\text{TSym}^k(\mathcal{T}_{N,n}))\right) \\ &\simeq g_{n,*} \circ g_n^*(\text{TSym}^k(\mathcal{T}_{N,n})) \xrightarrow{\text{trace}} \text{TSym}^k(\mathcal{T}_{N,n}). \end{aligned}$$

On the level of cohomology, one then obtains the commutative diagram

$$\begin{array}{ccc} H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \underline{\Delta}_{N,n}) & \xrightarrow{\simeq} & H_{\text{ét}}^1(Y_1(Np^n)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^n) \\ \downarrow \text{mom}_{N,n}^{[k]} & & \downarrow \cup e_{2,n}^{\otimes k} \\ H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^k(\mathcal{T}_{N,n})) & \longleftarrow & H_{\text{ét}}^1(Y_1(Np^n)_{\overline{\mathbb{Q}}}, g_n^*(\text{TSym}^k(\mathcal{T}_{N,n}))), \end{array}$$

where the bottom horizontal map is the trace map (c.f. [23], Proposition 2.6.8).

6.2.8. For each $r \in \mathbb{Z}$, we have the moment maps of sheaves on μ_m°

$$\text{mom}_{\mu_m^\circ}^{[r,n]} : \underline{\Delta}_{\mu_m^\circ}^{\text{cyc}} \longrightarrow h_{n,*} \circ h_n^*(\mathbb{Z}_p(-r)), \quad n \geq 0.$$

The stalks of $\underline{\Delta}_{\mu_m^\circ}^{\text{cyc}}$ are isomorphic to the Γ -module $\Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t$ at geometric points (c.f. [24], §6.3) and the moment map coincides with the map

$$\begin{aligned} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)}) &\longrightarrow \mathbb{Z}_p[G_n](-r), & G_n &:= \text{Gal}(\mathbb{Q}(\zeta_{mp^n})/\mathbb{Q}(\zeta_m)), \\ g &\longmapsto \chi^{-r}(g)\bar{g}_n \otimes \chi^{-r} \end{aligned}$$

where $\bar{g}_n \in G_n$ denotes the image of $g \in \Gamma_{\mathbb{Q}(\zeta_m)}$ under the natural projection $\Gamma_{\mathbb{Q}(\zeta_m)} \rightarrow G_n$. Let us define

$$\text{mom}_{N, \mu_m^\circ}^{[k,r,n]} := \text{mom}_N^{[k]} \boxtimes \text{mom}_{\mu_m^\circ}^{[r,n]} : \underline{\Delta}_{N, \mu_m^\circ}^{\text{cyc}} \longrightarrow \text{TSym}^k(\mathcal{T}_N)(-r).$$

Recall the elements ${}_{c,d}z_{1,N,m}^{(p)}(k, r, r', \xi, S)$ which Kato has constructed in [22, §8.9]. Set

$${}_{c,d}\mathbf{BK}_{N,m}(k, j, r, \xi) := {}_{c,d}z_{1,N,m}^{(p)}(j+k+2, r, j+1, \xi, S) \\ \in H^1\left(\mathbb{Z}[1/S, \zeta_{mp^n}], H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{Sym}^{j+k}(\mathcal{T}_N)(2-r)\right)\right).$$

Note that in his construction Kato makes use of the dual sheaf $\mathcal{T}_N^\vee$ together with the canonical isomorphism $\mathcal{T}_N^\vee \simeq \mathcal{T}_N(-1)$ provided by duality.

Proposition 6.3 *For all integers $j, k \geq 0$ and $r \in \mathbb{Z}$, the induced map*

$$\text{mom}_{N,j,\mu_m}^{[k,r,n]} : H^1\left(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_N) \otimes \underline{\Delta}_{N,\mu_m}^{\text{cyc}}(2)\right)\right) \\ \longrightarrow H^1\left(\mathbb{Z}[1/S, \zeta_{mp^n}], H_{\text{ét}}^1\left(Y_1(N)_{\overline{\mathbb{Q}}}, \text{TSym}^{j+k}(\mathcal{T}_N)(2-r)\right)\right)$$

sends ${}_{c,d}\mathbf{BK}_{N,m}(j, \xi)$ to ${}_{c,d}\mathbf{BK}_{N,mp^n}(k, j, r, \xi)$.

Proof This is clear by the construction. $\square$

6.3 Overconvergent étale sheaves

Until the end of this article, we set $Y = Y(1, N(p))$ unless we state otherwise and assume that N is coprime to p .

6.3.1. We denote by $\mathcal{F}$ the canonical sheaf on Y . Throughout this section, $\mathcal{W}$ and U denote an affinoid and a wide open disk such that

$$\mathcal{W} \subset U \subset W^*.$$

In what follows, we will allow ourselves to shrink both U and $\mathcal{W}$ as necessary. We adopt the notation and conventions of Sect. 5. In particular, O_U° denotes the ring of analytic functions on U that are bounded by 1. We let

$$\chi_U : \mathbb{Z}_p^\times \longrightarrow O_U^{\circ,\times}$$

denote the composition of the cyclotomic character $\chi : G_{\mathbb{Q}} \rightarrow \mathbb{Z}_p^\times$ with the canonical weight character $\kappa_U : \mathbb{Z}_p^\times \hookrightarrow O_U^{\circ,\times}$.

We review the theory of overconvergent sheaves introduced in [3]. See also [26, §4] and [14, §4] for further details concerning the material in this subsection.

Definition 6.4

- (i) We let $\mathcal{D}_U^\circ(\mathcal{T})$ (resp., $\mathcal{D}_U^\circ(\mathcal{T}')$, resp., $\mathcal{A}_U^0(\mathcal{T}')$) denote the pro-étale sheaf of O_U° -modules on Y , whose pullback to the pro-scheme $Y(p^\infty, Np^\infty)$ is the constant pro-sheaf $\mathbb{D}(U)^\circ$ (resp., $\mathbb{D}'(U)^\circ$, resp., $\mathbb{A}'(U)^0$).
- (ii) We set

$$M_U^\circ(\mathcal{T}') := H_{\text{ét}}^1\left(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U^\circ(\mathcal{T}')\right)(1), \quad M_U^\circ(\mathcal{T}) := H_{\text{ét}}^1\left(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U^\circ(\mathcal{T})\right)(\chi_U^{-1}), \\ N_U^1(\mathcal{T}') := H_{\text{ét},c}^1\left(Y_{\overline{\mathbb{Q}}}, \mathcal{A}_U^0(\mathcal{T}')\right)$$

and

$$M_U(\mathcal{T}') := M_U^\circ(\mathcal{T}')[1/p], \quad M_U(\mathcal{T}) := M_U^\circ(\mathcal{T})[1/p], \quad N_U(\mathcal{T}') := N_U^1(\mathcal{T}')[1/p].$$

The evaluation map

$$\mathrm{ev} : \mathbb{A}'(U)^\circ \otimes \mathbb{D}'(U)^\circ \longrightarrow O_U^\circ$$

induces a pairing

$$\mathcal{A}_U^0(\mathcal{T}') \otimes \mathcal{D}_U^\circ(\mathcal{T}') \longrightarrow \mathcal{O}_U^\circ,$$

where $\mathcal{O}_U^\circ$ is the sheaf associated to O_U° as in [14, §4.2]. Together with the trace map $H_{\text{ét},c}^2(Y_{\overline{\mathbb{Q}}}, \mathcal{O}_U^\circ(1)) \xrightarrow{\sim} O_U^\circ$, this pairing induces the pairing

$$N_U^\circ(\mathcal{T}') \otimes_{\Lambda_U} M_U^0(\mathcal{T}') \longrightarrow O_U^\circ. \quad (43)$$

We finally recall that Proposition 4.4.5 of [26] supplies us with a morphism of sheaves on Y :

$$\underline{\Delta}(\mathcal{T}(D')) \longrightarrow \mathcal{D}_U^\circ(\mathcal{T}'). \quad (44)$$

6.3.2. By GAGA, we have the canonical isomorphisms

$$\begin{aligned} M_U(\mathcal{T})(\chi_U) &\simeq H^1(\Gamma_p, \mathbb{D}(U)), & M_U(\mathcal{T}')(-1) &\simeq H^1(\Gamma_p, \mathbb{D}'(U)), \\ N_U(\mathcal{T}') &\simeq H_c^1(\Gamma_p, \mathbb{A}'(U)) \end{aligned} \quad (45)$$

(c.f. [3], Proposition 3.18). The isomorphisms in (45) allow us to define a Hecke action on the spaces on the left of each isomorphism. The submodules $M_U(\mathcal{T}')^{\leq v}$, $M_U(\mathcal{T})^{\leq v}$ and $N_U(\mathcal{T}')^{\leq v}$ are stable under the action of $G_{\mathbb{Q}}$ and the actions of prime-to- p Hecke operators.

For an affinoid $\mathcal{W}$ contained in a wide open U in the weight space, we set $M_{\mathcal{W}}(\mathcal{T}')^{\leq v} := M_U(\mathcal{T}')^{\leq v} \otimes_{O_U} O_{\mathcal{W}}$. We similarly define $M_{\mathcal{W}}(\mathcal{T})^{\leq v}$ and $N_{\mathcal{W}}(\mathcal{T}')^{\leq v}$.

Proposition 6.5 *Let x_0 denote a cuspidal point of the eigencurve $\mathcal{C}$. There exist a sufficiently small wide open U , an affinoid $\mathcal{W} \subset U$ and a slope v such that $x_0 \in \mathcal{C}_{\mathcal{W},v}$ and the following hold true.*

(i) *Let $\mathcal{X}$ denote the connected component of x_0 in $\mathcal{C}_{\mathcal{W},v}$. There is an isomorphism of $O_{\mathcal{X}}$ -modules*

$$\mathrm{comp} : \mathrm{Symb}_{\Gamma_p}(\mathbb{D}^\dagger(W))^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}} \xrightarrow{\sim} M_{\mathcal{W}}(\mathcal{T})^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}}$$

which interpolates Artin's comparison isomorphisms between Betti and étale cohomology.

(ii) *All three $O_{\mathcal{X}}$ -modules*

$$M_{\mathcal{W}}(\mathcal{T})^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}}, \quad M_{\mathcal{W}}(\mathcal{T}')^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}}, \quad N_{\mathcal{W}}(\mathcal{T}')^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}} \quad (46)$$

are free of rank 2.

We remark that it is absolutely crucial that the affinoid neighborhood $\mathcal{X}$ of x_0 falls within the cuspidal eigencurve.

Proof

- (i) This portion follows from (45), Lemma 5.2 and Proposition 5.8.
- (ii) It follows from (i) and Theorem 5.4 that $M_{\mathcal{W}}(\mathcal{T})^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}}$ is free of rank 2 over $O_{\mathcal{X}}$. By [26, Proposition 4.4.8.4], $M_{\mathcal{W}}(\mathcal{T}')^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}}$ is isomorphic to $M_{\mathcal{W}}(\mathcal{T})^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}}$ as $O_{\mathcal{W}}$ -modules and therefore, they both have the same $O_{\mathcal{X}}$ -rank. Finally, $N_{\mathcal{W}}(\mathcal{T}')^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} O_{\mathcal{X}}$ have rank 2 by the isomorphism (45) and Corollary 5.7.

□

6.3.3. Let $j \geq 1$ be a positive integer. The map $F(x) \mapsto F(x - j)$ defines an isomorphism $t_j : O_{U-j}^0 \simeq O_U^0$. If $F \in \mathbb{A}'(U - j)^0$ and $G \in \mathcal{P}_j(O_E)$ is a homogeneous polynomial of degree j , then $t_j \circ (FG) \in \mathbb{A}'(U)^0$, and we have a well defined map $\mathbb{A}'(U - j)^0 \otimes \mathcal{P}_j(O_E) \rightarrow \mathbb{A}'(U)^0$. Passing to the duals, we obtain a map

$$\beta_j^* : \mathbb{D}'(U)^\circ \longrightarrow \mathbb{D}'(U - j)^\circ \otimes \mathrm{TSym}^j(O_E^2)$$

and the associated morphism of sheaves:

$$\beta_j^* : \mathcal{D}_U^\circ(\mathcal{T}') \longrightarrow \mathcal{D}_{U-j}^\circ(\mathcal{T}') \otimes \mathrm{TSym}^j(\mathcal{T}). \quad (47)$$

Let us denote by

$$\delta_j : \mathbb{A}'(U)^\circ \longrightarrow \mathbb{A}'(U - j)^\circ \otimes \mathcal{P}_j(O_E)$$

the map given by $\delta_j(F) := \frac{1}{j!} \sum_{i+m=j} \frac{\partial^j F(x, y)}{\partial x^i \partial y^m} \otimes x^i y^m$. On transposing this map, we obtain

$$\delta_j^* : \mathbb{D}'(U - j)^\circ \otimes \mathrm{TSym}^j(O_E^2) \longrightarrow \mathbb{D}'(U)^\circ$$

and the induced morphism of sheaves

$$\delta_j^* : \mathcal{D}_{U-j}^\circ(\mathcal{T}') \otimes \mathrm{TSym}^j(\mathcal{T}) \longrightarrow \mathcal{D}_U^\circ(\mathcal{T}'). \quad (48)$$

6.4 Big Galois representations

6.4.1. For the convenience of the reader, we review Deligne's construction of p -adic representations associated to eigenforms. Let $\mathcal{T}_{\mathbb{Q}_p}^\vee$ be the dual sheaf of $\mathcal{T}_{\mathbb{Q}_p}$. Let f denote a p -stabilized eigenform of weight $k + 2 \geq 2$ and level Γ_p , which is new away from p (but could be new or old at p). Deligne's representation associated to f is defined as the f -isotypic Hecke submodule $V[f] \subset H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{T}_{\mathbb{Q}_p}^\vee)$ for the Hecke operators $\{T_\ell\}_{\ell \nmid Np}$ and $\{U_\ell\}_{\ell \mid Np}$, which are given as in [22, §4.9] (but denoted in op. cit. by $T(\ell)$ for all ℓ). Recall that this is a two dimensional vector space equipped with a continuous action of $G_{\mathbb{Q}}$, which is de Rham at p (crystalline iff f is p -old) with Hodge–Tate weights $(0, k + 1)$. By the theory of newforms, the Hecke module $H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{T}_{\mathbb{Q}_p}^\vee)$ is f -semisimple. In particular, $V[f]$ coincides with the *generalized* Hecke eigenspace $V[[f]] \subset H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{T}_{\mathbb{Q}_p}^\vee)$ in the sense of [10, §1.2]. We denote by V_f the f -isotypic quotient of $H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k \mathcal{T}_{\mathbb{Q}_p}^\vee)$. Then the natural projection $V[[f]] \rightarrow V_f$ is an isomorphism.

Dually, consider $H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathrm{TSym}^k \mathcal{T}_{\mathbb{Q}_p})(1)$. The Hecke algebra $\mathcal{H}$ acts on this space via the dual Hecke operators the dual Hecke operators $\{T'_\ell\}_{\ell \nmid Np}$ and $\{U'_\ell\}_{\ell \mid Np}$, which are given as in [22, §4.9] (denoted by $T'(\ell)$ for all ℓ). Let $V'[f]$ (respectively, V'_f) denote the f -isotypic Hecke submodule (respectively, the f -isotypic quotient) of $H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathrm{TSym}^k \mathcal{T}_{\mathbb{Q}_p})(1)$. Paralleling the discussion in the previous paragraph, the Hecke module $H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathrm{TSym}^k \mathcal{T}_{\mathbb{Q}_p})(1)$ is semisimple at f and $V'[f]$ coincides with the generalized Hecke eigenspace $V'[[f]]$ inside $H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathrm{TSym}^k \mathcal{T}_{\mathbb{Q}_p})(1)$. The canonical projection $V'[[f]] \rightarrow V'_f$ is an isomorphism. Poincaré duality induces a perfect Galois-equivariant pairing

$$(\cdot, \cdot)_f : V'_f \otimes V[f] \longrightarrow E. \quad (49)$$

6.4.2. For any normalized eigenform $f = \sum_{n \geq 1} a_n q^n \in S_{k+2}(\Gamma_p)$, we set

$$f^c = \sum_{n \geq 1} \overline{a}_n q^n.$$

Since f is an eigenform for the Hecke operators $\{T_\ell\}_{\ell \nmid Np}$ and $\{U_\ell\}_{\ell \mid Np}$ with eigenvalues $\{a_\ell\}$, it is also an eigenform for the dual operators $\{T'_\ell\}_{\ell \nmid Np}$ and $\{U'_\ell\}_{\ell \mid Np}$, with eigenvalues $\{\bar{a}_\ell\}$. The isomorphism $\mathcal{T}_{\mathbb{Q}_p}^\vee(1) \simeq \mathcal{T}_{\mathbb{Q}_p}$ induces a Hecke equivariant map

$$H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \text{Sym}^k \mathcal{T}_{\mathbb{Q}_p}^\vee)(k+1) \longrightarrow H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^k \mathcal{T}_{\mathbb{Q}_p})(1),$$

which gives rise to a canonical isomorphism

$$V[f^c](k+1) \simeq V'[f] \quad (50)$$

on the f -isotypic subspaces for the dual operators $\{T'_\ell\}_{\ell \nmid Np}$ and $\{U'_\ell\}_{\ell \mid Np}$. With obvious modifications, all these constructions also make sense for modular forms of level $\Gamma_1(N)$.

6.4.3. We maintain the notation and the conventions of Section 6.3.2. Fix a p -stabilized eigenform f as in §6.4.1 and denote by x_0 the corresponding point of the eigencurve $\mathcal{C}$. Let α_0 denote the U_p -eigenvalue on f .

Fix a slope $v \geq v_p(\alpha_0)$ a wide open affinoid U and an affinoid $\mathcal{W}$ as in Proposition 6.5. Let $\mathcal{X}$ denote the connected component of x_0 in $\mathcal{C}_{\mathcal{W},v}$.

In the remainder of this paper we will work with the $\mathcal{O}_{\mathcal{X}}$ -adic Galois representations

$$\begin{aligned} V_{\mathcal{X}} &:= N_{\mathcal{W}}(\mathcal{T}')^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} \mathcal{O}_{\mathcal{X}}, \\ V'_{\mathcal{X}} &:= M_{\mathcal{W}}(\mathcal{T}')^{\leq v} \otimes_{\mathcal{H}_{\mathcal{W}}} \mathcal{O}_{\mathcal{X}}. \end{aligned} \quad (51)$$

By Proposition 6.5, both $\mathcal{O}_{\mathcal{X}}$ -modules have rank two.

We denote by

$$(\cdot)_{\mathcal{X}} : V'_{\mathcal{X}} \otimes V_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{W}} \quad (52)$$

the $G_{\mathbb{Q},S}$ -equivariant cup-product pairing induced by (43).

6.4.4. For any E -valued classical point $x \in \mathcal{X}^{\text{cl}}(E)$ of $\mathcal{X}$, we let f_x denote the corresponding p -stabilized cuspidal eigenform of weight $w(x)+2$. In particular, f_{x_0} is our fixed p -stabilized eigenform f . By Proposition 5.3 we have $\mathcal{O}_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{W}}[X]/(X^e - Y)$ for some $e \geq 1$, so the formalism of Sect. 2 applies in this context.

We define the representations $V_{\mathcal{X},x}$, $V_{\mathcal{X},w(x)}$, $V_{\mathcal{X}}[x]$ and $V_{\mathcal{X}}[[x]]$ (respectively $V'_{\mathcal{X},x}$, $V'_{\mathcal{X},w(x)}$, $V'_{\mathcal{X}}[x]$ and $V'_{\mathcal{X}}[[x]]$) by plugging in $M = V_{\mathcal{X}}$ (respectively $M = V'_{\mathcal{X}}$) in Definition 2.2. In particular, $V_{\mathcal{X},x} := V_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X},x}} E$ and $V_{\mathcal{X},w(x)} := V_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{W},w(x)}} E$. We recall that $e = 1$ except when the p -stabilized form f_{x_0} is θ -critical.

Let $\mathcal{P}_k(\mathcal{O}_E)$ denote the space of homogeneous polynomials in two variables of degree k with coefficients in E . If $k \in U$, we have a canonical injection $\mathcal{P}_k(\mathcal{O}_E) \hookrightarrow A'(U)^\circ$, which induces a map $\text{Sym}^k(\mathcal{O}_E^{\oplus 2}) \hookrightarrow A'(U)^\circ$ and the dual map $D'(U)^\circ \rightarrow \text{TSym}^k(\mathcal{O}_E^{\oplus 2})$. We have the associated morphisms of sheaves

$$\mathcal{D}_U^\circ(\mathcal{T}') \longrightarrow \text{Sym}^k \mathcal{T}', \quad \text{Sym}^k \mathcal{T}^\vee \longrightarrow \mathcal{A}_U^\circ(\mathcal{T}') \quad (53)$$

which give rise to the pair of Hecke equivariant morphisms

$$\begin{aligned} H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \text{Sym}^k \mathcal{T}_{\mathbb{Q}_p}^\vee) &\longrightarrow H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{A}_U(\mathcal{T}')), \\ H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U(\mathcal{T}')) &\longrightarrow H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^k \mathcal{T}_{\mathbb{Q}_p}). \end{aligned} \quad (54)$$

We obtain the following morphisms by the definitions of $M_U(\mathcal{T}')$ and $N_U(\mathcal{T}')$:

$$\begin{aligned} H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \text{Sym}^k \mathcal{T}_{\mathbb{Q}_p}^\vee) &\longrightarrow N_U(\mathcal{T}'), \\ M_U(\mathcal{T}') &\longrightarrow H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^k \mathcal{T}_{\mathbb{Q}_p}(1)). \end{aligned}$$

On tensoring with $O_{\mathcal{X}}$ and taking slope $\leq \nu$ submodules, we deduce for any $x \in \mathcal{X}^{\text{cl}}(E)$ that there are canonical morphisms

$$\begin{aligned} V[f_x] &\xrightarrow{j_x} V_{\mathcal{X}}[[x]] \hookrightarrow V_{\mathcal{X}}, \\ V'_{\mathcal{X}} &\longrightarrow V'_{\mathcal{X}, w(x)} \xrightarrow{\pi_x} V'_{f_x}. \end{aligned} \quad (55)$$

It follows from definitions that

- $V_{\mathcal{X}, w(x)}$ and $V'_{\mathcal{X}, w(x)}$ are E -vector spaces of dimension $2e$, where e denotes the ramification degree of the weight map $w : \mathcal{X} \rightarrow \mathcal{W}$ at x_0 ;
- $V_{\mathcal{X}}[x]$ and $V'_{\mathcal{X}}[x]$ are the f_x -isotypic Hecke eigenspaces inside $V_{\mathcal{X}, w(x)}$ and $V'_{\mathcal{X}, w(x)}$ (in alternative wording, $V'_{\mathcal{X}}[x]$ is the largest subspace of $V'_{\mathcal{X}, w(x)}$ annihilated by $X - X(x) \in O_{\mathcal{X}}$); and these are both E -vector spaces of dimension 2.

For any $x \in \mathcal{X}^{\text{cl}}(E)$, let us also write $(,)_{w(x)}$ for the pairing

$$(,)_{w(x)} : V'_{\mathcal{X}, w(x)} \otimes_E V_{\mathcal{X}, w(x)} \longrightarrow E$$

induced from the pairing (52) by $O_{\mathcal{W}}$ -linearity. We summarize the basic properties of these objects (that we shall make use of in what follows) in Proposition 6.6 below.

Proposition 6.6 *Suppose $x \in \mathcal{X}^{\text{cl}}(E)$ is an E -valued classical point.*

(i) *The Hecke-equivariant map*

$$V'_{\mathcal{X}}[[x]] \hookrightarrow V'_{\mathcal{X}, w(x)} \xrightarrow{\pi_x} V'_{f_x}$$

is surjective. In particular, it induces an isomorphism

$$V'_{\mathcal{X}, x} \xrightarrow{\sim} V'_{f_x}$$

(ii) *The Hecke-equivariant map*

$$V[f_x] \xrightarrow{j_x} V_{\mathcal{X}}[[x]],$$

is injective. In particular, it induces an isomorphism

$$V[f_x] \xrightarrow{\sim} V_{\mathcal{X}}[x].$$

(iii) *The pairing $(,)_{\mathcal{X}}$ verifies the property **Adj**.*

(iv) *The restriction of $(,)_{w(x)}$ to $V'_{\mathcal{X}, w(x)} \otimes V_{\mathcal{X}}[x]$ factors as*

$$\begin{array}{ccc} V'_{\mathcal{X}, w(x)} \otimes V_{\mathcal{X}}[x] & \xrightarrow{(,)_{w(x)}} & E \\ \pi_x \downarrow & \parallel & \parallel \\ V'_{\mathcal{X}, x} \otimes V_{\mathcal{X}}[x] & \longrightarrow & E \end{array}$$

We shall denote the induced pairing on $V'_{\mathcal{X}, x} \otimes V_{\mathcal{X}}[x]$ by $(,)_x$.

(v) *The diagram*

$$\begin{array}{ccc} V'_{\mathcal{X}, x} \otimes V_{\mathcal{X}}[x] & \xrightarrow{(,)_x} & E \\ \pi_x \downarrow \sim & \sim \uparrow j_x & \parallel \\ V'_{f_x} \otimes V[f_x] & \xrightarrow{(,)_{f_x}} & E \end{array}$$

where the bottom pairing is the Poincaré duality (49), commutes. Namely, for all $v' \in V'_{\mathcal{X},x}$ and $v \in V[f_x]$ one has

$$(\pi_x(v'), v)_{f_x} = (v', j_x(v))_x.$$

Proof

- (i) The surjectivity of $V'_{\mathcal{X}}[[x]] \rightarrow V'_{f_x}$ follows from [10, Corollary 3.19], where the analogous statement is proved for the spaces of modular symbols, combined with Proposition 6.5(i).
- (ii) This is clear by definitions.
- (iii) This portion follows from the [14, §4.2] (see in particular the discussion following (69)).
- (iv) This is a particular case of the factorization (2).
- (v) This statement follows from the functoriality of the cup products.

□

6.5 Beilinson–Kato elements over the eigencurve

6.5.1. We maintain previous notation and conventions in Sects. 5 and 6. Let $x_0 \in \mathcal{C}^{\text{cusp}}(E)$ be a classical cuspidal point on the eigencurve. Fix a wide open U , an affinoid $\mathcal{W}$ and a slope $v \geq 0$ such that $x_0 \in \mathcal{W} \subset U$ and the conditions of Proposition 6.5 are satisfied. Let $\mathcal{X}$ denote the connected component of x_0 in $\mathcal{C}_{\mathcal{W},v}$. Consider the following maps:

- For each non-negative integer $k \in U$, the map (54) which we notate as

$$\rho_U^{[k]} : H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U(\mathcal{T}'_{\mathbb{Q}_p})) \longrightarrow H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^k(\mathcal{T}_{\mathbb{Q}_p})).$$

- For each classical $x \in \mathcal{X}$, the map (55), which we shall notate as

$$\rho_{\mathcal{X}}^{[x]} : V'_{\mathcal{X}} \rightarrow V'_{f_x}.$$

Moreover, we have the following commutative diagram, which follows from the slope decomposition:

$$\begin{array}{ccc} M_{\mathcal{W}}(\mathcal{T}') & \longrightarrow & H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^{w(x)} \mathcal{T}_{\mathbb{Q}_p}(1)) \\ \downarrow & & \downarrow \\ V'_{\mathcal{X}} & \longrightarrow & V'_{f_x}. \end{array} \quad (56)$$

Here, by definition, $M_{\mathcal{W}}(\mathcal{T}') := H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U(\mathcal{T}'))_{\mathcal{W}}$.

- The cyclotomic moment maps

$$\text{mom}_{\mu_m^{\circ}}^{[r,n]} : \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t \rightarrow E[G_n](-r), \quad r \in \mathbb{Z}, n \geq 0,$$

which have defined defined in §6.2.8. Together with the maps $\rho_U^{[k]}$ and $\rho_{\mathcal{X}}^{[x]}$, the cyclotomic moment maps give rise to the morphisms

$$\begin{aligned} \rho_{U,m}^{[k,r,n]} &: H^1(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U(\mathcal{T}') \widehat{\otimes} \underline{\Delta}_{\mu_m^{\circ}}^{\text{cyc}}(2))) \\ &\longrightarrow H^1(\mathbb{Z}[1/S, \zeta_{mp^n}], H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^k(\mathcal{T})(2-r))) \end{aligned} \quad (57)$$

and

$$\rho_{\mathcal{X},m}^{[x,r,n]} : H^1(\mathbb{Z}[1/S, \zeta_m], V'_{\mathcal{X}} \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t(1)) \longrightarrow H^1(\mathbb{Z}[1/S, \zeta_{mp^n}], V'_{f_x}(1-r)). \quad (58)$$

It follows from the commutativity of the diagram (56) that the maps (57) and (58) are compatible in the evident sense.

6.5.2. The map (44) induces a morphism

$$H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \underline{\Delta}(\mathcal{T}(D'))) \longrightarrow H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U^\circ(\mathcal{T}')).$$

The composition of this map with (35) gives rise to the map

$$H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \underline{\Delta}_{Np}(2)) \longrightarrow H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U^\circ(\mathcal{T}')(2)) =: M_U^\circ(\mathcal{T}')(1). \quad (59)$$

More generally, for each integer $j \geq 0$, the morphism (48) together with the trace map induce

$$\begin{aligned} H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{Np}) \otimes \underline{\Delta}_{Np}(2)) &\longrightarrow H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}) \otimes \mathcal{D}_{U-j}^\circ(\mathcal{T}')(2)) \\ &\xrightarrow{\delta_j^*} H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U^\circ(\mathcal{T}')(2)) =: M_U^\circ(\mathcal{T}')(1). \end{aligned} \quad (60)$$

Definition 6.7 Let S, ξ, j, c, d, m be as in Definition 6.1.

(i) We let

$${}_{c,d}\mathbb{BK}_{N,m}^{[U]}(j, \xi) \in H_{\text{ét}}^1(\mathbb{Z}[1/S, \zeta_m], M_U^\circ(\mathcal{T}') \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t(1))$$

denote the image of the Beilinson–Kato element ${}_{c,d}\mathbb{BK}_{N,m}(j, \xi)$ under the map

$$\begin{aligned} &H_{\text{ét}}^1(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1(Y_1(Np), \text{TSym}^j(\mathcal{T}_{Np}) \otimes \underline{\Delta}_{Np} \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t(2)) \\ &\longrightarrow H_{\text{ét}}^1(\mathbb{Z}[1/S, \zeta_m], M_U^\circ(\mathcal{T}') \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t(1)) \end{aligned}$$

induced by the map (60).

(ii) Let x_0 be a cuspidal point of the eigencurve $\mathcal{C}$. Let $\mathcal{W} \subset U$ be an affinoid disk centered at x_0 such that $\mathcal{W}$, U and $v \geq 0$ satisfy the conditions of Proposition 6.5. We denote by

$${}_{c,d}\mathbb{BK}_{N,m}^{[\mathcal{W}, \leq v]}(j, \xi) \in H^1(\mathbb{Z}[1/S, \zeta_m], M_{\mathcal{W}}^\circ(\mathcal{T}')^{\leq v} \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t(1))$$

the image of ${}_{c,d}\mathbb{BK}_{N,m}^{[U]}(j, \xi)$ under the map induced by the natural projection $M_U^\circ(\mathcal{T}') \rightarrow M_{\mathcal{W}}^\circ(\mathcal{T}')^{\leq v}$.

(iii) Let $\mathcal{X}$ denote the connected component of x_0 . We denote by

$${}_{c,d}\mathbb{BK}_{N,m}^{[\mathcal{X}]}(j, \xi) \in H^1(\mathbb{Z}[1/S, \zeta_m], V'_{\mathcal{X}} \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})^t(1))$$

the image of ${}_{c,d}\mathbb{BK}_{N,m}^{[\mathcal{W}, \leq v]}(j, \xi)$ under the map induced by the natural projection $M_{\mathcal{W}}^\circ(\mathcal{T}')^{\leq v} \rightarrow V'_{\mathcal{X}}$.

6.5.3. The following is the interpolation property for $O_{\mathcal{X}}$ -adic Beilinson–Kato elements ${}_{c,d}\mathbb{BK}_{N,m}^{[\mathcal{X}]}(j, \xi)$.

Theorem 6.8 In the setting of Definition 6.7, the following are valid:

(i) For any $k \geq 0$, the diagram

$$\begin{array}{ccc} H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{Np}) \otimes \underline{\Delta}_{Np}) & \xrightarrow{\text{mom}_{Np,j}^{[k]}} & H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^{j+k}(\mathcal{T}_{Np})) \\ \text{(60)} \downarrow & & \downarrow \text{pr}'_* \\ H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_U^\circ(\mathcal{T}')) & \xrightarrow{\rho_U^{[j+k]}} & H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^{j+k}(\mathcal{T})) \end{array} \quad (61)$$

commutes. Here, $\text{mom}_{Np,j}^{[k]}$ is the moment map induced from (42), and the right vertical map pr'_* is the trace map associated to the projection $\text{pr}' : Y_1(Np) \rightarrow Y$.

ii) For any $m \geq 1$ and $k \geq 0$, the diagram (61) induces the commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{Np}) \otimes \underline{\Delta}_{Np, \mu_m^{\circ}}^{\text{cyc}}(2)) & \xrightarrow{\text{mom}_{Np,j, \mu_m^{\circ}}^{[k,r,n]}} & H^1(\mathbb{Z}[1/S, \zeta_{mp^n}], H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^{j+k}(\mathcal{T}_{Np})(2-r))) \\ \downarrow \text{pr}'_* & & \downarrow \text{pr}'_* \\ H^1(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_{\mathcal{U}}^{\circ}(\mathcal{T}') \otimes \underline{\Delta}_{\mu_m^{\circ}}^{\text{cyc}}(2))) & \xrightarrow{\rho_{U,m}^{[j+k,r,n]}} & H^1(\mathbb{Z}[1/S, \zeta_{mp^n}], H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^{j+k}(\mathcal{T})(2-r))), \end{array} \quad (62)$$

where $\text{mom}_{Np,j, \mu_m^{\circ}}^{[k,r,n]}$ is the map defined in Proposition 6.3.

(iii) For any $x \in \mathcal{X}^{\text{cl}}(E)$, the diagram (62) induces the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathbb{Z}[1/S, \zeta_m], H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{Np}) \otimes \underline{\Delta}_{Np, \mu_m^{\circ}}^{\text{cyc}}(2)) & \xrightarrow{\text{mom}_{Np,j, \mu_m^{\circ}}^{[x,r,n]}} & H^1(\mathbb{Z}[1/S, \zeta_{mp^n}], H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^{w(x)}(\mathcal{T})(2-r)))_{f_x} \\ \downarrow \text{pr}'_* & & \downarrow \simeq \\ H^1(\mathbb{Z}[1/S, \zeta_m], V_{\mathcal{X}} \widehat{\otimes} \Lambda(\Gamma_{\mathbb{Q}(\zeta_m)})(1)) & \xrightarrow{\rho_{\mathcal{X},m}^{[x,r,n]}} & H^1(\mathbb{Z}[1/S, \zeta_{mp^n}], V_{f_x}^*(1-r)), \end{array}$$

where the right vertical isomorphism is induced by the canonical isomorphism between f_x -isotypic components of $H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^{w(x)}(\mathcal{T}_{Np, \mathbb{Q}_p}))$ and $H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^{w(x)}(\mathcal{T}_{\mathbb{Q}_p}))$ and the horizontal map $\text{mom}_{Np,j, \mu_m^{\circ}}^{[x,r,n]}$ on the first row is the composition of $\text{mom}_{Np,j, \mu_m^{\circ}}^{[w(x),r,n]}$ with the projection to the f_x -component.

iv) For any $x \in \mathcal{X}^{\text{cl}}(E)$, one has

$$\rho_{\mathcal{X},m}^{[x,r,n]} \left(c, d \mathbb{BK}_{N,m}^{[\mathcal{X}]}(j, \xi) \right) = c, d \mathbb{BK}_{Np,mp^n}(f_x^c, j, r, \xi)$$

where

$$c, d \mathbb{BK}_{Np,mp^n}(f_x^c, j, r, \xi) \in H^1(\mathbb{Z}[1/S, \zeta_{mp^n}], V_{f_x}^c(w(x) + 2 - r))$$

is the projection of the Beilinson–Kato element $c, d \mathbb{BK}_{Np,mp^n}(w(x) - j, j, r, \xi)$ introduced in §6.2.8 to the f_x^c -isotypic eigenspace for the action of Hecke operators $\{T_\ell\}_{\ell \nmid Np}$ and $\{U_\ell\}_{\ell \mid Np}$. Here we have used the canonical isomorphism (50) to identify $c, d \mathbb{BK}_{N,mp^n}(f_x^c, j, r, \xi)$ with an element of $H^1(\mathbb{Z}[1/S, \zeta_{mp^n}], V_{f_x}^*(1-r))$.

Proof

(i) Consider the following diagram:

$$\begin{array}{ccccc} & & H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{Np}) \otimes \underline{\Delta}_{Np}) & & \\ & & \downarrow & & \downarrow \\ & & H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{Np}) \otimes \Lambda(\mathcal{T}_{Np}(\mathcal{S}_{Np}))) & \xrightarrow{[Np]_*} & H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}_{Np}) \otimes \Lambda(\mathcal{T}_{Np})) & \xrightarrow{\text{mom}^{[k]}} & H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \text{TSym}^{j+k}(\mathcal{T}_{Np})) \\ & & \downarrow \text{pr}'_* & & \downarrow \text{pr}'_* & & \downarrow \text{pr}'_* \\ & & H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}) \otimes \Lambda(\mathcal{T}(D'))) & \xrightarrow{[p]_*} & H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}) \otimes \Lambda(\mathcal{T})) & \xrightarrow{\text{mom}^{[k]}} & H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^{j+k}(\mathcal{T})) \\ & & \downarrow & & \downarrow \rho_{U,m}^{[k]} & & \downarrow \rho_{U,m}^{[j+k]} \\ & & H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \text{TSym}^j(\mathcal{T}) \otimes \mathcal{D}_{U-j}^{\circ}(\mathcal{T}')) & \xrightarrow{s_j^*} & H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_{U-j}^{\circ}(\mathcal{T}')) & & H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{D}_{U-j}^{\circ}(\mathcal{T}')) \end{array} \quad (60)$$

It follows from definitions that the map (60) decomposes as explained in the diagram. The factorization of the moment map $\text{mom}_{Np,j}^{[k]}$ follows from its definition (c.f. [24], §4) and the fact that it agrees with the description given in Section 6.2.7 proved in [24, Theorem 4.5.2.2]. Therefore, it suffices to prove that this diagram commutes. The commutativity of the squares (1) and (2) follow from the functoriality of moment maps. The commutativity of the triangle (3) follows from [26, Proposition 4.2.10]. The commutativity of (4) is clear from definitions.

(ii–iii) These assertions follow immediately from (i).

(iv) This portion follows from (iii) and Proposition 6.3.

□

Corollary 6.9 *We retain the notation and hypotheses of Theorem 6.8. We let*

$${}_{c,d}\text{BK}_{Np,\text{Iw}}(f_x^c, j, \xi) \in H^1(\mathbb{Z}[1/S], V_{f_x^c}(w(x) + 2) \otimes \Lambda^t) = H^1(\mathbb{Z}[1/S], V'_{f_x} \otimes \Lambda^t(1))$$

denote the unique class which specializes to ${}_{c,d}\text{BK}_{Np,p^n}(f_x^c, j, r, \xi) \in H^1(\mathbb{Z}[1/S, \zeta_{p^n}], V_{f_x^c}(w(x) + 2 - r))$ for all $n \in \mathbb{N}$ under the evident morphisms. Then,

$${}_{c,d}\text{BK}_{N,1}^{[\mathcal{X}]}(j, \xi, x) = {}_{c,d}\text{BK}_{Np,\text{Iw}}(f_x^c, j, \xi)$$

where ${}_{c,d}\text{BK}_{N,1}^{[\mathcal{X}]}(j, \xi, x)$ is the specialization of ${}_{c,d}\text{BK}_{N,1}^{[\mathcal{X}]}(j, \xi)$ to x .

Remark 6.10 We retain the notation and hypotheses of Theorem 6.8. Suppose that $x \in \mathcal{X}^{\text{cl}}(E)$ is a classical point and f_x is p -old, arising as the p -stabilization of a newform g of level N coprime to p , with respect to the root $a_p(f_x)$ of the Hecke polynomial of g at p . Then f_x^c is the p -stabilization of the newform g^c with respect to the root $\bar{a}_p(f_x)$ of the Hecke polynomial of g^c at p . In this remark, we shall explain the relation between ${}_{c,d}\text{BK}_{Np,\text{Iw}}(f_x^c, j, \xi)$ and the Beilinson–Kato element ${}_{c,d}\text{BK}_{N,\text{Iw}}(g^c, j, \xi)$ that one associates to the newform g^c . The natural projection $Y(1, N(p)) =: Y \xrightarrow{\text{pr}} Y_1(N)$ induces the trace map

$$\text{pr}_* : H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \text{Sym}^k \mathcal{T}_{\mathbb{Q}_p}^{\vee}) \longrightarrow H_{\text{ét},c}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \text{Sym}^k \mathcal{T}_{\mathbb{Q}_p}^{\vee}),$$

which restricts to an isomorphism (since this map commutes with the action of Hecke operators $\{T'_\ell\}_{\ell \nmid Np}$)

$$\text{pr}_* : V_{f_x^c} \simeq V_{g^c}. \quad (63)$$

Then by construction, the image of the Beilinson–Kato element ${}_{c,d}\text{BK}_{N,\text{Iw}}(g^c, j, \xi)$ under the natural morphism induced from the isomorphism (63) coincides with ${}_{c,d}\text{BK}_{Np,\text{Iw}}(f_x^c, j, \xi)$ (c.f. [22], Proposition 8.7).

6.6 Normalized Beilinson–Kato elements

In Sect. 6.6, we normalize the $\mathcal{O}_{\mathcal{X}}$ -adic Beilinson–Kato elements (in a weak sense). These normalized elements interpolate Kato’s normalized elements in [22, §13].

6.6.1. We recall that $\Lambda(\Gamma_1) := \mathbb{Z}_p[[\Gamma_1]]$, where $\Gamma_1 = \text{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p(\zeta_p))$ and $\Lambda = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \Lambda(\Gamma_1)$, where $\Delta = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. One then has

$$\Lambda = \bigoplus_{\eta \in X(\Delta)} \Lambda(\Gamma_1)e_\eta.$$

Let us set $\Lambda_{\mathcal{X}}(\Gamma_1) = \mathcal{O}_{\mathcal{X}} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(\Gamma_1)$ and put $\Lambda_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda$.

We start with the following auxiliary lemma.

Lemma 6.11 *Let M be a finitely generated $\Lambda_{\mathcal{X}}(\Gamma_1)$ -module. Suppose that $M_{x_0} := M/XM$ is torsion-free as a $\Lambda(\Gamma_1)[1/p]$ -module and that $M[X] = 0$. Then there exist an affinoid disk $\mathcal{X}' \subset \mathcal{X}$ such that $M_{\mathcal{O}_{\mathcal{X}'}} := M \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}'}$ is torsion-free as a $\Lambda_{\mathcal{X}'}(\Gamma_1)$ -module.*

Proof Suppose on the contrary that $0 \neq m \in M_{\text{tor}}$ is a torsion element. For sufficiently small $\mathcal{X}' \subset \mathcal{X}$, the affinoid domain $\mathcal{O}_{\mathcal{X}'}$ is a PID. Let us choose $\mathcal{X}'$ with that property and assume until the end of our proof that $\mathcal{X} = \mathcal{X}'$. Since $\Lambda_{\mathcal{X}}$ is a UFD, we may choose m so that m is annihilated by an irreducible element $0 \neq f \in \Lambda_{\mathcal{X}}$. Note that since $M[X] = 0$, the irreducible element f cannot be an associate of X and moreover, we can assume without loss of generality that $m \in M \setminus XM$.

Let us write $f = \sum_{n=0}^{\infty} a_n(X)(\gamma_1 - 1)^n$ where $a_n(X) \in \mathcal{O}_{\mathcal{X}}$. Since M_{x_0} is torsion-free as a $\Lambda(\Gamma_1)[1/p]$ -module, we infer that $a_0(0) = 0$. Indeed, since we have $f \cdot (m + XM) = a_0(0)(m + XM) \in (M_{x_0})_{\text{tor}} = \{0\}$ and as $m + XM \neq 0$ as an element of the torsion-free module $M/XM = M_{x_0}$, the required conclusion that $a_0(0) = 0$ follows. This in turn shows that $X \mid f$. Since f is irreducible, this in turn implies that f is an associate of X , contrary to the discussion in the previous paragraph. This concludes the proof of our lemma. $\square$

6.6.2. Fix a p -stabilized normalized eigenform $f \in S_{k_0+2}(\Gamma_1(N) \cap \Gamma_0(p))$ where $(N, p) = 1$. Let us denote by x_0 the corresponding point of the cuspidal eigencurve and set

$$\begin{aligned} H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) &:= H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^t(1)), \\ H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_f(1)) &:= H^1(\mathbb{Z}[1/S], V'_f \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^t(1)). \end{aligned}$$

Lemma 6.12 *There exists a neighborhood $\mathcal{X}$ of x_0 such that $H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1))$ is torsion-free as a $\Lambda_{\mathcal{X}}(\Gamma_1)$ -module.*

Proof We apply Lemma 6.11 to $M = H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1))$. The long exact sequence of Galois cohomology associated to the short exact sequence

$$0 \rightarrow V'_{\mathcal{X}} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^t(1) \xrightarrow{X} V'_{\mathcal{X}} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^t(1) \rightarrow V'_f \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^t(1) \rightarrow 0$$

shows that we have an injection

$$M/XM \hookrightarrow H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_f(1)).$$

By Kato's fundamental result (c.f. [22], Theorem 12.4), the $\Lambda[1/p]$ -module $H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_f(1))$ is free of rank one. Therefore $M_{x_0} = M/XM$ is also torsion-free. The same long exact sequence together with the fact that $H_{\text{Iw}}^0(\mathbb{Z}[1/S], V'_f(1)) = 0$ gives

$$M[X] = H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1))[X] = \text{im} \left(H_{\text{Iw}}^0(\mathbb{Z}[1/S], V'_f(1)) \longrightarrow H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) \right) = 0.$$

Thus the hypotheses of Lemma 6.11 are satisfied and one can shrink $\mathcal{X}$ so as to ensure that M is torsion-free. $\square$

Let $Q(\Lambda_{\mathcal{X}})$ (resp., $Q(\Lambda)$) denote the total quotient field of $\Lambda_{\mathcal{X}}$ (resp., of Λ). It follows from Lemma 6.11 that on shrinking $\mathcal{X}$ as necessary, one can ensure that the natural

$$H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) \longrightarrow H^1(\mathbb{Z}[1/S], V'_f(1)) \otimes_{\Lambda_{\mathcal{X}}} Q(\Lambda_{\mathcal{X}})$$

is an injection.

6.6.3. We consider the product

$$\mathcal{E}_N(x) = \prod_{\ell|N} (1 - a_\ell(\mathbf{f})\sigma_\ell^{-1}) \in \Lambda_{\mathcal{X}} \quad (64)$$

as a two variable function in the weight and cyclotomic variables.

Lemma 6.13 *For each classical point x and integer m , we have $\mathcal{E}_N(x, \chi^m) \neq 0$ provided that $m \notin \{\frac{w(x)}{2}, \frac{w(x)+1}{2}\}$. In particular, we have $\mathcal{E}_N(x, \chi^{\frac{w(x)}{2}+1}) \neq 0$.*

Proof According to [33, Corollary 2] (the Weight-Monodromy Conjecture for modular forms), $a_\ell(f_x)$ is an algebraic integer with complex absolute value either $\ell^{\frac{w(x)}{2}}$ or $\ell^{\frac{w(x)+1}{2}}$. In particular, we have $\mathcal{E}_N(x, \chi^m) = (1 - a_\ell(f_x)\ell^{-m}) \neq 0$ whenever $m \notin \{\frac{w(x)}{2}, \frac{w(x)+1}{2}\}$. $\square$

6.6.4. For an integer c , we let $\sigma_c \in \Gamma$ denote the element defined by the requirement that $\chi(\sigma_c) = c$. We shall consider σ_c also as an element of Λ in the obvious manner. Following Kato, we define the partial normalization factors

$$\begin{aligned} \mu_1(c, j) &= c^2 - c^{-j}\sigma_c, \\ \mu_2(d, j, x) &= d^2 - d^{j-w(x)}\sigma_d, \\ \mu_0(c, d, j, x) &:= \mu_1(c, j)\mu_2(d, j, x) \in \Lambda_{\mathcal{X}}. \end{aligned} \quad (65)$$

Here, for any $x \in \mathcal{X}$, we define $d^{w(x)} := d^{k_0} \langle d \rangle^{w(x)-k_0}$, where $\langle d \rangle^{w(x)-k_0} := \exp((w(x) - k_0) \log \langle m \rangle)$. We remark that $\langle d \rangle^{w(x)-k_0} = d^{w(x)-k_0}$ whenever $x \in \mathcal{X}^{\text{cl}}$, since $w(x) - k_0$ is divisible by $p - 1$. We also define the full normalization factor:

$$\mu(c, d, j, x) := \mu_0(c, d, j, x) \mathcal{E}_N(x) \in \Lambda_{\mathcal{X}}. \quad (66)$$

6.6.5. To simplify notation, we shall write ${}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi)$ in place of ${}_{c,d}\mathbb{BK}_{N,1}^{[\mathcal{X}]}(j, \xi)$. When c, d and j are understood, we will write $\mu_?$ in place of $\mu_?(c, d, j, x)$ for $? \in \{0, \emptyset\}$. We denote by $\Lambda_{\mathcal{X}}[\mu_0^{-1}] \subset Q(\Lambda_{\mathcal{X}})$ the $\Lambda_{\mathcal{X}}$ -subalgebra generated by $\mu_0(c, d, j, x)^{-1}$. We also denote by $\Lambda[\mu_{x_0}^{-1}] \subset Q(\Lambda)$ the $\Lambda[\frac{1}{p}]$ -subalgebra generated by $\mu_0(c, d, j, x_0)^{-1}$.

Definition 6.14 Let us fix $\mathcal{W}$ so that $H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1))$ is torsion-free as a $\Lambda_{\mathcal{X}}$ -module (we can choose such $\mathcal{W}$ thanks to our discussion in §6.6.2). Let us fix integers $c \equiv 1 \equiv d \pmod{N}$ with $(cd, 6p) = 1$ and $c^2 \neq 1 \neq d^2$, and an integer $j \in [0, k_0]$. We define the partially normalized Beilinson–Kato elements on setting

$$\begin{aligned} \mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) &:= \mu_0(c, d, j, x)^{-1} {}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) \otimes_{\Lambda_{\mathcal{X}}} \Lambda_{\mathcal{X}}[\mu_0^{-1}], \\ \xi &\in \text{SL}_2(\mathbb{Z}). \end{aligned}$$

The integrality properties of the partial normalizations of the Beilinson–Kato elements are established in Proposition 6.15, whose proof has been relegated to Appendix A.

Proposition 6.15 *Suppose c, d and j are as in Definition 6.14.*

i) *The partially normalized Beilinson–Kato element*

$$\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) := \mu_0(c, d, j, x)^{-1} {}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi)$$

is independent of the choice of c and d .

ii) *There exists an affinoid neighborhood $\mathcal{W}$ of k_0 so that $\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1))$.*

Proof The first assertion is Proposition A.2, whereas the second is Proposition A.4 combined with (i). $\square$

6.6.6. We check that $\mu(c, d, j, x)$ agrees with the normalization factor considered in [22, §13.9], whenever $x \in \mathcal{X}^{\text{cl}}$. For each newform g of weight k Kato normalizes the elements that he denotes by

$$\left({}_{c,d}\mathbf{z}_{p^n}^{(p)}(g, k, j, \xi, \text{prime}(pN)) \right)_{n \geq 1} \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V_g).$$

These elements are given as the Hecke equivariant projection of the elements $\left({}_{c,d}\mathbf{z}_{p^n}^{(p)}(k, k, j, \xi, \text{prime}(pN)) \right)_{n \geq 1}$ recalled at the end of Sect. 6.2 to the g -isotypic Hecke eigenspace. He introduces the normalization factor

$$\mu_{\text{Kato}}(c, d, j, g) := \underbrace{(c^2 - c^{k+1-j}\sigma_c)(d^2 - d^{j+1}\sigma_d)}_{\mu_{\text{Kato}}^{(0)}} \underbrace{\prod_{\ell|N} (1 - \bar{a}_\ell(g)\ell^{-k}\sigma_\ell^{-1})}_{\mathcal{E}_{\text{Kato}}},$$

see [22, §13.9].

Suppose $x \in \mathcal{X}^{\text{cl}}(E)$ is a classical point. As we have explained in Corollary 6.9 (see also Remark 6.10), our big Beilinson–Kato element ${}_{c,d}\mathbb{BK}_{N,1}^{[\mathcal{X}]}(j, \xi)$ interpolates the elements

$${}_{c,d}\mathbf{BK}_{Np,\text{Iw}}(f_x^c, j, \xi) \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V_{f_x^c}(w(x) + 2)), \quad x \in \mathcal{X}^{\text{cl}}(E)$$

which coincide with the f_x^c -isotypic projection of Kato's element $\left({}_{c,d}\mathbf{z}_{p^n}(k + j + 2, 0, j + 1, \xi, \text{prime}(pN)) \right)_{n \geq 1}$, where $w(x) = k + j$. Kato's element $\left({}_{c,d}\mathbf{z}_{p^n}(k + j + 2, 0, j + 1, \xi, \text{prime}(pN)) \right)_{n \geq 1}$ differs from

$$\left({}_{c,d}\mathbf{z}_{p^n}(k + j + 2, k + j + 2, j + 1, \xi, \text{prime}(pN)) \right)_{n \geq 1} \quad (67)$$

by the twist by $\chi^{w(x)+2} = \chi^{k+j+2}$. Recall that we denote by $\text{Tw}_m : \Lambda \rightarrow \Lambda$ the twisting morphism which is given on the group like elements by $\gamma \mapsto \chi^{-m}(\gamma)\gamma$.

Since f is new away from p , it follows from [10, Lemma 2.7] that one can shrink $\mathcal{X}$ as necessary to ensure that f_x is new away from p and p -old for all $x \in \mathcal{X}^{\text{cl}}(E) \setminus \{x_0\}$. There are two scenarios:

- (i) f_x is a newform (so $x = x_0$ and $f_x = f$, owing to our choice of $\mathcal{X}$). Then f_x^c is a newform too.
- (ii) f_x is the p -stabilization of a newform $f_x^\circ =: g$ of level N coprime to p with respect to the root $a_p(f_x)$ of the Hecke polynomial of g at p . As we have explained in Remark 6.10, f_x^c is a p -stabilization of the newform g^c and one naturally identifies ${}_{c,d}\mathbf{BK}_{Np,\text{Iw}}(f_x^c, j, \xi)$ with ${}_{c,d}\mathbf{BK}_{N,\text{Iw}}(g^c, j, \xi)$. In this scenario, we have $\mu_{\text{Kato}}(c, d, j, f_x^c) = \mu_{\text{Kato}}(c, d, j, g^c)$.

We then have, in both scenarios,

$$\begin{aligned} & \text{Tw}_{k+j+2}(\mu_{\text{Kato}}(c, d, j, f_x^c)) \\ &= \text{Tw}_{k+j+2} \left((c^2 - c^{k+2}\sigma_c)(d^2 - d^{j+2}\sigma_d) \prod_{\ell|N} (1 - a_\ell(f_x)\ell^{-k-j-2}\sigma_\ell^{-1}) \right) \\ &= (c^2 - c^{-j}\sigma_c)(d^2 - d^{j-w(x)}\sigma_d) \prod_{\ell|N} (1 - a_\ell(f_x)\sigma_\ell^{-1}) =: \mu(c, d, j, x). \end{aligned}$$

In [22, Section 13], Kato shows that there exists an element $\mathbf{z}(g^c, j, \xi) \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V_{g^c}(w(x) + 2))$ such that

$$c, d \text{BK}_{N, \text{Iw}}(g^c, j, \xi) = \mu(c, d, j, x) \mathbf{z}(g^c, j, \xi).$$

We remark that in Kato's notation, we have $\mathbf{z}(g^c, j, \xi) = \text{Tw}_{w(x)+2}(\mathbf{z}_\gamma^{(p)})$, where γ is the cohomology class associated to j and ξ . Combining this fact with the preceding remarks, we deduce that

$$\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi, x) = \mathcal{E}_N(x) \mathbf{z}(g^c, j, \xi). \quad (68)$$

7 “Étale” construction of p -adic L -functions

Our objective in Sect. 7 is to use the $\mathcal{O}_{\mathcal{X}}$ -adic Beilinson–Kato element we have introduced in Definition 6.7(iii) and 6.14 to give an “étale” construction of Stevens’ two-variable p -adic L -function in neighborhoods of non- θ -critical cuspidal points on the eigencurve. In particular, we assume throughout Sect. 7 that $\mathcal{X}$ is étale over $\mathcal{W}$ (i.e. $e = 1$); see our companion article [5] for the treatment of the complementary case (which concerns the neighborhoods of θ -critical points on the eigencurve).

Recall the representations $V_{\mathcal{X}}$ and $V'_{\mathcal{X}}$ given as in (51). We will always assume that $\mathcal{X}$ is sufficiently small to satisfy the conditions of Proposition 6.15. For each classical $x \in \mathcal{X}^{\text{cl}}(E)$, we denote by f_x the corresponding eigenform. The $\mathcal{O}_{\mathcal{X}}$ -adic Beilinson–Kato elements $\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi)$ take coefficients in the cohomology of the universal cyclotomic twist of $V'_{\mathcal{X}}$.

Recall also that these representations are equipped with the $\mathcal{O}_{\mathcal{X}}$ -linear pairing

$$(\cdot, \cdot) : V'_{\mathcal{X}} \otimes V_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{X}}$$

(c.f. (52), bearing in mind that we have assumed $e = 1$), which satisfies the formalism of Sect. 2. We will use repeatedly the properties of this pairing summarized in Proposition 6.6.

Let $f = f_0^{\alpha_0}$ be a p -stabilized non- θ -critical cuspidal eigenform of weight $k_0 + 2 \geq 2$ and as before, which corresponds to the point x_0 on the cuspidal eigencurve. The eigencurve is étale at x_0 over the weight space (c.f. [10], Lemma 2.8). In other words, $e = 1$ and $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{W}}$. As we have explained in §3.3.2, the simplification of the local behaviour of the eigencurve in this scenario exhibits itself also in the the p -local study (namely, the properties of the triangulation over the eigencurve). The reader might find it convenient to identify $\mathcal{X}$ with a Coleman family $\mathbf{f}$ through $f_0^{\alpha_0}$, so that the overconvergent eigenform f_x that corresponds to a point $x \in \mathcal{X}(E)$ is simply the specialization $\mathbf{f}(x)$ of the Coleman family $\mathbf{f}$.

To simplify notation, we write $\mathcal{R}_{\mathcal{X}}$ for the relative Robba ring $\mathcal{R}_{\mathcal{O}_{\mathcal{X}}}$ and $\mathcal{H}_{\mathcal{X}}(\Gamma)$ for the relative Iwasawa algebra $\mathcal{H}_{\mathcal{O}_{\mathcal{X}}}(\Gamma)$. The p -adic representation $V_{\mathcal{X}}$ (given as in (51)) verifies the properties (C1)–(C3) and therefore comes equipped with a triangulation $\mathbf{D} \subset \mathbf{D}_{\text{rig}, \mathcal{O}_{\mathcal{X}}}^{\dagger}(V_{\mathcal{X}})$ over the relative Robba ring $\mathcal{R}_{\mathcal{X}}$, verifying the properties $\varphi\Gamma_1$ – $\varphi\Gamma_3$). Fix an $\mathcal{O}_{\mathcal{X}}$ -basis $\{\eta\}$ of $\mathcal{D}_{\text{cris}}(\mathbf{D})$ and let $\eta_x \in \mathcal{D}_{\text{cris}}(\mathbf{D}_x) = \mathcal{D}_{\text{cris}}(\mathbf{D}[x])$ denote its specialization at $x \in \mathcal{X}(E)$ (equivalently, at weight $w(x)$).

Since $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{W}}$ in this setting (therefore also $\mathcal{H}_{\mathcal{X}}(\Gamma) = \mathcal{H}_{\mathcal{O}_{\mathcal{W}}}(\Gamma)$), the $G_{\mathbb{Q}, S}$ -equivariant pairing $(\cdot, \cdot)$ in (52) is $\mathcal{O}_{\mathcal{X}}$ -linear. Recall from (8) that this pairing induces a canonical $\mathcal{H}_{\mathcal{X}}(\Gamma)$ -linear pairing

$$(\cdot, \cdot) : (\mathcal{H}_{\mathcal{X}}(\Gamma) \otimes_{\Lambda_{\mathcal{X}}} H_{\text{Iw}}^1(\mathbb{Q}_p, V'_{\mathcal{X}}(1))) \otimes (\mathcal{H}_{\mathcal{X}}(\Gamma) \otimes_{\Lambda_{\mathcal{X}}} H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\mathcal{X}})^{\iota}) \longrightarrow \mathcal{H}_{\mathcal{X}}(\Gamma). \quad (69)$$

Recall the Perrin-Riou exponential map

$$\mathrm{Exp}_{\mathbf{D},0} : \mathcal{R}_{\mathcal{X}}^{\psi=0} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathrm{cris}}(\mathbf{D}) \longrightarrow H_{\mathrm{Iw}}^1(\mathbb{Q}_p, \mathbf{D})$$

as well as the canonical embedding $H_{\mathrm{Iw}}^1(\mathbb{Q}_p, \mathbf{D}) \hookrightarrow \mathcal{H}_{\mathcal{X}}(\Gamma) \otimes_{\Lambda_{\mathcal{X}}} H_{\mathrm{Iw}}^1(\mathbb{Q}_p, V_{\mathcal{X}})$ that the Fontaine–Herr complex gives rise to, through which we treat the image of $\mathrm{Exp}_{\mathbf{D},0}$ as a submodule of $\mathcal{H}_{\mathcal{X}}(\Gamma) \otimes_{\Lambda_{\mathcal{X}}} H_{\mathrm{Iw}}^1(\mathbb{Q}_p, V_{\mathcal{X}})$.

Consider the action of the complex conjugation on $H_{\mathrm{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1))$. For all integers c, d as in Definition 6.14, $0 \leq j \leq k_0$ and $\xi \in \mathrm{SL}_2(\mathbb{Z})$, we denote by ${}_{c,d}\mathbb{B}_N^{[\mathcal{X}],\pm}(j, \xi) \in H_{\mathrm{Iw}}^1(\mathbb{Q}_p, V_{\mathcal{X}}(1))$ the $\pm$ -parts of the Beilinson–Kato element ${}_{c,d}\mathbb{B}_N^{[\mathcal{X}]}(j, \xi)$. We similarly define $\mathbb{B}_N^{[\mathcal{X}],\pm}(j, \xi) \in H_{\mathrm{Iw}}^1(\mathbb{Q}_p, V_{\mathcal{X}}(1))$ (c.f. Proposition 6.15(ii)) for the partially normalized Beilinson–Kato element $\mathbb{B}_N^{[\mathcal{X}]}(j, \xi) := \mu_0(c, d, j, x)^{-1} {}_{c,d}\mathbb{B}_N^{[\mathcal{X}]}(j, \xi)$

Let us also denote by $\epsilon(j)$ the sign of $(-1)^j$.

Definition 7.1 (Arithmetic non- θ -critical p -adic L -function in two-variables) We set (sic!)

$$\begin{aligned} L_{p,\eta}^+(\mathbf{f}; j, \xi) &:= \left\langle \mathrm{res}_p \left(\mathbb{B}_N^{[\mathcal{X}],\epsilon(k_0-1)}(j, \xi) \right), c \circ \mathrm{Exp}_{\mathbf{D},0}(\tilde{\eta})^t \right\rangle \in \mathcal{H}_{\mathcal{X}}(\Gamma) \\ L_{p,\eta}^-(\mathbf{f}; j, \xi) &:= \left\langle \mathrm{res}_p \left(\mathbb{B}_N^{[\mathcal{X}],\epsilon(k_0)}(j, \xi) \right), c \circ \mathrm{Exp}_{\mathbf{D},0}(\tilde{\eta})^t \right\rangle \in \mathcal{H}_{\mathcal{X}}(\Gamma), \end{aligned}$$

where $\tilde{\eta} = \eta \otimes (1 + \pi)$.

Definition 7.2 For any $x \in \mathcal{X}(E)$, we put $L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi, x) = x \circ L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi, x) \in \mathcal{H}_E(\Gamma)$. For a character $\rho \in X(\Gamma)$, we set $L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi, x, \rho) := \rho \circ L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi, x)$.

For x above, we let f_x° denote the newform associated to f_x . We write $\alpha(x)$ for the U_p -eigenvalue on f_x , so that when f_x is p -old, it is the p -stabilization of the newform f_x° with respect to the root $\alpha(x)$ of the Hecke polynomial of f_x° at p . We let $L_{p,\alpha(x)}(f_x^{\circ})$ denote the Manin–Višik p -adic L -function associated to an unspecified choice of (a pair of) Shimura periods (see, however, item c) in the proof of Theorem 7.3 below, where this choice is made explicit).

The following result (in varying level of generality) has been previously announced in the independent works of Hansen, Ochiai and Wang.

Theorem 7.3 *There exists a neighborhood $\mathcal{X}$ of x_0 such that the following hold true.*

- (i) *There exist (j_+, ξ_+) and (j_-, ξ_-) such that $L_{p,\eta}^{\pm}(\mathbf{f}; j_{\pm}, \xi_{\pm}, x_0)$ are nonzero elements of $\mathcal{H}_E(\Gamma)$.*
- (ii) *Assume that $(j_{\pm}, \xi_{\pm})$ satisfy the conditions in (i). Then for each $x \in \mathcal{X}^{\mathrm{cl}}(E)$ such that $v_p(\alpha(x)) < w(x) + 1$, the p -adic L -functions $L_{p,\eta}^{\pm}(\mathbf{f}; j_{\pm}, \xi_{\pm}, x)$ agree with the Manin–Višik p -adic L -function $L_{p,\alpha(x)}(f_x^{\circ})$, up to multiplication by $D^{\pm} \mathcal{E}_N(x)$ where $D^{\pm} \in E^{\times}$ and $\mathcal{E}_N(x)$ is the product of Euler factors at bad primes, given as in (64).*
- (iii) *Assume that $v_p(\alpha_0) = k_0 + 1$, but $f = \mathbf{f}(x_0)$ is not θ -critical. Then $L_{p,\eta}^{\pm}(\mathbf{f}; j_{\pm}, \xi_{\pm}, x_0)$ agree with the one variable p -adic L -functions of Pollack–Stevens [32] up to multiplication by $D^{\pm} \mathcal{E}_N(x)$, where $D^{\pm} \in E^{\times}$ is a constant.*
- (iv) *Let $L_p(\mathbf{f}, \Phi^{\pm})$ denote the two-variable p -adic L -functions of Bellaïche and Stevens associated to modular symbols $\Phi^{\pm} \in \mathrm{Symb}_{\Gamma_p}^{\pm}(\mathcal{X})$ (c.f. [10], Theorem 3). Then there exist functions $u^{\pm}(x) \in \mathcal{O}_{\mathcal{X}}^{\times}$ such that*

$$L_{p,\eta}^{\pm}(\mathbf{f}; j_{\pm}, \xi_{\pm}) = u^{\pm}(x) \mathcal{E}_N(x) L_p(\mathbf{f}, \Phi^{\pm}).$$

Proof Let $x \in \mathcal{X}^{\text{cl}}(E)$ be a classical point as in the statement of the theorem. Let g denote the unique newform which admits f_x as a p -stabilization if f_x is p -old and $g = f_x$ if f_x is new. By (68), the specialization of the big Beilinson–Kato element at x can be compared with the normalized Beilinson–Kato element for g :

$$\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi, x) = \mathcal{E}_N(x) \mathbf{z}(g^c, j, \xi).$$

Since Perrin-Riou’s exponential map commutes with base-change, the specializations $L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi, x)$ of p -adic L -functions $L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi)$ coincide with one variable Perrin-Riou’s L -functions constructed using the element $\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi, x)$:

$$\begin{aligned} L_{p,\eta}^+(\mathbf{f}; j, \xi, x) &:= \left\langle \text{res}_p \left(\mathbb{BK}_N^{[\mathcal{X}], \epsilon(k_0-1)}(j, \xi, x) \right), c \circ \text{Exp}_{\mathbf{D}_x, 0}(\tilde{\eta}_x)^t \right\rangle_x \in \mathcal{H}_E(\Gamma), \\ L_{p,\eta}^-(\mathbf{f}; j, \xi, x) &:= \left\langle \text{res}_p \left(\mathbb{BK}_N^{[\mathcal{X}], \epsilon(k_0)}(j, \xi, x) \right), c \circ \text{Exp}_{\mathbf{D}_x, 0}(\tilde{\eta}_x)^t \right\rangle_x \in \mathcal{H}_E(\Gamma). \end{aligned}$$

We identify $\mathbf{z}(g^c, j, \xi)$ with an element of $H_{1\text{w}}^1(\mathbb{Z}[1/S], V'_x(1))$ via the isomorphism $V'_x(1) \simeq V_g(w(x) + 2)$ and define

$$\begin{aligned} L_{p,\eta}^+(g, j, \xi) &:= \left\langle \text{res}_p \left(\mathbf{z}(g^c, j, \xi)^{\epsilon(k_0-1)} \right), c \circ \text{Exp}_{\mathbf{D}_x, 0}(\tilde{\eta}_x)^t \right\rangle_x \in \mathcal{H}_E(\Gamma), \\ L_{p,\eta}^-(g, j, \xi) &:= \left\langle \text{res}_p \left(\mathbf{z}(g^c, j, \xi)^{\epsilon(k_0)} \right), c \circ \text{Exp}_{\mathbf{D}_x, 0}(\tilde{\eta}_x)^t \right\rangle_x \in \mathcal{H}_E(\Gamma). \end{aligned}$$

Therefore

$$L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi, x) = \mathcal{E}_N(x) L_{p,\eta}^{\pm}(g, j, \xi). \quad (70)$$

The following statements concerning $L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi, x)$ are proved by Kato:

- a)** There exist (j_+, ξ_+) and (j_-, ξ_-) such that $L_{p,\eta}^+(\mathbf{f}; j_+, \xi_+, x)$ and $L_{p,\eta}^-(\mathbf{f}; j_-, \xi_-, x)$ are non-zero functions. This follows from Kato’s explicit reciprocity law [22, Theorem 6.6] together with Ash–Stevens theorem (see [22], Theorem 13.6), the non-vanishing results of Jacquet–Shalika and Rohrlich (see [22], Theorem 13.5) and the fact that the product $\mathcal{E}_N(x) \in \mathcal{H}_E(\Gamma)$ of bad Euler factors is a non-zero-divisor (c.f. Lemma 6.13).
- b)** For any pair (j, ξ) , there exist constants $C^{\pm} = C^{\pm}(j, \xi) \in E$ such that $L_{p,\eta}^{\pm}(g; j, \xi, x) = C^{\pm} L_{p,\eta}^{\pm}(g; j_{\pm}, \xi_{\pm}, x)$. This follows from Kato’s explicit reciprocity law and the fact that $H_{1\text{w}}^1(\mathbb{Z}[1/S], V'_x(1))$ is free of rank one over $\Lambda[1/p]$. See [22, §13.9].
- c)** If $v_p(\alpha(x)) < w(x) + 1$, the functions $L_{p,\eta}^{\pm}(g; j_{\pm}, \xi_{\pm})$ agree with the Manin–Višik p -adic L -functions associated to the Shimura periods that come attached to the choice of the classes $\delta(g, j_{\pm}, \xi_{\pm})^{\pm}$ (see [22] for precise definitions).

Plugging in $x = x_0$ in (70) we have

$$L_{p,\eta}^{\pm}(\mathbf{f}; j, \xi, x_0) = \mathcal{E}_N(x) L_{p,\eta}^{\pm}(g, j, \xi).$$

Applying **a)** and **b)**, and noticing that $\mathcal{E}_N(x_0) \in \mathcal{H}_E(\Gamma)$ is a non-zero-divisor (c.f. Lemma 6.13), the proof of Part (i) follows.

Let us choose (j_+, ξ_+) as in (i), so that $L_{p,\eta}^+(\mathbf{f}; j_+, \xi_+, x_0)$ is non-zero. In particular, for each even integer $1 \leq i \leq p-1$, there exists a character $\chi_i \in X(\Gamma_1)$ so that $L_{p,\eta}^+(\mathbf{f}; j_+, \xi_+, x_0, \omega^i \chi_i) \neq 0$. Since $L_{p,\eta}^+(\mathbf{f}; j_+, \xi_+, x, \omega^i \chi_i)$ is analytic in the variable x (where ω is the Teichmüller character), there exists a neighborhood $\mathcal{X}$ of x_0 such that $L_{p,\eta}^+(\mathbf{f}; j_+, \xi_+, x)$ is a non-zero-divisor of $\mathcal{H}_E(\Gamma)^+$, for all $x \in \mathcal{X}$. Similarly, one checks that $L_{p,\eta}^-(\mathbf{f}; j_-, \xi_-, x)$ is a non-zero-divisor of $\mathcal{H}_E(\Gamma)^-$. Combining these observations with **(a)**, **(b)** and **(c)** above, we conclude the proof of (ii).

We next prove Part (iv). Using the Amice transform, we can consider the functions $L_p(\mathbf{f}, \Phi^\pm)$ as elements of $\mathcal{H}_{\mathcal{X}}(\Gamma)$. It follows from [10, Theorem 3] for classical non-critical points $x \in \mathcal{X}(E)$ that $L_p(\mathbf{f}, \Phi^+, x)$ coincides, up to multiplication by a non-zero constant, with the Manin–Višik p -adic L -function. We infer from (i) that for classical non-critical points $x \in \mathcal{X}^{\text{cl}}(E)$, we have

$$L_{p,\eta}^+(\mathbf{f}; j_+, \xi_+, x) = u_x \mathcal{E}_N(x) L_p(\mathbf{f}, \Phi^+, x)$$

for some constant $u_x \in E$, with $u_{x_0} \neq 0$ by the choice of (j_+, ξ_+) . On expressing $L_{p,\eta}^+(\mathbf{f}; j_+, \xi_+, x)$ and $\mathcal{E}_N(x) L_p(\mathbf{f}, \Phi^+, x)$ as power series with coefficients in $O_{\mathcal{X}}$, we immediately deduce that there exists a function $u^+(x) \in O_{\mathcal{X}}$ such that $u^+(x_0) \neq 0$ and

$$L_{p,\eta}^+(\mathbf{f}; j_+, \xi_+, x) = u^+(x) \mathcal{E}_N(x) L_p(\mathbf{f}, \Phi^+, x).$$

On shrinking $\mathcal{X}$ as necessary, we can ensure that $u^+(x)$ does not vanish on $\mathcal{X}$. The same argument proves that $L_{p,\eta}^-(\mathbf{f}; j_-, \xi_-, x) = u^-(x) L_p(\mathbf{f}, \Phi^-, x)$ for some $u^-(x) \in O_{\mathcal{X}}$. This concludes the proof of (iv).

Part (iii) is a direct consequence of (iv) (as per the definition of the Pollack–Stevens p -adic L -function) and our theorem is proved. $\square$

We note that in the particular case when $\mathbf{f}$ is the Coleman family passing through a newform $f_0 = f_0^{\alpha_0}$ of level $\Gamma_0(Np)$ and weight $k_0 + 2$ with $\alpha_0 = a_p(f_0) = p^{k_0/2}$, the conclusions of Theorem 7.3 play a crucial role in [6]. When $f = f_0^{\alpha_0}$ has critical slope but not θ -critical, it is also crucially used in the proof of a conjecture of Perrin-Riou in [11].

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Appendix A: Integrality of normalizations

For any $x \in \mathcal{X}$, let us define $d^{w(x)} := d^{k_0} \langle d \rangle^{w(x)-k_0}$. We define the partial normalization factor

$$\mu_0(c, d, j, x) := \underbrace{(c^2 - c^{-j} \sigma_c)}_{\mu_1(c, j)} \underbrace{(d^2 - d^{j-w(x)} \sigma_d)}_{\mu_2(d, j, x)} \in \Lambda_{\mathcal{X}} \quad (71)$$

so that we have

$$\mu(c, d, j, x) = \mu_0(c, d, j, x) \underbrace{\prod_{\ell|N} (1 - a_\ell(\mathbf{f})\sigma_\ell^{-1})}_{\mathcal{E}_N}$$

for the normalization factor $\mu(c, d, j, x)$ given as in (66).

A.1 The normalization factors revisited

Let $\mathbb{X}_{\text{cyc}} : G_{\mathbb{Q}} \rightarrow \Gamma \hookrightarrow \Lambda^\times$ denote the universal cyclotomic character, so that $\mathbb{X}_{\text{cyc}}^{-1}$ gives the action on Λ^ι . We also define the universal cyclotomic character parametrized by the weight space $\mathbb{X}_{\text{wt}}$ as the compositum

$$\mathbb{X}_{\text{wt}} : G_{\mathbb{Q}} \rightarrow \Gamma \xrightarrow[\chi]{\sim} \mathbb{Z}_p^\times \hookrightarrow \mathbb{Z}_p[[\mathbb{Z}_p^\times]]^\times \hookrightarrow \mathcal{O}_{\mathcal{W}}^\times.$$

Let $\gamma \in \Gamma$ denote a fixed topological generator.

Lemma A.1 *Suppose c and d are integers coprime to p .*

(i) *We have*

$$-c^{-j}\mu_1(c, j) = \sigma_c - \chi^{j+2}(\sigma_c).$$

In particular, if c is a primitive root modulo p^2 (so that σ_c generates Γ), then $\mu_1(c, j) \doteq \gamma - \chi^{j+2}(\gamma)$, where “ $\doteq$ ” means equality up to a unit multiple.

(ii) *We have*

$$d^{j-w(x)}\mu_2(d, j, x) = \sigma_d - \mathbb{X}_{\text{wt}}\chi^{-j+2}(\sigma_d).$$

In particular, if d is a primitive root modulo p^2 (so that σ_d generates Γ), then $\mu_2(d, j, x) \doteq \gamma - \mathbb{X}_{\text{wt}}\chi^{-j+2}(\gamma)$.

(iii) *Suppose c and d are primitive roots modulo p^2 . For each $y \in \mathcal{X}^{\text{cl}}(E)$ with $w(y) \neq 2j$, the elements $\mu_1(c, j)$ and $\mu_2(d, j, y)$ of $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ are coprime.*

Proof Direct calculation. □

A.2 Dependence on c and d

In this subsection, we prove the first part of Proposition 6.15.

Proposition A.2 *Assume that $\mathcal{X}$ is sufficiently small.*

(i) *Suppose c, c', d, d', j are auxiliary integers so that (c, d, j) and (c', d', j) are as in Definition 6.1. Then*

$$\mu_0(c', d', j, x)_{c,d} \mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) = \mu_0(c, d, j, x)_{c',d'} \mathbb{BK}_N^{[\mathcal{X}]}(j, \xi).$$

(ii) *For c, d, j as above, the partially normalized Beilinson–Kato element*

$$\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) := \mu_0(c, d, j, x)^{-1}_{c,d} \mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) \otimes_{\Lambda_{\mathcal{X}}} \Lambda_{\mathcal{X}}[\mu_0^{-1}]$$

is independent of the choice of c and d . Here and elsewhere, μ_0 is a shorthand for $\mu_0(c, d, j)$ when c, d, j are understood.

Before we proceed with the proof of Proposition A.2, we record the following auxiliary lemma.

Lemma A.3 *Suppose R is an integral domain and M is a finitely generated torsion-free R -module. Let $\{P_i\}_{i \in I}$ be an infinite collection of prime ideals of R such that $\bigcap_{i \in I} P_i = \{0\}$. Then,*

$$\bigcap_{i \in I} P_i M = \{0\}.$$

Proof of Lemma A.3 Let r be the rank of M . Then M contains a free submodule $M_0 := \bigoplus_{k=1}^r R e_k$ of rank r , and M/M_0 is annihilated by multiplication by some $a \in R$. Let $m \in \bigcap_{i \in I} P_i M$ be any element. Then am belongs to $\bigcap_{i \in I} P_i M_0$. Since am can be written in a unique way in the form

$$am = \sum_{k=1}^r \alpha_k e_k, \quad \alpha_k \in R,$$

we see that $\alpha_k \in \bigcap_{i \in I} P_i = \{0\}$ for all k . $\square$

Proof of Proposition A.2 We will proceed in two steps.

(a) We consider $H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1))$ as a module over $\Lambda_{\mathcal{X}}(\Gamma_1)$. For any $x \in \mathcal{X}^{\text{cl}}(E)$, let $\mathfrak{m}_x \subset \Lambda_{\mathcal{X}}(\Gamma_1)$ denote the kernel of the evaluation-at- x map

$$\Lambda_{\mathcal{X}}(\Gamma_1) \longrightarrow \Lambda_E(\Gamma_1), \quad \sum_{i=0}^{\infty} a_i(X)(\gamma_1 - 1)^i \longmapsto \sum_{i=0}^{\infty} a_i(x)(\gamma_1 - 1)^i.$$

It is clear from its definition that $\mathfrak{m}_x$ is the principal ideal generated by $(X - x)$. Then $\bigcap_{x \in \mathcal{X}^{\text{cl}}(E)} \mathfrak{m}_x = 0$. Combining Lemmas 6.12 and A.3, we deduce for sufficiently small $\mathcal{X}$ that

$$\bigcap_{x \in \mathcal{X}^{\text{cl}}(E)} \mathfrak{m}_x H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) = \{0\}.$$

b) It follows from [22, §13.9] that

$$\begin{aligned} & \mu_0(c', d', j, x) {}_{c,d} \mathbb{B}_N^{[\mathcal{X}]}(j, \xi) - \mu_0(c, d, j, x) {}_{c',d'} \mathbb{B}_N^{[\mathcal{X}]}(j, \xi) \\ & \in \ker \left(H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) \rightarrow H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)) \right) \\ & = \mathfrak{m}_x H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) \end{aligned}$$

for all $x \in \mathcal{X}^{\text{cl}}$. Now the part (i) of the proposition follows from (a). The part (ii) is clear. $\square$

A.3 Integrality of partial normalizations

A.3.1. We will next analyze the regularity of the partially normalized Beilinson–Kato element $\mathbb{B}_N^{[\mathcal{X}]}(j, \xi)$. This amounts to an analysis of the divisibility of ${}_{c,d} \mathbb{B}_N^{[\mathcal{X}]}(j, \xi)$ by $\mu_0(c, d, j, x)$. In view of Proposition A.2, we may (and henceforth will) work with c and d which are both primitive roots mod p^2 , without any loss.

Proposition A.4 *Suppose c and d are primitive roots modulo p^2 . There exists an affinoid neighborhood $\mathcal{W}$ of k_0 and a unique element $\mathbb{B}\mathbb{K}_N^{[\mathcal{X}]}(j, \xi) \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1))$ so that*

$$c, d \mathbb{B}\mathbb{K}_N^{[\mathcal{X}]}(j, \xi) = \mu_0(d, j, x) \cdot \mathbb{B}\mathbb{K}_N^{[\mathcal{X}]}(j, \xi).$$

The uniqueness of $\mathbb{B}\mathbb{K}_N^{[\mathcal{X}]}(j, \xi)$ follows from Lemma 6.12 (on shrinking $\mathcal{X}$ as necessary). The existence will be proved in the §A.3.2 and §A.3.2 below.

A.3.2. We first prove that there exists an affinoid neighborhood $\mathcal{W}$ of k_0 such that

$$c, d \mathbb{B}\mathbb{K}_N^{[\mathcal{X}]}(j, \xi) \in \mu_1(c, j) \cdot H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) \quad (72)$$

for all $\xi \in \text{SL}_2(\mathbb{Z})$.

Considering the long exact sequence of Galois cohomology induced from

$$0 \longrightarrow V'_{\mathcal{X}}(-j-1) \xrightarrow{X} V'_{\mathcal{X}}(-j-1) \longrightarrow V'_f(-j-1) \longrightarrow 0$$

together with the vanishing of $H^0(\mathbb{Z}[1/S], V'_f(-j-1))$ (which follows from the fact that V_f is irreducible), we infer that the $\mathcal{O}_{\mathcal{X}}$ -module $H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(-j-1))$ has no X -torsion. On shrinking $\mathcal{X}$ as necessary, we may therefore ensure that it is torsion free, and therefore free (since $\mathcal{O}_{\mathcal{X}}$ is a PID) as a $\mathcal{O}_{\mathcal{X}}$ -module. As a matter of fact, on shrinking $\mathcal{X}$, we can ensure that the L -value $L(f_x^c, w(x) - j)$ is non-central for any $x \in \mathcal{X}^{\text{cl}}(E) \setminus \{x_0\}$. For all such x , it follows from [22, Theorem 14.5] and the isomorphism $V'_x(-j-1) \simeq V'_{f_x}(w(x) - j)$ that $H^1(\mathbb{Z}[1/S], V'_x(-j-1))$ is a 1-dimensional E -vector space. This in turn shows that $H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(-j-1))$ is a free $\mathcal{O}_{\mathcal{X}}$ -module of rank one.

The specialization map $\gamma \mapsto \chi^{j+2}(\gamma)$ induces an exact sequence

$$0 \longrightarrow V'_{\mathcal{X}}(1) \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^t \xrightarrow{\gamma - \chi^{j+2}(\gamma)} V'_{\mathcal{X}}(1) \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^t \rightarrow V'_{\mathcal{X}}(-j-1) \longrightarrow 0,$$

which gives rise to an exact sequence

$$H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) \xrightarrow{\gamma - \chi^{j+2}(\gamma)} H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) \longrightarrow H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(-j-1)).$$

For each $x \in \mathcal{X}^{\text{cl}}(E)$, let $\mathfrak{m}_x \subset \mathcal{O}_{\mathcal{X}}$ denote the maximal ideal $(X - x)$. We have a commutative diagram

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ & & & & \mathfrak{m}_x H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(-j-1)) \\ & & & & \downarrow \\ H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) & \xrightarrow{\gamma - \chi^{j+2}(\gamma)} & H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(1)) & \longrightarrow & H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(-j-1)) \\ \downarrow \text{sp}_x & & \downarrow \text{sp}_x & & \downarrow \text{sp}_x \\ H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)) & \xrightarrow{\gamma - \chi^{j+2}(\gamma)} & H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)) & \longrightarrow & H^1(\mathbb{Z}[1/S], V'_x(-j-1)), \end{array}$$

with exact rows and columns, where the vertical maps that are denoted by sp_x are induced by the specialization at x . Let us denote by

$$c, d \mathbb{B}\mathbb{K}_N^{[\mathcal{X}]}(j, \xi)|_{s=j+2} \in H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(-j-1))$$

the image of ${}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi)$ under the indicated cyclotomic specialization, and by

$${}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi, x) \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1))$$

its image under sp_x . Note that by Corollary 6.9, ${}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi, x)$ coincides with the classical Beilinson–Kato element ${}_{c,d}\text{BK}_{Np, \text{Iw}}(f_x^c, j, \xi)$. Now Kato's integrality results in [22, §13.12] (combined with the discussion in Remark 6.10 when f_x is p -old) show that

$${}_{c,d}\mathbb{BK}_N^{[\mathcal{X}, x]}(j, \xi) \in \mu(c, d, j, x) \cdot H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)) \subset \mu_1(c, j) H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1))$$

for all c, d as before, integers $j \in [0, k_0]$ and $\xi \in \text{SL}_2(\mathbb{Z})$. Since $\mu_1(c, j) \doteq \gamma - \chi^{j+2}(\gamma)$ by Lemma A.1, an easy diagram chase shows that

$${}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi)|_{s=j+2} \in \mathfrak{m}_x H^1(\mathbb{Z}[1/S], V'_x(-j-1)).$$

It follows from Lemma A.3 that ${}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi)|_{s=j+2} = 0$. Therefore

$${}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) \in (\gamma - \chi^{j+2}(\gamma)) \cdot H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)),$$

and (72) is proved.

A.3.3. Let $\mathfrak{X}_{1, \mathcal{X}}^{(j, \xi)} \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1))$ denote an element such that ${}_{c,d}\mathbb{BK}_N^{[\mathcal{X}]}(j, \xi) = \mu_1(c, j) \mathfrak{X}_{1, \mathcal{X}}^{(j, \xi)}$. We shall prove that there exists an affinoid neighborhood $\mathcal{W}$ of k_0 such that

$$\mathfrak{X}_{1, \mathcal{X}}^{(j, \xi)} \in \mu_2(d, j, x) \cdot H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)). \quad (73)$$

The proof of this part is very similar to the verification of (72) in §A.3.2, after obvious modifications. We consider the Galois representation $V'_{\mathcal{X}}(\mathbb{Z}_{\text{wt}}^{-1} \chi^{j-1})$, in place of $V'_x(-j-1)$. Note that the specialization of $V'_{\mathcal{X}}(\mathbb{Z}_{\text{wt}}^{-1} \chi^{j-1})$ at x is

$$V'_x(\chi^{-w(x)+j-1}) \simeq V_{f_x^c}(j).$$

Employing the same argument as in the first paragraph of our proof of (72), one observes that the $\mathcal{O}_{\mathcal{X}}$ -module $H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(\mathbb{Z}_{\text{wt}}^{-1} \chi^{j-1}))$ is free of rank one, on shrinking $\mathcal{X}$ if necessary. The cyclotomic specialization $\gamma \mapsto \mathbb{Z}_{\text{wt}} \chi(\gamma)^{2-j}$ gives an exact sequence

$$0 \longrightarrow V'_x(1) \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^t \xrightarrow{\gamma - \mathbb{Z}_{\text{wt}} \chi^{-j+2}(\gamma)} V'_x(1) \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^t \longrightarrow V'_x(\mathbb{Z}_{\text{wt}}^{-1} \chi^{j-1}) \longrightarrow 0. \quad (74)$$

Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ & & & & \mathfrak{m}_x H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(\mathbb{Z}_{\text{wt}}^{-1} \chi^{j-1})) \\ & & & & \downarrow \\ H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)) & \xrightarrow{\gamma - \mathbb{Z}_{\text{wt}} \chi^{-j+2}(\gamma)} & H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)) & \longrightarrow & H^1(\mathbb{Z}[1/S], V'_{\mathcal{X}}(\mathbb{Z}_{\text{wt}}^{-1} \chi^{j-1})) \\ \downarrow \text{sp}_x & & \downarrow \text{sp}_x & & \downarrow \text{sp}_x \\ H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)) & \xrightarrow{\gamma - \mathbb{Z}_{\text{wt}} \chi^{-j+2}(\gamma)} & H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)) & \longrightarrow & H^1(\mathbb{Z}[1/S], V'_x(\mathbb{Z}_{\text{wt}}^{-1} \chi^{j-1})). \end{array}$$

Let $\mathfrak{X}_{1,\mathcal{X}}^{(j,\xi)}(x)$ denote the specialization of $\mathfrak{X}_{1,\mathcal{X}}^{(j,\xi)}$ at $x \in \mathcal{X}^{\text{cl}}(E)$. Since

$$\mu_1(c, j) \mathfrak{X}_{1,\mathcal{X}}^{(j,\xi)}(x) = {}_{c,d}\mathbb{B}\mathbb{K}_N^{[\mathcal{X},x]}(j, \xi) \in H^1(\mathbb{Z}[1/S], V'_x(1))$$

coincides with the classical Beilinson–Kato element

$${}_{c,d}\mathbf{BK}_{Np,\text{Iw}}(f_y^c, j, \xi) \in H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)). \quad (75)$$

Kato's integrality results in [22, §13.12] (combined with the discussion in Remark 6.10 when f_x is p -old) show for such x that

$${}_{c,d}\mathbb{B}\mathbb{K}_N^{[\mathcal{X}]}(j, \xi, x) \in \mu_0(c, d, j, x) \cdot H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)),$$

for all c, d as before, integers $j \in [0, k_0]$ and $\xi \in \text{SL}_2(\mathbb{Z})$. Since

$$\mu_0(c, d, j, x) = \mu_1(c, j)\mu_2(d, j, x),$$

and $H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1))$ is torsion-free (and even free) $\Lambda[1/p]$ -module, we deduce that

$$\mathfrak{X}_{1,\mathcal{X}}^{(j,\xi)}(x) \in \mu_2(d, j, x) \cdot H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_x(1)).$$

Let $\mathfrak{X}_{1,\mathcal{X}}^{(j,\xi)}|_{s=w-j+2} \in H^1(\mathbb{Z}[1/S], V'_\mathcal{X}(\mathbb{X}_{\text{wt}}^{-1}\chi^{j-1}))$ denote the cyclotomic specialization of $\mathfrak{X}_{1,\mathcal{X}}^{(j,\xi)}$. Recall that $\mu_2(d, j, x) = \gamma - \mathbb{X}_{\text{wt}}\chi^{-j+2}(\gamma)$. Now a diagram chase similar to the one employed in §A.3.2 shows that

$$\mathfrak{X}_{1,\mathcal{X}}^{(j,\xi)}|_{s=w-j+2} \in \mathfrak{m}_x H^1(\mathbb{Z}[1/S], V'_\mathcal{X}(\mathbb{X}_{\text{wt}}^{-1}\chi^{j-1})).$$

Since the intersection of any infinite family of ideals $\mathfrak{m}_x$ is zero, we deduce using Lemma A.3 that

$$\mathfrak{X}_{1,\mathcal{X}}^{(j,\xi)}|_{s=w-j+2} \in \bigcap_{x \in \mathcal{X}^{\text{cl}}(E)} \mathfrak{m}_x H^1(\mathbb{Z}[1/S], V'_\mathcal{X}(\mathbb{X}_{\text{wt}}^{-1}\chi^{j-1})) = \{0\}$$

This shows that

$$\mathfrak{X}_{1,\mathcal{X}}^{(j,\xi)} \in (\gamma - \mathbb{X}_{\text{wt}}\chi^{-j+2}(\gamma))H_{\text{Iw}}^1(\mathbb{Z}[1/S], V'_\mathcal{X}(1)),$$

and (73) is proved. This also completes the proof of Proposition A.4. $\square$

Remark A.5 One reason why we need the rather tortuous analysis in Proposition A.4 is that we don't know if the following statement holds true: In the notation of [22, Theorem 6.6], we have

$$\delta(f, j, \xi)^\pm \neq 0$$

for some $j \neq \frac{k_0}{2} \pm 1$, where $\pm$ is the sign of $(-1)^{\frac{k_0}{2}}$.

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