A dominated convergence theorem for Eisenstein series

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Abstract

Based on the new approach to modular forms presented in [6] that uses rational functions, we prove a dominated convergence theorem for certain modular forms in the Eisenstein space. It states that certain rearrangements of the Fourier series will converge very fast near the cusp $\tau = 0$. As an application, we consider L -functions associated to products of Eisenstein series and present natural generalized Dirichlet series representations that converge in an expanded half plane.

Résumé

En nous appuyant sur la nouvelle approche des formes modulaires présentée dans [6] qui utilise des fonctions rationnelles, nous prouvons un théorème de convergence dominé pour certaines formes modulaires dans l'espace d'Eisenstein. Il énonce que certains réarrangements de la série de Fourier convergeront très rapidement vers le point de rebroussement $\tau = 0$. En tant qu'application, nous considérons les fonctions L associées aux produits des séries d'Eisenstein et présentons des représentations de séries de Dirichlet généralisées et naturelles qui convergent dans un demi-plan étendu.

Keywords L -functions · Eisenstein series · Mellin transform · Partial summation

Mathematics Subject Classification Primary 11F11; Secondary 11M41

Introduction

In this paper we prove a dominated convergence theorem for Eisenstein series. Roughly speaking, it states an upper bound for specific partial sums of Eisenstein series at purely imaginary arguments near the cusp $\tau = 0$. This can be applied to questions involving L -functions assigned to products of Eisenstein series, since we obtain better control of the corresponding Mellin integral. One of the main ingredients we use is an alternative elementary approach to modular forms [6]. It relies on a class of very simple functions which we will call *weak functions*. A weak function ω is a 1-periodic meromorphic function in the entire plane, which has the following properties:

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- (i) All poles of ω are simple and lie in $\mathbb{Q}$.
- (ii) The function ω tends to 0 rapidly as the absolute value of the imaginary part increases, so

$$\omega(x + iy) = O(|y|^{-M})$$

for all $M > 0$ as $|y| \rightarrow \infty$.

By Liouville's theorem one quickly sees that each weak ω is essentially just a rational function $R \in \mathbb{C}(X)$ with (only simple) poles only in roots of unity, such that $R(0) = R(\infty) = 0$. Here we put $\omega(z) := R(e(z))$, where $e(z) := e^{2\pi iz}$. One defines W_N to be the space of weak functions with the property, that $\omega(z/N)$ only has poles in $\mathbb{Z}$. We associate to ω a periodic divisor function $\beta_\omega(x) := -2\pi i \operatorname{res}_{z=x} \omega(z)$. Now one can show the following construction theorem for modular forms for the congruence subgroup

$$\Gamma(N_1 N_2) \subset \Gamma_1(N_1, N_2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_1, N_2) \mid a \equiv d \equiv 1 \pmod{N_1 N_2} \right\}.$$

Theorem 0.1 (cf. Theorem 4.7 in [6]) *Let $k \geq 3$ and $N_1, N_2 > 1$ be integers. There is a homomorphism*

$$\begin{aligned} W_{N_1} \otimes W_{N_2} &\longrightarrow M_k(\Gamma_1(N_1, N_2)) \\ \omega \otimes \eta &\longmapsto \vartheta_k(\omega \otimes \eta; \tau) := \sum_{x \in \mathbb{Q}^\times} x^{k-1} \beta_\eta(x) \omega(x\tau). \end{aligned}$$

In the case that $k = 1$ and $k = 2$ the map stays well-defined under the restriction that the function $z \mapsto z^{k-1} \eta(z) \omega(z\tau)$ is removable in $z = 0$.

The main tools for the proof are Weil's converse theorem and the following observation.

Theorem 0.2 (cf. Theorem 3.5 in [6]) *We define the involution $\omega \mapsto \hat{\omega}$ by $\hat{\omega}(z) := \omega(-z)$. Let $k \in \mathbb{Z}$ be an integer. For all weak ω and η we have the following transformation property.*

$$\vartheta_k(\omega \otimes \eta; \tau)|_k S = \vartheta_k(\eta \otimes -\hat{\omega}; \tau) + g_{\omega, \eta}(\tau)|_k S,$$

where $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $g_{\omega, \eta}$ is a rational function which can be evaluated explicitly by

$$g_{\omega, \eta}(\tau) = 2\pi i \operatorname{res}_{z=0} \left(z^{k-1} \eta(z) \omega(z\tau) \right).$$

In this paper, we continue the study of this new perspective to modular forms and apply it to Dirichlet series. We first want to investigate the space $\vartheta_k(W_{N_1} \otimes W_{N_2})$ and it will turn out, that it is generated by Eisenstein series.

Theorem 0.3 (cf. Theorem 1.1) *Let $k \geq 3$. The image space $\vartheta_k(W_{N_1} \otimes W_{N_2})$ is generated by the elements $E_k(\chi, \psi; \frac{N_1 d_2}{N_2 d_1} \tau)$ where χ and ψ run over all non-trivial characters modulo $d_1 | N_1$ and $d_2 | N_2$, respectively, such that $\chi(-1)\psi(-1) = (-1)^k$.*

The cases $k = 1$ and $k = 2$ can be treated similarly. We want to apply the series representations of ϑ_k in terms of rational functions to Dirichlet series. To every modular form $f(\tau) = \sum_{n \geq 0} a(n) q^{n/N}$ of weight k for some congruence subgroup $\Gamma \subset \operatorname{SL}_2(\mathbb{Z})$ we can associate an L -function $L(f, s)$ given by

$$L(f, s) = \sum_{n=1}^{\infty} a(n) n^{-s}$$

One can show that this function converges absolutely on the half plane $\operatorname{Re}(s) > k$, has meromorphic continuation to the entire plane and satisfies a certain functional equation. The complete L -function $\Lambda(f, s) = (2\pi/N)^{-s} \Gamma(s) L(f, s)$ can be written as an integral

$$\Lambda(f, s) = \int_0^\infty (f(ix) - a(0)) x^{s-1} dx$$

In the case that f is a cuspidal Hecke eigenform its L -function is entire, has an Euler product expansion and encodes deep arithmetic information.

We give a proof for a dominated convergence theorem for Eisenstein series arising from rational functions. In order to formulate it, we need the concept of the height of a N -periodic function β . The height of such a β is defined to be the largest integer d such that for all integers $\alpha \geq 0$:

$$\sum_{n=1}^{NT} \beta(n) n^\alpha = \sum_{u=0}^{\alpha-d} \gamma_{\alpha, \beta}(u) T^u = O(T^{\alpha-d}), \quad T \rightarrow \infty.$$

Theorem 0.4 (cf. Theorem 2.14) *Let $\omega \otimes \eta \in W_{N_1} \otimes W_{N_2}$ be a pair of weak functions such that ω is removable in $z = 0$ and $\kappa_{N_2} \beta_\eta$ has height d . Then for all $\alpha \in \mathbb{N}_0$ there is a constant $C_{\beta, \omega, \alpha} > 0$ such that uniformly for all $T \in \mathbb{N}$ and $y \in [0, 1]$*

$$\left| \sum_{n=1}^{N_2 T} n^\alpha \beta_\eta(n/N_2) \omega(niy) \right| \leq C_{\beta, \omega, \alpha} y^{d-\alpha}.$$

An application of this theorem is a new, in some sense more natural, representation of L -functions associated to products of Eisenstein series in terms of a generalized Dirichlet series. Modifying the sum a bit leads to convergence in a much wider region. In particular, we have the following theorem.

Theorem 0.5 (cf. Corollary 3.18) *Let $\ell \in \mathbb{N}$, $\mathbf{k} = (k_1, \dots, k_\ell) \in \mathbb{N}^\ell$ a vector of positive integers and $\chi_j, \psi_j, j = 1, \dots, \ell$, be non-principal, primitive characters modulo M and N , respectively, such that $\chi_j(-1)\psi_j(-1) = (-1)^{k_j}$. Then, if we put $|\mathbf{k}| = k_1 + \dots + k_\ell$, we have for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > |\mathbf{k}| - \ell - \frac{1}{2} \sum_{j=1}^\ell (\psi_j(-1) + 1)$*

$$L\left(\prod_{j=1}^\ell E_{k_j}(\chi_j, \psi_j; \tau), s\right) = \left(-\frac{2\pi i}{N}\right)^{|\mathbf{k}|} \prod_{j=1}^\ell \frac{\mathcal{G}(\psi_j)}{(k_j - 1)!} \sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell} \Pi_{\mathbf{k}}(\mathbf{u}) \overline{\psi}(\mathbf{u}) \chi(\mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s},$$

where $\Pi_{\mathbf{k}}(\mathbf{u}) = u_1^{k_1} \dots u_\ell^{k_\ell}$, $\psi(\mathbf{u}) = \psi_1(u_1) \dots \psi_\ell(u_\ell)$ and $\chi(\mathbf{v}) = \chi_1(v_1) \dots \chi_\ell(v_\ell)$. Here, $\mathcal{G}(\psi_j)$ is the usual Gauss sum of ψ_j . For convergence in the extended half plane the sum has to be modified slightly, this is explained below.

Note that this representation of the L -function of the considered product is more natural since it is a direct generalization of the formula in the case $\ell = 1$, where the series directly splits into a product of two Dirichlet L -functions. An important question, which is still unsolved in the very general case, is that which modular forms can be written as sums of products of Eisenstein series. But there is a lot of progress in this field. Dickson and Neururer have shown in [5], that, if $k \geq 4$, $N = p^a q^b N'$ where p^a, q^b are powers of primes and N' is square free, the space $M_k(\Gamma_0(N))$ is generated by $E_k(\Gamma_0(N), \chi_{0,N})$ and a subspace containing products of two Eisenstein series. A similar result for $M_k(\Gamma_0(p))$ and $k \geq 4$,

where p is prime, is due to Imamoglu and Kohnen [7] (for $p = 2$) and Kohnen and Martin [8] for $p > 2$. Recently, Raum and Xia proved in [10] that essentially all modular forms of weight 2 can be represented by products of Eisenstein series. For a correspondence between values of L -functions for products of pairs of different Eisenstein series see [3].

The paper is organized as follows. In the first section we identify generators for the space of modular forms that arise from rational functions. In the second section we prove a Dominated convergence theorem for Eisenstein series, which provides an upper bound for several partial sums of the series involving weak functions for modular forms near the cusp $\tau = 0$. In the last section we apply this theorem to L -functions associated to products of Eisenstein series.

Notation. We use the introduced notations W_N for the vector space of weak functions of level $d|N$, and $W_N^\pm$ for the odd and even function part.

As usually, $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout ℓ is a positive integer. We briefly define $\mathbf{k} = (k_1, \dots, k_\ell) \in \mathbb{N}^\ell$ to be a vector of positive integers. We write $|\mathbf{k}| = k_1 + \dots + k_\ell$.

For real valued vectors $\mathbf{u} = (u_1, \dots, u_\ell) \in \mathbb{R}^\ell$ we briefly write $\max(\mathbf{u}) := \max\{u_1, \dots, u_\ell\}$.

We sometimes use the notation $\operatorname{sgn}(f) = \pm 1$ to indicate that f is an even or odd function, respectively.

For any set L we define $L^{\mathbb{C}_0}$ to be the space of all functions $f : L \rightarrow \mathbb{C}$, that are zero everywhere (except finitely many $x \in L$). The subspace $L_0^{\mathbb{C}_0} \subset L^{\mathbb{C}_0}$ is given by all f satisfying $\sum_{x \in L} f(x) = 0$. For positive integers N we abbreviate $\mathbb{F}_N := \mathbb{Z}/N\mathbb{Z}$ and $\mathbb{F}_{\frac{1}{N}} := \mathbb{Z}[\frac{1}{N}]/\mathbb{Z}$. Especially when going over to Fourier series it will be useful to identify functions in $\mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_0}$ with those in $\mathbb{F}_N^{\mathbb{C}_0}$ via the obvious map

$$\begin{aligned} \kappa_N : \mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_0} &\xrightarrow{\sim} \mathbb{F}_N^{\mathbb{C}_0} \\ (\kappa_N f)(x) &:= f\left(\frac{x}{N}\right). \end{aligned}$$

We will identify functions $f \in \mathbb{F}_N^{\mathbb{C}_0}$ with N -periodic functions $f : \mathbb{Z} \rightarrow \mathbb{C}$. For integers M we will set $f[M](x) := f(Mx)$ when $f : \mathbb{Z} \rightarrow \mathbb{C}$.

For any Dirichlet character ψ modulo N we define the Gauss sum $\mathcal{G}(\psi) := \sum_{n=0}^{N-1} \psi(n)e^{2\pi i n/N}$. For the generalized Gauss sum it will be more convenient to use the more general notion of a discrete Fourier transform

$$\begin{aligned} \mathcal{F}_N : \mathbb{F}_N^{\mathbb{C}_0} &\xrightarrow{\sim} \mathbb{F}_N^{\mathbb{C}_0} \\ (\mathcal{F}_N f)(j) &:= \sum_{n=0}^{N-1} f(n)e^{-2\pi i j n/N}. \end{aligned}$$

Note that we have an inverse transformation

$$(\mathcal{F}_N^{-1} g)(j) := \frac{1}{N} \sum_{n=0}^{N-1} g(n)e^{2\pi i j n/N}.$$

We use the same notation for functions $f \in \mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_0}$ and have $\kappa_N \mathcal{F}_N f = \mathcal{F}_N \kappa_N f$. For $d|N$ we also use the trivial injection

$$\iota_N^d : (\mathbb{F}_d)^{\mathbb{C}_0} \longrightarrow (\mathbb{F}_N)^{\mathbb{C}_0}$$

$$(\iota_N^d f)(x) := \begin{cases} f(\frac{xd}{N}), & x \equiv 0 \pmod{\frac{N}{d}} \\ 0, & \text{else} \end{cases}$$

for purposes of notation. Note that if $f \in \mathbb{F}_d^{\mathbb{C}_0}$ and $d|N$ we have $\mathcal{F}_N \iota_N^d f = \mathcal{F}_d f$, where the left hand side is d -periodic and can be seen as a function of $\mathbb{F}_d$.

For the complex variable $z = x + iy$ we write $e(z) := e^{2\pi iz}$ and for the complex variable τ we define $q := e^{2\pi i\tau}$. We also use the notation $\zeta_M := e^{\frac{2\pi i}{M}}$ for roots of unity.

We denote by $\mathfrak{C}_N$ the group of all characters modulo N . Also we write $\overline{\mathfrak{C}_N}$ for the set of all characters modulo d , where d divides N . We write $\chi_{0,d}$ for the principal character modulo d . In particular, $\chi_{0,1}$ denotes the *trivial* character.

1 The space of weak modular forms

For Dirichlet characters χ and ψ modulo positive integers M and N , respectively, and some integer $k \geq 3$ one defines the corresponding Eisenstein series for $\tau \in \mathbb{H}$ (= upper half plane) via

$$E_k(\chi, \psi; \tau) := \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \chi(m)\psi(n)(m\tau + n)^{-k}. \quad (1.1)$$

This series converges absolutely and uniformly on compact subsets of the upper half plane and defines a holomorphic function in that region. One can show that (1.1) leads to a non-zero function if and only if $\chi(-1)\psi(-1) = (-1)^k$ and that the E_k are modular forms of weight k for the congruence subgroups

$$\Gamma_0(M, N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid b \equiv 0 \pmod{M}, c \equiv 0 \pmod{N} \right\}$$

with Nebentypus character $\chi\overline{\psi}$ of $\Gamma_0(M, N)$. The cases $k = 1, 2$ are treated differently, see also [9] on p. 274 ff. or [4].

Every Eisenstein series admits a Fourier series. The coefficients are well-known and given by

$$2L(\psi, k)\chi(0) + \frac{2(-2\pi i)^k}{N^k(k-1)!} \sum_{m=1}^{\infty} \left(\sum_{d|m} d^{k-1} (\mathcal{F}_N \psi)(-d) \chi\left(\frac{m}{d}\right) \right) q^{m/N}, \quad (1.2)$$

where as usual $q := e^{2\pi i\tau}$ and $L(\psi, s)$ is the Dirichlet L -function. Note that in the case that ψ is primitive one has $(\mathcal{F}_N \psi)(a) = \overline{\psi(a)}(\mathcal{F}_N \psi)(1)$ and one obtains the simpler expression $\sum_{d|m} d^{k-1} \overline{\psi}(d) \chi(n/d)$ for the coefficients up to a constant.

It is clear that every $\vartheta_k(\omega \otimes \eta; \tau)$ admits a Fourier expansion. Since we only focus on the non-trivial cases we assume $\omega \otimes \eta \in (W_M \otimes W_N)^\pm$ if $(-1)^k = \pm 1$. It is given by

$$\vartheta_k(\omega \otimes \eta; \tau) = 2N^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left(d^{k-1} (\kappa_N \beta_\eta)(d) (\mathcal{F}_M \kappa_M \beta_\omega)\left(\frac{m}{d}\right) \right) q^{m/N}. \quad (1.3)$$

According to (1.2) we conclude for non-principal characters

$$E_k(\chi, \psi; \tau) = \frac{\psi(-1)(-2\pi i)^k}{N(k-1)!} \vartheta_k(\omega_{\mathcal{F}_M^{-1}(\chi)} \otimes \omega_{\mathcal{F}_N(\psi)}; \tau). \quad (1.4)$$

In particular, if χ and ψ are primitive and hence conjugate up to a constant under the Fourier transform, this simplifies to

$$E_k(\chi, \psi; \tau) = \frac{\chi(-1)(-2\pi i)^k \mathcal{G}(\psi)}{N(k-1)! \mathcal{G}(\overline{\chi})} \vartheta_k(\omega_{\overline{\chi}} \otimes \omega_{\overline{\psi}}; \tau). \quad (1.5)$$

Already here the connection between Eisenstein series and weak functions is intuitively clear.

In this section we want to find generators for the space $\vartheta_k(W_{N_1} \otimes W_{N_2})$. We call their elements weak modular forms. In other words, the vector space $V_k(\Gamma_1(N_1, N_2))$ of all weak modular forms is the image of the linear map

$$W_{N_1} \otimes W_{N_2} \longrightarrow M_k(\Gamma_1(N_1, N_2)).$$

Let $(\mathbb{F}_N)_{*,0}^{\mathbb{C}_0} \subset \mathbb{F}_N^{\mathbb{C}_0}$ be the subspace of all functions with $f(0) = 0$. It is an easy exercise to verify that the discrete Fourier transform defines an isomorphism

$$\mathcal{F}_N : (\mathbb{F}_N)_{*,0}^{\mathbb{C}_0} \xrightarrow{\sim} (\mathbb{F}_N)_0^{\mathbb{C}_0}.$$

With this we conclude that $(\omega_{\mathcal{F}_{N_1}^{-1}\chi} \otimes \omega_{\mathcal{F}_{N_2}\psi})_{(\chi,\psi) \in \overline{\mathbb{C}_{N_1}} \setminus \{\chi_{0,1}\} \times \overline{\mathbb{C}_{N_2}} \setminus \{\chi_{0,1}\}}$ is a basis for $W_{N_1} \otimes W_{N_2}$. The next theorem provides generators for the space $V_k(\Gamma_1(N_1, N_2))$.

Theorem 1.1 *Let $k \geq 3$. The space $V_k(\Gamma_1(N_1, N_2))$ is generated by the elements $E_k(\chi, \psi; \frac{N_1 d_2}{N_2 d_1} \tau)$ where χ and ψ run over all non-trivial characters modulo $d_1|N_1$ and $d_2|N_2$, respectively, such that $\chi(-1)\psi(-1) = (-1)^k$.*

Proof It is clear that the Fourier transform preserves the subspaces of odd and even functions. Hence, for characters $\chi(-1)\psi(-1) = (-1)^k$, we have the Fourier expansion

$$\begin{aligned} \vartheta_k(\omega_{\mathcal{F}_{N_1}^{-1}\chi} \otimes \omega_{\mathcal{F}_{N_2}\psi}; \tau) &= 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left(d^{k-1} (\mathcal{F}_{N_2} \iota_{N_2}^{d_2} \psi)(d) (\mathcal{F}_{N_1} \mathcal{F}_{N_1}^{-1} \iota_{N_1}^{d_1} \chi) \left(\frac{m}{d} \right) \right) q^{m/N_2} \\ &= 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left(d^{k-1} (\mathcal{F}_{d_2} \iota_{N_2}^{d_2} \psi \left[\frac{N_2}{d_2} \right])(d) \iota_{N_1}^{d_1} \chi \left(\frac{m}{d} \right) \right) q^{m/N_2} \\ &= 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left(d^{k-1} (\mathcal{F}_{d_2} \psi)(d) \iota_{N_1}^{d_1} \chi \left(\frac{m N_1}{d d_1} \right) \right) q^{m N_1 / N_2 d_1} \\ &= 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left(d^{k-1} (\mathcal{F}_{d_2} \psi)(d) \chi \left(\frac{m}{d} \right) \right) q^{m(N_1 d_2 / N_2 d_1) / d_2}. \end{aligned}$$

This proves the theorem. $\square$

For our investigations we are especially interested in a subspace of V_k which we will denote by U_k and which contains all weak modular forms which arise from weak functions that are removable in $z = 0$. In the following we shall give generators for U_k . Let $H_{N_i} \subset W_{N_i}$ be the subspace of weak functions that are removable in $z = 0$. Then we have

$$W_{N_i} = \mathbb{C} \omega_{\mathcal{F}_{N_i}^{\pm 1} \chi_{0, N_i}} \oplus H_{N_i}.$$

In other words, the space H_N is given by weak elements $\omega(z)$ such that $\beta_{\omega}(0) = 0$. On the periodic function side, we define the subspace of these coefficients by $(\mathbb{F}_{\frac{1}{N_i}})_{0,0}^{\mathbb{C}_0}$. Note that the Fourier transform $\mathcal{F}_{N_i}$ defines an automorphism on the subspace $(\mathbb{F}_{N_i})_{0,0}^{\mathbb{C}_0}$. So firstly,

consider the basis $(\omega_{\mathcal{F}_{N_1}^{-1}\chi} \otimes \omega_{\mathcal{F}_{N_2}\psi})_{\chi,\psi}$ of $H_{N_1} \otimes H_{N_2}$, where χ and ψ are either non-principal characters modulo $d_1|N_1$ and $d_2|N_2$ or functions $\frac{\varphi(N_i)}{\varphi(d_i)} \iota_{N_i}^{d_i} \chi_{0,d_i} - \chi_{0,N_i}$ for $i = 1, 2$.

Theorem 1.2 *Let $k \geq 1$. The space $U_k = \vartheta_k(H_{N_1} \otimes H_{N_2})$ is generated by the elements $E_k(\chi, \psi; \frac{N_1 d_2}{N_2 d_1} \tau)$ and the linear combinations*

$$\begin{aligned} & \frac{\varphi(N_1)}{\varphi(d_1)} E_k(\chi_{0,d_1}, \psi; \frac{N_1 d_2}{N_2 d_1} \tau) - E_k(\chi_{0,N_1}, \psi; \frac{d_2}{N_2} \tau), \\ & \frac{\varphi(N_2)}{\varphi(d_2)} E_k(\chi, \chi_{0,d_2}; \frac{N_1 d_2}{N_2 d_1} \tau) - \left(\frac{N_2}{d_2}\right)^k E_k(\chi, \chi_{0,N_2}; \frac{N_1}{d_1} \tau), \end{aligned}$$

and

$$\begin{aligned} & \frac{\varphi(N_1)}{\varphi(d_1)} \frac{\varphi(N_2)}{\varphi(d_2)} E_k(\chi_{0,d_1}, \chi_{0,d_2}; \frac{N_1 d_2}{N_2 d_1} \tau) - \frac{\varphi(N_1)}{\varphi(d_1)} \left(\frac{N_2}{d_2}\right)^k E_k(\chi_{0,d_1}, \chi_{0,N_2}; \frac{N_1}{d_1} \tau) \\ & - \frac{\varphi(N_2)}{\varphi(d_2)} E_k(\chi_{0,N_1}, \chi_{0,d_2}; \frac{d_2}{N_2} \tau) + \left(\frac{N_2}{d_2}\right)^k E_k(\chi_{0,N_1}, \chi_{0,N_2}; \tau), \end{aligned}$$

where $1 < d_i < N_i$ and χ, ψ are non-principal characters modulo d_1 and d_2 , respectively, such that $\text{sgn}(\chi\psi) = (-1)^k$.

Proof Since all considered weak functions are removable in $z = 0$, we can apply the theorem to all positive weights $k = 1, 2, \dots$. The proof works similarly to the one of Theorem 1.1 and we omit it. $\square$

Theorem 1.3 *We have the following.*

- (i) *The space of weak modular forms of weight $k = 1$ is given by $V_1(\Gamma_1(N_1, N_2)) = \vartheta_1(H_{N_1} \otimes H_{N_2})$. In particular, it is generated by the elements given in Theorem 1.2 for $k = 1$.*
- (ii) *The space of weak modular forms of weight $k = 2$ is given by $V_2(\Gamma_2(N_1, N_2)) = \vartheta_2(H_{N_1} \otimes H_{N_2} \oplus \mathbb{C}\omega_{\mathcal{F}_{N_1}^{-1}\chi_{0,N_1}} \otimes H_{N_2} \oplus H_{N_1} \otimes \mathbb{C}\omega_{\mathcal{F}_{N_2}\chi_{0,N_2}})$. In particular, it is generated by the elements in Theorem 1.2 for $k = 2$ and $E_2(\chi_{0,N_1}, \psi; \frac{d_2}{N_2} \tau), E_2(\chi, \chi_{0,N_2}; \frac{N_1}{d_1} \tau)$, where χ and ψ are non-principal characters modulo $d_1|N_1$ and $d_2|N_2$, respectively.*

In the last section we would like to investigate L -functions of products of weak functions. To formalize this, we give the following final definition.

Definition 1.4 Let $\mathbf{k} = (k_1, \dots, k_\ell)$ be a vector of weights. We then define $V_{\mathbf{k}}(\Gamma_1(N_1, N_2))$ to be the vector space of all modular forms that can be written as a sum $\sum_j c_j f_{1,j} \cdots f_{\ell,j}$, where each $f_{r,j}$ is an element of $V_{k_r}(\Gamma_1(N_1, N_2))$. Analogously, we define the subspace $U_{\mathbf{k}}(\Gamma_1(N_1, N_2)) \subset V_{\mathbf{k}}(N_1, N_2)$ by demanding $f_{r,j} \in U_{k_r}(\Gamma_1(N_1, N_2))$. We will call the modular forms in $U_{\mathbf{k}}(\Gamma_1(N_1, N_2))$ higher weak modular forms.

2 A dominated convergence theorem

In this section we provide a Dominated convergence theorem, which will be applied to L -series associated to products of Eisenstein series in the following section. The idea is to

investigate finite sums of the form

$$\sum_{n=1}^T n^{\alpha} \beta(n) \omega(n\tau) \quad (2.1)$$

on the upper half plane in detail, where $\alpha \geq 0$ is an integer, β is some N -periodic function ($N \in \mathbb{N}_{>1}$) and $\omega(z)$ is some weak function of level M with a removable singularity in $z = 0$. By Theorems 1.1, 1.2 and 1.3, expression (2.1) will converge to a linear combination of Eisenstein series as T tends to infinity, if $\beta = \beta_{\eta}$ comes from a weak function. The purpose of the Dominated convergence theorem is now to give a condition providing a non-trivial upper bound for the sum (2.1). In general, there will be no non-trivial “small” upper bound of (2.1) in terms of T , τ and α . However, when replacing T by NT and τ by iy , where $1 \geq y > 0$, it is possible, but quite technical, to give a “small” uniform upper-bound in the sense that it is independent of the choice of T . This upper bound is of the form Cy^w with some integer w . This is summarized in Theorem 2.14.

Before going into the proofs, we sketch the idea why dominated convergence of Eisenstein series is useful. When considering L -functions of modular forms (vanishing in the cusps $\tau \in \{0, i\infty\}$), we first look at the Mellin transform

$$\int_0^{\infty} f(iy) y^{s-1} dy = \int_0^{\infty} \sum_{n=1}^{\infty} a(n) e^{-2\pi \frac{n}{N} y} y^{s-1} dy.$$

While convergence of integral and sum is no problem on the interval $[1, \infty]$, the situation looks different for $(0, 1]$. A priori, we will only be allowed to switch integral and sum in the obvious region of absolute convergence. In this “trivial region” it is well-known that we end up with the ordinary Dirichlet series for the L -function. But if we can rearrange the Fourier series to a series of Lambert type and give “small” upper bounds for the partial sums (2.1), we may use Lebesgue’s dominated convergence theorem to switch integral and sum also in non-trivial regions. As a result, we obtain a generalized form of Dirichlet series that also converges in a wider region to $L(f; s)$. All of this will be explained in Sect. 3.

We will start this section with a classical result.

Theorem 2.1 (Faulhaber’s formula) *We have for all $\alpha \in \mathbb{N}_0$ and $T \in \mathbb{N}$:*

$$\sum_{j=1}^T j^{\alpha} = \sum_{k=0}^{\alpha} \frac{B_k}{k!} \frac{\alpha!}{(\alpha - k + 1)!} T^{\alpha - k + 1}.$$

Here, the B_k denote the Bernoulli numbers.

It is a trivial but very important observation for us that the left sum defines a unique polynomial in T by interpolation, which is given on the right hand side. We will not prove Theorem 2.1. It can be verified, for example, by using Euler-MacLaurin summation. For more details on this topic, the reader is advised to consult [2] on p. 21–31.

Definition 2.2 Let N be a positive integer and $\beta : \mathbb{Z} \rightarrow \mathbb{C}$ a function. We say that β has height d (with respect to N), if for all $\alpha \in \mathbb{N}_0$ and $T \in \mathbb{N}$:

$$\sum_{j=1}^{NT} \beta(j) j^{\alpha} = \sum_{u=0}^{\alpha-d} \gamma_{\alpha, \beta}(u) T^u = O(T^{\alpha-d}), \quad T \rightarrow \infty.$$

Here, the complex numbers $\gamma_{\alpha,\beta}(u)$ only depend on α, β and u . The height of the zero function is always defined to be ∞ . We denote by $[N, d]$ the vector space of functions with height (with respect to N) at least d .

Like in Theorem 2.1, the key property of functions in Definition 2.2 is that the left side defines a polynomial. We easily see that the constant sequence $\beta(j) = 1$ and more generally, $\beta(j) = j^d$ will have heights -1 and $-d - 1$, respectively, where $d \geq 0$ is some integer. But while here the negative height causes an increase in the growth of the considered sums, we are rather interested in the opposite phenomenon of a non-negative height. In this case we obtain a decrease in the growth. Periodic functions with this feature play the key role when looking for “small” upper bounds of partial sums (2.1). Of course, not all functions β do have a height.

Remark 2.3 If $d_1 \leq d_2$ we have the natural embedding

$$[N, d_2] \longrightarrow [N, d_1].$$

We are only interested in periodic functions. The next proposition guarantees that they have a height.

Proposition 2.4 We have $\mathbb{F}_N^{\mathbb{C}_0} \subset [N, -1]$.

Proof Since β is periodic, we can rewrite the sum over $\beta(j)j^\alpha$ as

$$\sum_{j=1}^{NT} \beta(j)j^\alpha = \sum_{c=1}^N \beta(c) \sum_{j=0}^{T-1} (Nj + c)^\alpha.$$

It is clear by Theorem 2.1 that for any c the expressions

$$\beta(c) \sum_{j=0}^{T-1} (Nj + c)^\alpha$$

are polynomials in T with degree up to $\alpha + 1$. This proves $\mathbb{F}_N^{\mathbb{C}_0} \subset [N, -1]$. $\square$

Proposition 2.5 Let $d \geq 0$ be an integer and $\beta : \mathbb{Z} \rightarrow \mathbb{C}$ be a N -periodic function, such that

$$\sum_{j=1}^N \beta(j)j^u = 0$$

for all $0 \leq u \leq d$. Then $\beta \in [N, d]$.

Proof Since β is N -periodic we know by Proposition 2.4 that the expressions

$$\sum_{j=1}^{NT} \beta(j)j^\alpha$$

define polynomials for all integers values $0 \leq \alpha$. We need to show, that these have degree at most $\alpha - d$. We obtain

$$\sum_{j=1}^{NT} \beta(j)j^\alpha = \sum_{\ell=0}^{T-1} \sum_{q=1}^N \beta(N\ell + q)(N\ell + q)^\alpha = \sum_{\ell=0}^{T-1} \beta(q)(N\ell + q)^\alpha$$

$$\begin{aligned}
&= \sum_{\ell=0}^{T-1} \sum_{q=1}^N \beta(q) \sum_{v=0}^{\alpha} \binom{\alpha}{v} (N\ell)^{\alpha-v} q^v = \sum_{\ell=0}^{T-1} \sum_{v=0}^{\alpha} \binom{\alpha}{v} (N\ell)^{\alpha-v} \sum_{q=1}^N \beta(q) q^v \\
&= \sum_{\ell=0}^{T-1} \sum_{v=d+1}^{\alpha} \binom{\alpha}{v} (N\ell)^{\alpha-v} \sum_{q=1}^N \beta(q) q^v = \sum_{v=d+1}^{\alpha} \binom{\alpha}{v} N^{\alpha-v} \sum_{q=1}^N \beta(q) q^v \sum_{\ell=0}^{T-1} \ell^{\alpha-v}.
\end{aligned}$$

Since the sum over v starts at $d+1$, by Theorem 2.1 this defines a polynomial of degree at most $\alpha-d$. Hence, $\beta \in [N, d]$. $\square$

Remark 2.6 Each non-principal Dirichlet character mod N has height at least 0 with respect to N , since

$$\sum_{j=1}^N \chi(j) = 0,$$

and each (non-principal) even character has height at least 1, since then we additionally have

$$\sum_{j=1}^N \chi(j)j = 0.$$

Proposition 2.7 Let $\beta : \mathbb{Z} \rightarrow \mathbb{C}$ be in $[N, d]$ for $d \geq 0$. Then, for all $u \geq 0$, there are coefficients $\gamma_{\beta,u}$ such that

$$(1-x)^{u-d} \sum_{p=1}^N \left(\sum_{r=1}^p \beta(r)r^u \right) x^p = \sum_{j=0}^{N+u-d} \gamma_{\beta,u}(j)x^j.$$

Proof For $d \leq u$ the proposition is clear, so we assume $d \geq 1$ and $0 \leq u < d$. Let $0 \leq \ell \leq d-u-1$ be an integer. Let

$$P(x) := \sum_{p=1}^N \left(\sum_{r=1}^p \beta(r)r^u \right) x^p.$$

Then we obtain for the value $P^{(\ell)}(1)$:

$$\begin{aligned}
&\sum_{p=1}^N \left(\sum_{r=1}^p \beta(r)r^u p(p-1)\cdots(p-\ell+1) \right) = \sum_{r=1}^N \beta(r)r^u \sum_{p=r}^N [p^\ell + b_{\ell-1}p^{\ell-1} + \cdots + b_1p] \\
&= \sum_{r=1}^N \beta(r)r^u (Q_\ell(N) - Q_\ell(r-1)) = 0,
\end{aligned}$$

since Q_ℓ is some polynomial of degree $\ell+1 \leq d-u$. This proves $P(x) = (1-x)^{d-u} Q(x)$ with some polynomial Q . $\square$

Our investigations rest on the properties of some explicit polynomials. They are similar, but simpler as the sums in (2.1). For a fixed non-negative integer α we define a sequence by

$$p_T(\alpha; x) = (1-x)^{\alpha+1} \sum_{\ell=1}^T \ell^\alpha x^\ell, \quad T = 1, 2, 3, \dots$$

For example we have $p_T(0; x) = x - x^{T+1}$ for $T = 1, 2, \dots$

Lemma 2.8 *The sequence $(p_T(\alpha; x))_{T \in \mathbb{N}}$ converges to some polynomial function on the interval $[0, 1)$ from below for all $\alpha \geq 0$. In particular, the terms p_T are uniformly bounded in the sense*

$$\sup_{T \in \mathbb{N}} \sup_{x \in [0, 1]} |p_T(\alpha; x)| \leq C_\alpha$$

for some constant $C_\alpha > 0$.

This uniform boundedness is a very important property as we will see later.

Proof It is clear that $p_T(\alpha; x)$ is an increasing sequence in T for fixed $0 < x < 1$. The power series

$$\sum_{\ell=1}^{\infty} \ell^\alpha x^\ell$$

converges for $x \in [0, 1)$ to a rational function $\frac{Q_\alpha(x)}{(1-x)^{\alpha+1}}$, where $Q_\alpha(x)$ is some polynomial which is non-negative in $[0, 1]$. This follows inductively by $\sum_{\ell=1}^{\infty} x^\ell = \frac{x}{1-x}$ and the fact that

$$x \frac{d}{dx} \left(\frac{Q_{\alpha-1}(x)}{(1-x)^\alpha} \right) = \frac{Q_\alpha(x)}{(1-x)^{\alpha+1}}$$

with polynomials $Q_{\alpha-1}$ and Q_α . Put $C_\alpha = \sup_{x \in [0, 1]} Q_\alpha(x)$. $\square$

Remark 2.9 In fact, one can give an explicit formula for the Q_α in terms of Eulerian numbers, but we will not need such a precise description for our applications.

Lemma 2.10 *For each fixed $T \geq 1$ there is some number $0 < \xi_{\alpha, T} < 1$ such that the function $p_T(\alpha; x)$ is increasing in the interval $[0, \xi_{\alpha, T}]$ and decreasing in the interval $[\xi_{\alpha, T}, 1]$, with respect to the variable x .*

Proof Since we have $p_T(\alpha; x) \geq 0$ for $0 \leq x \leq 1$ (with equality if $x = 0$ or $x = 1$), it is sufficient to show that $p'_T(\alpha; x) = 0$ has exactly one solution $0 < \xi_{\alpha, T} < 1$. For values $0 < x < 1$ we obtain

$$p'_T(\alpha; x) = -(\alpha+1)(1-x)^\alpha \sum_{\ell=1}^T \ell^\alpha x^\ell + (1-x)^{\alpha+1} \sum_{\ell=1}^T \ell^{\alpha+1} x^{\ell-1} = 0$$

which is equivalent to

$$\sum_{\ell=1}^T \left(-(\alpha+1)x^\ell + \ell^{\alpha+1}x^{\ell-1} - \ell^{\alpha+1}x^\ell \right) = 0,$$

and after further manipulations

$$\frac{1}{x^T} + \sum_{\ell=1}^{T-1} \left(\sum_{j=2}^{\alpha+1} \binom{\alpha+1}{j} \ell^{\alpha+1-j} \right) x^{\ell-T} = (\alpha+1)T^\alpha + T^{\alpha+1}.$$

The right hand side is greater than the left hand side for $x = 1$, since

$$1 + \sum_{\ell=1}^{T-1} \left(\sum_{j=2}^{\alpha+1} \binom{\alpha+1}{j} \ell^{\alpha+1-j} \right) = 1 + \sum_{\ell=1}^{T-1} ((1+\ell)^{\alpha+1} - (\alpha+1)\ell^\alpha - \ell^{\alpha+1})$$

$$= 1 + (T^{\alpha+1} - 1) - (\alpha + 1) \sum_{\ell=1}^{T-1} \ell^{\alpha} \leq T^{\alpha+1} < T^{\alpha+1} + (\alpha + 1)T^{\alpha}.$$

On the other hand, the left hand side is unbounded and monotonically decreasing in the interval $(0, 1]$. Hence, there is exactly one solution for the above equation in this area and the claim follows. $\square$

Before we can go on to the next lemma of this section we recall:

Lemma 2.11 *Let a_k be a sequence of complex numbers and b_k and c_k sequences of positive real numbers such that $0 \leq b_{k+1} \leq b_k$ and $c_{k+1} \geq c_k \geq 0$ for all k . Then we have for all $n \geq 1$:*

$$\left| \sum_{k=1}^n a_k b_k \right| \leq b_1 \max_{r=1, \dots, n} \left| \sum_{k=1}^r a_k \right|$$

and

$$\left| \sum_{k=1}^n a_k c_k \right| \leq (2c_n - c_1) \max_{r=1, \dots, n} \left| \sum_{k=1}^r a_k \right|.$$

Proof The first statement is called Abel's inequality, so we will only prove the second one. We set $A_n = \sum_{k=1}^n a_k$ and obtain by partial summation

$$\begin{aligned} \left| \sum_{k=1}^n a_k c_k \right| &= \left| A_n c_n + \sum_{k=1}^{n-1} A_k (c_k - c_{k+1}) \right| \leq |A_n| c_n + \sum_{k=1}^{n-1} |A_k| |c_k - c_{k+1}| \\ &\leq \max_{r=1, \dots, n} |A_r| \left(c_n + \sum_{k=1}^{n-1} (c_{k+1} - c_k) \right) = (2c_n - c_1) \max_{r=1, \dots, n} |A_r|. \end{aligned}$$

Hence the lemma is proved. $\square$

Our strategy will be to expand $\omega(z)$ in (2.1) into a Fourier series. With this we will obtain a double series, which is on the one hand more complicated. On the other hand, this simplifies the occurring summands drastically. Partial summation and Abel's inequalities are then the key tools when estimating sums of this type, as the next boundedness lemma shows.

Lemma 2.12 *Let $M, L, T > 1$ and $w \geq 0$ be integers, $\zeta_M^j \neq 1$ be a root of unity, $0 \leq X, Y \leq 1$ be real numbers and c_k be a monotonically increasing (or decreasing) sequence (that may depend on X and Y), which is bounded by $0 \leq c_k \leq B$ and B does not depend on X, Y, L and j . Then we have uniformly for L, X, Y, j ,*

$$\left| \sum_{k=1}^L (\zeta_M^j X)^k c_k p_T(w; Y^k) \right| \leq 6BC_w M,$$

where C_w is the constant defined in Lemma 2.8.

Proof Without loss of generality, we assume c_k to be an increasing sequence. In the case that c_k is decreasing the proof works similar. By Lemma 2.11 we first obtain

$$\left| \sum_{k=1}^L (\zeta_M^j X)^k c_k p_T(w; Y^k) \right| \leq 2B \max_{1 \leq l \leq L} \left| \sum_{k=1}^l (\zeta_M^j X)^k p_T(w; Y^k) \right|. \quad (2.2)$$

In the case c_k is decreasing we could switch $2B$ by B , but since $B \leq 2B$ the estimate works in both cases. To estimate the inner sum for any value I with $1 \leq I \leq L$, we will use the fact, that the p_T are monotonically increasing first in some interval $[0, \xi_{w,T}]$ and then monotonically decreasing in $[\xi_{w,T}, 1]$, as it was shown in Lemma 2.10. For any I choose the unique $1 \leq I(w, T, Y) \leq I$ such that $Y^k > \xi_{w,T}$ for all $1 \leq k \leq I(w, T, Y)$ and $Y^k \leq \xi_{w,T}$ for $I(w, T, Y) < k \leq I$. Note that in the case $Y = 1$ the second condition is empty. Then, using the triangle inequality, we see

$$\left| \sum_{k=1}^I (\zeta_M^j X)^k p_T(w; Y^k) \right| \leq \left| \sum_{k=1}^{I(w,T,Y)} (\zeta_M^j X)^k p_T(w; Y^k) \right| + \left| \sum_{k=I(w,T,Y)+1}^I (\zeta_M^j X)^k p_T(w; Y^k) \right|.$$

We apply Lemma 2.11 on the first sum to obtain

$$\left| \sum_{k=1}^{I(w,T,Y)} (\zeta_M^j X)^k p_T(w; Y^k) \right| \leq 2C_w \max_{1 \leq J \leq I(w,T,Y)} \left| \sum_{k=1}^J (\zeta_M^j X)^k \right|,$$

where C_w is the constant given in Lemma 2.8. The inner sum can be estimated again with Lemma 2.11, since $0 \leq X^{k+1} \leq X^k \leq 1$ by

$$\left| \sum_{k=1}^J (\zeta_M^j X)^k \right| \leq \max_{1 \leq H \leq J} \left| \sum_{k=1}^H \zeta_M^{jk} \right| \leq M,$$

hence

$$\left| \sum_{k=1}^{I(w,T,Y)} (\zeta_M^j X)^k p_T(w; Y^k) \right| \leq 2C_w \max_{1 \leq J \leq I(w,T,Y)} M = 2C_w M.$$

Similarly, we obtain with Lemma 2.11

$$\left| \sum_{k=I(w,T,Y)+1}^I (\zeta_M^j X)^k p_T(w; Y^k) \right| \leq C_w \max_{I(w,T,Y)+1 \leq J \leq I} \left| \sum_{k=I(w,T,Y)+1}^J (\zeta_M^j X)^k \right| \leq C_w M.$$

Finally, with (2.2) we obtain

$$\left| \sum_{k=1}^L (\zeta_M^j X)^k c_k p_T(w; Y^k) \right| \leq 2B \max_{1 \leq I \leq L} 3C_w M = 6BC_w M.$$

This proves the lemma. $\square$

The next lemma can be seen as an analogous result to the previous lemma.

Lemma 2.13 Let $M, N, L, T > 1$ be integers, $0 \leq y \leq 1$ any real number, $\zeta_M^j \neq 1$ a root of unity and $p(X) = \sum_{u=0}^d \gamma(u) X^u$ a polynomial of degree at most d , with coefficients independent of L, T and y . Then there is a constant $D_{j,M,N,p} > 0$ only depending on j, M, N and p such that uniformly in L, T and y :

$$\left| y^d p(T) \sum_{k=1}^L \zeta_M^{jk} e^{-NTky} \right| \leq D_{j,M,N,p}.$$

Proof The constant

$$U_{j,M} := \sup_{0 \leq x \leq \infty} \frac{1}{|1 - e^{-Nx} \zeta_M^j|}$$

exists and only depends on j and M . Put $x := yT$. We obtain with the geometric summation formula

$$y^d p(T) \sum_{k=1}^L \zeta_M^{jk} e^{-NTky} = \sum_{u=0}^d \gamma(u) x^u y^{d-u} \frac{e^{-Nx} \zeta_M^j - e^{-Nx(L+1)} \zeta_M^{j(L+1)}}{1 - e^{-Nx} \zeta_M^j}$$

and hence

$$\left| y^d p(T) \sum_{k=1}^L \zeta_M^{jk} e^{-NTky} \right| \leq U_{j,M} \cdot 2e^{-Nx} \sum_{u=0}^d |\gamma(u)| x^u.$$

The right hand side is obviously bounded for $0 \leq x \leq \infty$ and only depends on j , M , N and p , so we have found a possible $D_{j,M,N,p}$. $\square$

We now have all the tools to prove the main theorem of this section.

Theorem 2.14 (Dominated convergence theorem) *Let β be a N -periodic function in $[N, d]$, $d \geq 0$, and $\omega \in W_M$ be a weak function that has a removable singularity in $z = 0$. Then for all $\alpha \in \mathbb{N}_0$ there is a constant $C_{\beta,\omega,\alpha} > 0$ such that uniformly for all $T \in \mathbb{N}$ and $y \in [0, 1]$*

$$\left| \sum_{n=1}^{NT} n^\alpha \beta(n) \omega(niy) \right| \leq C_{\beta,\omega,\alpha} y^{d-\alpha}.$$

Remark 2.15 Note that, by Theorem 2.14, in the case $\alpha \leq d$ the left hand side is bounded uniformly for values T and $y \in [0, 1]$. Since the series converges absolutely and uniformly on $[1, \infty]$, we obtain dominated convergence on $[0, \infty]$.

Proof For $y = 0$ the inequality holds since in the case $\alpha \leq d$ the left hand side is always zero (note that $\omega(0)$ exists) and otherwise the right hand side is $+\infty$ from the right. Let $y > 0$. We then have

$$\omega(z) = \sum_{j \in \mathbb{F}_M} \beta_\omega(j) \frac{e\left(z - \frac{j}{M}\right)}{1 - e\left(z - \frac{j}{M}\right)} = \sum_{j \in \mathbb{F}_M} \beta_\omega(-j) \lim_{L \rightarrow \infty} \sum_{k=1}^L e(kz) \zeta_M^{jk}$$

and hence

$$\sum_{n=1}^{NT} n^\alpha \beta(n) \omega(niy) = \lim_{L \rightarrow \infty} \sum_{j \in \mathbb{F}_M} \beta_\omega(-j) \sum_{k=1}^L \sum_{n=1}^{NT} n^\alpha \beta(n) \zeta_M^{kj} e^{-2\pi kny}. \quad (2.3)$$

In the first step we will only deal with the inner sums. We obtain with partial summation

$$\sum_{n=1}^{NT} n^\alpha \beta(n) e^{-2\pi kny} = e^{-2\pi kNTy} \sum_{n=1}^{NT} n^\alpha \beta(n) + \sum_{n=1}^{NT-1} \left(\sum_{r=1}^n \beta(r) r^\alpha \right) (e^{-2\pi kny} - e^{-2\pi k(n+1)y}).$$

Since β has height d , there is a polynomial $p_{\alpha,\beta}$ with degree at most $\alpha - d$ such that

$$e^{-2\pi kNTy} \sum_{n=1}^{NT} n^\alpha \beta(n) = e^{-2\pi kNTy} p_{\alpha,\beta}(T).$$

By Lemma 2.13 there is a constant $D_{\alpha,\beta,\omega} > 0$ only depending on α , β and ω (note that N belongs to β and M to ω , and that $\beta_\omega(0) = 0$ which implies $\zeta_M^j \neq 1$), such that

$$\left| \sum_{j \in \mathbb{F}_N} \beta_\omega(-j) \sum_{k=1}^L \zeta_M^{jk} e^{-2\pi k N T y} \sum_{n=1}^{NT} n^\alpha \beta(n) \right| \leq y^{d-\alpha} \sum_{j \in \mathbb{F}_N} |\beta_\omega(-j)| D_{j,M,N,p_{\alpha,\beta}}. \quad (2.4)$$

where we put

$$D_{\alpha,\beta,\omega} := \sum_{j \in \mathbb{F}_N} |\beta_\omega(-j)| D_{j,M,N,p_{\alpha,\beta}}.$$

On the other hand, we have

$$\begin{aligned} \sum_{n=1}^{NT-1} \left(\sum_{r=1}^n \beta(r) r^\alpha \right) (e^{-2\pi k n y} - e^{-2\pi k(n+1)y}) &= (1 - e^{-2\pi k y}) \sum_{n=1}^{NT-1} \left(\sum_{r=1}^n \beta(r) r^\alpha \right) e^{-2\pi k n y} \\ &= (1 - e^{-2\pi k y}) \sum_{q=1}^N \sum_{\ell=0}^{T-1} \sum_{r=1}^{N\ell+q} \beta(r) r^\alpha e^{-2\pi k(N\ell+q)y} - (1 - e^{-2\pi k y}) \sum_{r=1}^{NT} \beta(r) r^\alpha e^{-2\pi k N T y}. \end{aligned}$$

For the right sum we obtain with Lemma 2.11 and (2.4) (note that $1 - e^{-2\pi k y}$ is monotonous):

$$\sum_{j \in \mathbb{F}_N} |\beta_\omega(-j)| \left| \sum_{k=1}^L \zeta_M^{jk} (1 - e^{-2\pi k y}) e^{-2\pi k N T y} \sum_{r=1}^{NT} \beta(r) r^\alpha \right| \leq 2y^{d-\alpha} D_{\alpha,\beta,\omega}. \quad (2.5)$$

So we are left to give an estimate for the left sum. Here we obtain

$$\begin{aligned} &(1 - e^{-2\pi k y}) \sum_{q=1}^N \sum_{\ell=0}^{T-1} \sum_{r=1}^{N\ell+q} \beta(r) r^\alpha e^{-2\pi k(N\ell+q)y} \\ &= (1 - e^{-2\pi k y}) \left(\sum_{q=1}^N \sum_{\ell=0}^{T-1} \sum_{r=1}^{N\ell} \beta(r) r^\alpha e^{-2\pi k N \ell y} e^{-2\pi k q y} \right. \\ &\quad \left. + \sum_{q=1}^N \sum_{\ell=0}^{T-1} \sum_{r=N\ell+1}^{N\ell+q} \beta(r) r^\alpha e^{-2\pi k N \ell y} e^{-2\pi k q y} \right). \end{aligned} \quad (2.6)$$

The final estimate will be given by the sum of two separate estimates of both of these sums. Without loss of generality we assume $\alpha > d$, since otherwise the left sum vanishes, which now equals to

$$\begin{aligned} &(1 - e^{-2\pi k y}) \sum_{q=1}^N \sum_{\ell=0}^{T-1} \sum_{r=1}^{N\ell} \beta(r) r^\alpha e^{-2\pi k N \ell y} e^{-2\pi k q y} \\ &= (e^{-2\pi k y} - e^{-2\pi k(N+1)y}) \sum_{\ell=0}^{T-1} p_{\alpha,\beta}(\ell) e^{-2\pi k N \ell y} \\ &= (e^{-2\pi k y} - e^{-2\pi k(N+1)y}) \sum_{u=0}^{\alpha-d} \gamma_{\alpha,\beta}(u) \sum_{\ell=0}^{T-1} \ell^u e^{-2\pi k N \ell y} \end{aligned}$$

$$= \left(e^{-2\pi k y} - e^{-2\pi k(N+1)y} \right) \sum_{u=0}^{\alpha-d} \gamma_{\alpha,\beta}(u) \frac{p_{T-1}(u; e^{-2\pi k N y})}{(1 - e^{-2\pi k N y})^{u+1}}.$$

After multiplying and dividing by $(1 - e^{-2\pi k N y})^{\alpha-d+1}$, this equals

$$\frac{(e^{-2\pi k y} - e^{-2\pi k(N+1)y})}{(1 - e^{-2\pi k N y})^{\alpha-d+1}} \sum_{u=0}^{\alpha-d} \gamma_{\alpha,\beta}(u) (1 - e^{-2\pi k N y})^{\alpha-d-u} p_{T-1}(u; e^{-2\pi k N y}).$$

Put $Y := e^{-2\pi N y}$. There is a constant $A > 0$ not depending on y and k such that $|y(1 - Y^k)^{-1}| \leq A$ for $0 < y \leq 1$. Note that we have

$$\frac{(e^{-2\pi k y} - e^{-2\pi k(N+1)y})}{(1 - e^{-2\pi k N y})^{\alpha-d+1}} = y^{d-\alpha} \frac{y^{\alpha-d}}{(1 - Y^k)^{\alpha-d}} e^{-2\pi k y}.$$

For $k > 0$ the sequence

$$c_k := e^{-2\pi k y} \left(\frac{y}{1 - Y^k} \right)^{\alpha-d}$$

is decreasing and bounded between 0 and $A^{\alpha-d}$. Also put

$$\sum_{u=0}^{\alpha-d} \gamma_{\alpha,\beta}(u) (1 - Y^k)^{\alpha-d-u} = \sum_{u=0}^{\alpha-d} \tilde{\gamma}_{\alpha,d}(u) Y^{ku}.$$

This gives us

$$\begin{aligned} & y^{d-\alpha} \left| \sum_{k=1}^L \zeta_M^{kj} c_k \sum_{u=0}^{\alpha-d} \gamma_{\alpha,\beta}(u) (1 - Y^k)^{\alpha-d-u} p_{T-1}(u; Y^k) \right| \\ & \leq y^{d-\alpha} \sum_{u=0}^{\alpha-d} |\tilde{\gamma}_{\alpha,\beta}(u)| \left| \sum_{k=1}^L \left(\zeta_M^j Y^u \right)^k c_k p_{T-1}(u; Y^k) \right| \leq 6y^{d-\alpha} M A^{\alpha-d} \sum_{u=0}^{\alpha-d} |\tilde{\gamma}_{\alpha,\beta}(u)| C_u \end{aligned} \quad (2.7)$$

when putting $X := Y^u$ and using Lemma 2.12.

On the other hand, when putting $Z := e^{-2\pi y}$, we obtain for the right sum in (2.6)

$$\begin{aligned} & (1 - Z^k) \sum_{q=1}^N \sum_{\ell=0}^{T-1} \sum_{r=N\ell+1}^{N\ell+q} \beta(r) r^\alpha Z^{\ell N k} Z^{k q} \\ & = (1 - Z^k) \sum_{\ell=0}^{T-1} \sum_{q=1}^N \sum_{r=1}^q \beta(N\ell + r) (N\ell + r)^\alpha Z^{\ell N k} Z^{k q} \end{aligned}$$

and since β is N -periodic this equals

$$\begin{aligned} & (1 - Z^k) \sum_{\ell=0}^{T-1} \sum_{q=1}^N \sum_{r=1}^q \beta(r) \sum_{u=0}^{\alpha} \binom{\alpha}{u} (N\ell)^u r^{\alpha-u} Z^{\ell N k} e^{-2\pi k q y} \\ & = (1 - Z^k) \sum_{u=0}^{\alpha} \binom{\alpha}{u} N^u \left(\sum_{q=1}^N \sum_{r=1}^q \beta(r) r^{\alpha-u} Z^{k q} \right) \frac{p_{T-1}(u; Z^{N k})}{(1 - Z^{N k})^{u+1}} \end{aligned}$$

$$\begin{aligned}
&= (1 - Z^k)^{\sum_{u=0}^{\alpha} \binom{\alpha}{u} N^u \left(\sum_{q=1}^N \sum_{r=1}^q \beta(r) r^{\alpha-u} Z^{kq} \right)} \frac{(1 - Z^k)^{u+1} p_{T-1}(u; Z^{Nk})}{(1 - Z^k)^{u+1} (1 - Z^{Nk})^{u+1}} \\
&= \sum_{u=0}^{\alpha} \binom{\alpha}{u} \frac{N^u}{(1 - Z^k)^u} \left(\sum_{q=1}^N \sum_{r=1}^q \beta(r) r^{\alpha-u} Z^{kq} \right) c_k(u) p_{T-1}(u; Z^{Nk}),
\end{aligned}$$

with

$$c_k(u) := \left(\frac{1 - Z^k}{1 - Z^{Nk}} \right)^{u+1} = \frac{1}{(1 + Z^k + Z^{2k} + \dots + Z^{(N-1)k})^{u+1}}.$$

Note that we always have $0 \leq c_k(u) \leq c_{k+1}(u) \leq 1$. Since β has height d , by Lemma 2.7, there are coefficients $\delta_{\alpha, \beta, u}(w)$ such that

$$(1 - Z^k)^{\alpha-d-u} \left(\sum_{q=1}^N \sum_{r=1}^q \beta(r) r^{\alpha-u} Z^{kq} \right) = \sum_{w=0}^{N+|\alpha-d|} \delta_{\alpha, \beta, u}(w) (Z^k)^w.$$

We conclude

$$\begin{aligned}
&\left| \sum_{k=1}^L \zeta_M^{kj} \sum_{u=0}^{\alpha} \binom{\alpha}{u} \frac{N^u}{(1 - Z^k)^u} \left(\sum_{q=1}^N \sum_{r=1}^q \beta(r) r^{\alpha-u} Z^{kq} \right) c_k(u) p_{T-1}(u; Z^{Nk}) \right| \\
&\leq y^{d-\alpha} \sum_{u=0}^{\alpha} \binom{\alpha}{u} N^u \left| \sum_{k=1}^L y^{\alpha-d} (1 - Z^k)^{d-\alpha} \sum_{w=0}^{N+|\alpha-d|} \delta_{\alpha, \beta, u}(w) (Z^k)^w c_k(u) p_{T-1}(u; Z^{Nk}) \right|.
\end{aligned}$$

The sequence $y^{\alpha-d} (1 - Z^k)^{d-\alpha}$ in k is bounded by some $V^{\alpha-d}$ and monotonous. Hence we obtain with Lemma 2.11 that the above estimate is smaller or equal to

$$2V^{\alpha-d} y^{d-\alpha} \sum_{u=0}^{\alpha} \binom{\alpha}{u} N^u \sum_{w=0}^{N+|\alpha-d|} |\delta_{\alpha, \beta}(w)| \max_{1 \leq l \leq L} \left| \sum_{k=1}^L (\zeta_M^j Z^w)^k c_k(u) p_{T-1}(u; Z^w) \right|$$

and by Lemma 2.12 this is smaller or equal to

$$2V^{\alpha-d} y^{d-\alpha} \sum_{u=0}^{\alpha} \binom{\alpha}{u} N^u \sum_{w=0}^{N+|\alpha-d|} |\delta_{\alpha, \beta}(w)| \times 6C_u M \leq F_{\alpha, \beta, \omega} y^{d-\alpha}, \quad (2.8)$$

for some $F_{\alpha, \beta, \omega} > 0$ only depending on α, β and ω . By considering (2.4), (2.5), (2.7) and (2.8) and using the triangle inequality in (2.3) (note that the constants do not depend on L), the theorem is proved. $\square$

Since we have assumed β to be N -periodic it might come from a weak function $\eta \in W_N$, i.e., $\beta := \beta_\eta$. The purpose of the next section will be to use the Dominated convergence theorem to improve regions of convergence of L -functions assigned to products of weak modular forms.

3 Application to L -functions of modular forms

Let $S = \{t_1, t_2, \dots\}$ be a countable, totally ordered set (the direction is simply given by $t_\ell \leq t_j$ if and only if $\ell \leq j$) equipped with an integer map $|\cdot|_S : S \rightarrow \mathbb{N}$ such that for some

$L \geq 0$:

$$\#\{t \in S \mid |t|_S = n\} = O\left(n^L\right). \quad (3.1)$$

In the case the set S is clear, we simply write $|\cdot|$. For example, S could be the set of integral ideals of a number field and $|\cdot|$ their norm. Let $a(t_m)_{m \in \mathbb{N}}$ a sequence of complex numbers. We define the corresponding formal Dirichlet series by

$$F(s) := \sum_{t \in S} a(t) |t|^{-s} := \sum_{m=1}^{\infty} a(t_m) |t_m|^{-s}.$$

In the case that the series

$$\sum_{n=1}^{\infty} \left| \frac{1}{|t_n|^s} - \frac{1}{|t_{n+1}|^s} \right|$$

converges for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, one can check using partial summation that such Dirichlet series converge (if they do) on half planes and represent holomorphic functions in these regions. This is for example the case, if the $|t_n|$ increase monotonously. Since we have (3.1), one can show that $F(s)$ will converge in some point s_0 if and only if $a(t) = O(|t|^\nu)$ for some $\nu \in \mathbb{R}$.

Definition 3.1 Let $F(s) = \sum_{t \in S} a(t) |t|^{-s}$ be a Dirichlet series, Q a totally ordered countable set together with a surjective map $w : Q \rightarrow S$ with finite fibres. We also assume that F converges to a holomorphic function on some half plane $\{\operatorname{Re}(s) > \sigma_0\}$. The order of Q shall respect the order of S , this means $u_1 \leq_Q u_2$ implies $w(u_1) \leq_S w(u_2)$ for all $u_1, u_2 \in Q$. We define an integer map on Q via $|u|_Q := |w(u)|_S$. In other words, all elements in the same fibre of a $t \in S$ are associated to the same integer. By a splitting of F we mean a Dirichlet series $\tilde{F}(s) = \sum_{u \in Q} b(u) |u|_Q^{-s}$ that has the following properties:

- (i) $\tilde{F}(s)$ converges to a holomorphic function in some half plane $\{\operatorname{Re}(s) > \tilde{\sigma}_0\}$.
- (ii) We have for all $t \in S$ the summation formula $\sum_{u \in w^{-1}(t)} b(u) = a(t)$.

We may think of splittings in the following way: we have $Q = \bigcup_{t \in S} \sigma^{-1}(t)$ and therefore

$$\sum_{t \in S} a(t) |t|^{-s} = \sum_{t \in S} \sum_{u \in w^{-1}(t)} b(u) |u|_Q^{-s}.$$

Remark 3.2 Consider the number theoretic function $A_4(n) := \#\{(x_1, x_2, x_3, x_4) \in \mathbb{N}_0^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = n\}$. Note that normally one considers tuples in $\mathbb{Z}^4$ but to keep things simple in this example we use $\mathbb{N}_0^4$. Then the ordinary Dirichlet series

$$D_4(s) := \sum_{n=1}^{\infty} A_4(n) n^{-s}$$

converges for $\operatorname{Re}(s) > 2$. Here we have $S = \mathbb{N}$ and $|\cdot|_{\mathbb{N}}$ is simply given by $|n|_{\mathbb{N}} := n$. Now put $Q := \mathbb{N}_0^4 \times \mathbb{N}_0^4 \setminus \{(\mathbf{x}, \mathbf{y}) \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0\} \cap \{(\mathbf{x}, \mathbf{x}) \mid \mathbf{x} \in \mathbb{N}_0^4\}$ and consider the surjective map $w : Q \rightarrow \mathbb{N}$ with $w(\mathbf{x}, \mathbf{x}) := \langle \mathbf{x}, \mathbf{x} \rangle$. There are lots of orders we can define on Q as long as $(\mathbf{x}, \mathbf{x}) \leq_Q (\tilde{\mathbf{x}}, \tilde{\mathbf{x}})$ implies $\langle \mathbf{x}, \mathbf{x} \rangle \leq \langle \tilde{\mathbf{x}}, \tilde{\mathbf{x}} \rangle$. Since $w^{-1}(n)$ consists of all $(\mathbf{x}, \mathbf{x})$ satisfying $\langle \mathbf{x}, \mathbf{x} \rangle = n$ and $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 + x_3^2 + x_4^2$, we obtain

$$\sum_{(\mathbf{x}, \mathbf{x}) \in w^{-1}(n)} 1 = A_4(n)$$

As a result, the series

$$\sum_{(x,x) \in Q} \langle x, x \rangle^{-s}$$

is a possible splitting of D_4 . Note, that this series also converges (independent from the chosen order) on $\operatorname{Re}(s) > 2$ and represents a holomorphic function in this region, which shows that also condition (i) of Definition 3.1 is satisfied.

Splittings that are obtained by maps $\mathbb{N}^\ell \times \mathbb{N}^\ell \rightarrow \mathbb{N}$ and $(x, y) \mapsto \langle x, y \rangle$ will play the key role in the rest of this section. Throughout, we will omit the construction details as they were presented in the last example.

The next definition provides kind of an inverse concept for splittings.

Definition 3.3 Let $S = \bigcup_{j=1}^{\infty} S_j$ be a disjoint covering with finite S_j . We say that a Dirichlet series $F(s) = \sum_{t \in S} a(t)|t|^{-s}$ respects the rearrangement $(S_j)_{j \in \mathbb{N}}$, if the series is given by the partial sums

$$F_n(s) = \sum_{j=1}^n \sum_{t \in S_j} a(t)|t|^{-s}.$$

If there might be danger with confusion we simply write

$$(F, (S_j)_{j \in \mathbb{N}})(s) = \sum_{j=1}^{\infty} \sum_{t \in S_j} a(t)|t|^{-s}.$$

Obviously, $F(s)$ and $(F, (S_j)_{j \in \mathbb{N}})(s)$ coincide in all regions of absolute convergence. In the case of $S_j = \{t \in S \mid |t| = j\}$, $(F, (S_j)_{j \in \mathbb{N}})(s)$ is an ordinary Dirichlet series $\sum b(n)n^{-s}$ – we call this the standard rearrangement. The next proposition makes clear why rearrangements makes splitting undone in some situations.

Proposition 3.4 Let $\tilde{F}$ be a splitting of F over Q . Define the disjoint union $Q_j := \sigma^{-1}(t_j)$. If we now sum $\tilde{F}$ with respect to $(Q_j)_{j \in \mathbb{N}}$ we obtain F .

Proof This follows directly from the definitions. $\square$

Definition 3.5 We call $(T_j)_{j \in \mathbb{N}}$ a sub-rearrangement of $(S_j)_{j \in \mathbb{N}}$, if there is a sequence of integers $0 < k_1 < k_2 < k_3 < \dots$ such that $T_1 = S_1 \cup \dots \cup S_{k_1}$, $T_2 = S_{k_1+1} \cup \dots \cup S_{k_2}$ and so on.

In the following we define for any rearrangement the abscissa of convergence

$\sigma((F, (S_j)_{j \in \mathbb{N}}))$ to be the infimum real value σ_0 , such that for all complex values $s \in \mathbb{C}$ with $\operatorname{Re}(s) > \sigma_0$ the series converges and represents a holomorphic function in this region.

Remark 3.6 One easily checks $\sigma((F, (T_j)_{j \in \mathbb{N}})) \leq \sigma((F, (S_j)_{j \in \mathbb{N}}))$ for $(T_j)_{j \in \mathbb{N}}$ a sub-rearrangement of $(S_j)_{j \in \mathbb{N}}$. Hence Proposition 3.4 shows that splitting does not improve the area of convergence. However, when rearranging a split series the situation might look different.

Let $\mathfrak{R}(F)$ the set of all rearrangements of F . We define an equivalence relation on $\mathfrak{R}(F)$ by putting two coverings in the same class if the resultant series have the same abscissa of convergence. We collect this data in $\mathfrak{R}(F)/\sim$. We would like to study $\mathfrak{R}(F)/\sim$, in particular, we are interested in the following question:

Question 3.7 What is the value $\tilde{\sigma}(F) := \inf_{G \in \mathfrak{R}(F)/\sim} \sigma(G)$?

There is no simple answer to this question. It rather strongly depends on the Dirichlet series itself, as the next examples demonstrate.

- (i) If $a(t) \geq 0$ globally, the region of convergence can not be improved by rearranging the Dirichlet series. Hence $|\mathfrak{R}(F)/\sim| = 1$ and $\tilde{\sigma}(F) = \sigma(F)$.
- (ii) Although the set of possible rearrangements is large, $-\tilde{\sigma}(F)$ does not have to be unbounded even in the case that F is entire. If χ is an *even* real non-principal character modulo M , one can show that $\tilde{\sigma}(L(\chi; s)) = -1$ if $L(\chi; -1) \notin \mathbb{Z}$. In this case the “best” rearrangement of $L(\chi; s)$ is given by $\mathbb{N} = \bigcup_{j \in \mathbb{N}} \{M(j-1) + k \mid 1 \leq k \leq M\}$ and we have

$$L(\chi; s) = \sum_{j=1}^{\infty} \left(\sum_{m=1}^M \chi(m)(M(j-1) + m)^{-s} \right), \quad \operatorname{Re}(s) > -1.$$

We conclude $L(\chi, 0) = 0$. Since all inner summands in the rearrangements are integers when $s = -1$, there is indeed no better choice if $L(\chi, -1) \notin \mathbb{Z}$, as the reader may easily check.

A similar argument shows $\tilde{\sigma}(L(\chi; s)) = \sigma_0 = 0$ if χ is real, odd and $L(\chi, 0) \notin \mathbb{Z}$.

- (iii) The identity $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ for $\operatorname{Re}(s) > 1$ is well-known and elementary. Here $\mu(n)$ is the Möbius function. Since $\mu(n)$ has sign changes, it makes sense to look at possible rearrangements. However, it seems to be extremely difficult to find improvements of $\sigma = 1$, since there is no progress in this area until today! We have $\frac{1}{2} \leq \tilde{\sigma}(\zeta^{-1}) \leq 1$ and $\tilde{\sigma}(\zeta^{-1}) = \frac{1}{2}$ implies the Riemann hypothesis.

Remark 3.8 In the case of (ii), where the coefficients are well-studied, there are of course even more powerful tools for analytic continuation using series transformations, that can be seen as generalized rearrangements in the sense that we allow the splitting sets S_n to have infinite order. For example, when using Euler summation, we find the right hand series

$$L(\chi; s) = \sum_{n=0}^{\infty} 2^{-n-1} \sum_{v=0}^n \binom{n}{v} \chi(v+1)(v+1)^{-s},$$

will converge globally for non-principal characters χ .

Let $\mathbf{k} = (k_1, \dots, k_\ell)$ and $f \in U_{\mathbf{k}}(\Gamma_1(M, N))$ be a weak modular form. In the following we give a natural splitting for $L(f; s)$ in terms of the overset $\mathcal{Q} = \mathbb{N}^\ell \times \mathbb{N}^\ell$. After this, when applying the Dominated convergence theorem from the last section we can find good rearrangements of these splittings to give estimates for the size defined in Question 3.7. Let $G_N^{(\ell)} = \mathbb{F}_N^\times \times \dots \times \mathbb{F}_N^\times$ be the ℓ -fold product of the residue class groups modulo N . Then $G_N^{(\ell)}$ is a multiplicative group and there are $\varphi(N)^\ell$ characters $\psi : G_N^{(\ell)} \rightarrow \mathbb{C}^\times$ given by $\psi(\mathbf{n}) = \prod_{j=1}^{\ell} \psi_j(n_j)$, where $\psi_1, \dots, \psi_\ell$ are characters modulo N . We further call a character $\psi : G_N^{(\ell)} \rightarrow \mathbb{C}^\times$ *non-principal*, if no component ψ_j with $1 \leq j \leq \ell$ is principal and *principal* else. Analogously we say that ψ is *primitive* if and only if all components are primitive. Note that each ψ extends multiplicatively to a map $\psi : \mathbb{Z}^\ell \rightarrow \mathbb{C}^\times$. For $\mathbf{k} \in \mathbb{N}^\ell$ also define the (multiplicative) map $\Pi_{\mathbf{k}}(\mathbf{n}) = n_1^{k_1-1} \dots n_\ell^{k_\ell-1}$.

Lemma 3.9 *Let h be a M -periodic function in $[M, d]$, where $0 \leq d$. Then there is some constant $C_h > 0$, only depending on h , such that we have uniformly for $x \in [0, 1]$:*

$$\left| \sum_{v=1}^{MT} h(v)x^v \right| \leq C_h(1-x)^d.$$

Proof We use partial summation again. We obtain:

$$\begin{aligned} \sum_{v=1}^{MT} h(v)x^v &= \sum_{v=1}^{MT} h(v)x^{MT} + \sum_{r=1}^{MT-1} \sum_{u=1}^r h(u)(x^r - x^{r+1}) \\ &= (1-x) \sum_{r=1}^{MT} \left(\sum_{u=1}^r h(u) \right) x^r = (1-x) \sum_{k=0}^{T-1} \sum_{u=1}^M \sum_{r=1}^{Mk+u} h(r)x^{Mk+u} \\ &= (1-x) \sum_{u=1}^M x^u \sum_{r=1}^u h(r) \sum_{k=0}^{T-1} x^{Mk} = (1-x) \sum_{u=1}^M \left(\sum_{r=1}^u h(r) \right) x^u \frac{1-x^{MT}}{1-x^M} \\ &= \frac{(1-x)(1-x^{MT})}{1-x^M} \sum_{u=1}^M \left(\sum_{r=1}^u h(r) \right) x^u. \end{aligned}$$

By Proposition 2.7 there is a constant C_h such that uniformly on $[0, 1]$:

$$\left| \sum_{u=1}^M \left(\sum_{r=1}^u h(r) \right) x^u \right| \leq C_h(1-x)^d.$$

On the other hand, we uniformly have

$$\frac{(1-x)(1-x^{MT})}{1-x^M} \leq 1.$$

This proves the lemma. $\square$

In the following, consider the subsets $T_{M,p} \subset \mathbb{N}^\ell$ with $T_{M,p} = \{\mathbf{v} \in \mathbb{N}^\ell \mid (p-1)M < \max(\mathbf{v}) \leq Mp\}$. It is clear that we have a disjoint covering of $\mathbb{N}^\ell$ by all $T_{M,1}, T_{M,2}, \dots$

Lemma 3.10 *Let $h_1, \dots, h_\ell$ be functions in $(\mathbb{F}_{M^{-1}})_0^{\mathbb{C}_0}$, such that the associated weak functions ω_{h_j} have a removable singularity in $z = 0$, and $\mathcal{F}_M h_j \in [M, c_j]$ for some $c_j \geq 0$. Then we have for all vectors $\mathbf{u} \in \mathbb{N}^\ell$ and s with $\operatorname{Re}(s) > -\sum_{j=1}^\ell c_j$:*

$$\int_0^\infty \omega_{h_1} \left(\frac{u_1 x i}{N} \right) \cdots \omega_{h_\ell} \left(\frac{u_\ell x i}{N} \right) x^{s-1} dx = \Gamma(s) \left(\frac{N}{2\pi} \right)^s \sum_{\mathbf{v} \in \mathbb{N}^\ell} \mathcal{F}_N^{(\ell)} h(\mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s},$$

where $\mathcal{F}_N^{(\ell)} h(\mathbf{v}) = (\mathcal{F}_N h_1)(v_1) \cdots (\mathcal{F}_N h_\ell)(v_\ell)$ is the vector valued Fourier transform. Here, the order of summation respects the rearrangement $(T_{M,p})_{p \in \mathbb{N}}$.

Proof We have

$$\omega_{h_1} \left(\frac{u_1 x i}{N} \right) \cdots \omega_{h_\ell} \left(\frac{u_\ell x i}{N} \right) = \sum_{\mathbf{q} \in G_M^{(\ell)}} h_1(q_1) \cdots h_\ell(q_\ell) \prod_{j=1}^\ell \frac{e(u_j \frac{ix}{N})}{e(\frac{q_j}{M}) - e(u_j \frac{ix}{N})}$$

$$\begin{aligned}
&= \sum_{\mathbf{q} \in G_M^{(\ell)}} h_1(q_1) \cdots h_\ell(q_\ell) \prod_{j=1}^{\ell} \sum_{v_j=1}^{\infty} e^{-\frac{2\pi u_j v_j x}{N} - \frac{2\pi i q_j v_j}{M}} \\
&= \lim_{T \rightarrow \infty} \sum_{\mathbf{q} \in G_M^{(\ell)}} h_1(q_1) \cdots h_\ell(q_\ell) \sum_{v_1=1}^{MT} \cdots \sum_{v_\ell=1}^{MT} e^{-\frac{2\pi(v_1 u_1 + \cdots + v_\ell u_\ell)x}{N} - \frac{2\pi i(v_1 q_1 + \cdots + v_\ell q_\ell)}{M}} \\
&= \prod_{j=1}^{\ell} \left(\sum_{v_j=1}^{MT} (\mathcal{F}_M h_j)(v_j) e^{-\frac{2\pi u_j v_j x}{M}} \right).
\end{aligned}$$

We obtain with Lemma 3.9:

$$\left| \prod_{j=1}^{\ell} \left(\sum_{v_j=1}^{MT} (\mathcal{F}_M h_j)(v_j) e^{-\frac{2\pi u_j v_j x}{M}} \right) x^{s-1} \right| \leq C_{\mathbf{h}, \mathbf{u}} x^{\sigma-1+c_1+\cdots+c_\ell}$$

uniformly for $x \in [0, 1]$, where $C_{\mathbf{h}, \mathbf{u}} > 0$ only depends on the functions $h_1, \dots, h_\ell$ and the vector $\mathbf{u}$. As a result, the integral

$$\int_0^\infty \omega_{h_1} \left(\frac{u_1 x i}{N} \right) \cdots \omega_{h_\ell} \left(\frac{u_\ell x i}{N} \right) x^{s-1} dx$$

converges absolutely to a holomorphic function for $\operatorname{Re}(s) > -\sum_{j=1}^{\ell} c_j$ and we may switch it with summation in this region:

$$\begin{aligned}
&\int_0^\infty \omega_{h_1} \left(\frac{u_1 x i}{N} \right) \cdots \omega_{h_\ell} \left(\frac{u_\ell x i}{N} \right) x^{s-1} dx = \int_0^\infty \lim_{T \rightarrow \infty} \prod_{j=1}^{\ell} \left(\sum_{v_j=1}^{MT} (\mathcal{F}_M h_j)(v_j) e^{-\frac{2\pi u_j v_j x}{M}} \right) x^{s-1} dx \\
&= \lim_{T \rightarrow \infty} \sum_{\substack{v_j=1 \\ 1 \leq j \leq \ell}}^{MT} \prod_{j=1}^{\ell} (\mathcal{F}_M h_j)(v_j) \int_0^\infty e^{-\frac{2\pi(u_1 v_1 + u_2 v_2 + \cdots + u_\ell v_\ell)x}{N}} x^{s-1} dx = \Gamma(s) \left(\frac{N}{2\pi} \right)^{-s} \sum_{\mathbf{v} \in \mathbb{N}^\ell} \frac{\mathcal{F}_N^{(\ell)} h(\mathbf{v})}{\langle \mathbf{u}, \mathbf{v} \rangle^s},
\end{aligned}$$

where we respect the rearrangement $(T_{M,p})_{p \in \mathbb{N}}$ in the last sum. This proves the lemma. $\square$

In the following, we will look at L -functions corresponding to higher weak modular forms. Let f be a modular form in $M_k(\Gamma_1(M, N))$ with Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^{\frac{n}{N}}.$$

Then we remember that its corresponding L -function is given by

$$L(f; s) = \sum_{n=1}^{\infty} a(n) n^{-s}.$$

One can show that this series converges on some half plane and we have the relation

$$\left(\frac{2\pi}{N} \right)^{-s} \Gamma(s) L(f; s) = \int_0^\infty (f(ix) - a(0)) x^{s-1} dx.$$

Following this type of Mellin transformation, one can show that each $L(f; s)$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation. The next proposition makes a statement about the L -functions of certain products of weak modular forms defined in Definition 1.4.

Proposition 3.11 *Let $f \in U_k(\Gamma_1(M, N))$ be a higher weak modular form, such that*

$$f = \sum_{\alpha=1}^R \mu_{\alpha} \vartheta_{k_1}(\omega_{h_{\alpha,1}} \otimes \omega_{t_{\alpha,1}}) \cdots \vartheta_{k_{\ell}}(\omega_{h_{\alpha,\ell}} \otimes \omega_{t_{\alpha,\ell}}).$$

Here we assume that $\text{sgn}(h_{\alpha,j} t_{\alpha,j}) = (-1)^{k_j}$ for all $j = 1, \dots, \ell$. Then, for all complex numbers s with $\text{Re}(s) > |\mathbf{k}|$, we have

$$L(f; s) = \sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^{\ell} \times \mathbb{N}^{\ell}} a(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s}, \quad (3.2)$$

where the coefficients $a(\mathbf{u}, \mathbf{v})$ are given by

$$a(\mathbf{u}, \mathbf{v}) = 2^{\ell} N^{\ell-|\mathbf{k}|} \Pi_{\mathbf{k}}(\mathbf{u}) \sum_{\alpha=1}^R \mu_{\alpha} \prod_{j=1}^{\ell} t_{\alpha,j}(u_j) (\mathcal{F}_M h_{\alpha,j})(v_j). \quad (3.3)$$

Proof The series on the right of (3.2) converges absolutely on the half plane $\{s \in \mathbb{C} \mid \sigma > |\mathbf{k}|\}$, since

$$|a(\mathbf{u}, \mathbf{v})| \ll u_1^{k_1-1} \cdots u_{\ell}^{k_{\ell}-1}$$

and on the other hand, for all $\varepsilon > 0$,

$$(u_1 v_1 + \cdots u_{\ell} v_{\ell})^{|\mathbf{k}|+\varepsilon} = \prod_{j=1}^{\ell} (u_1 v_1 + \cdots u_{\ell} v_{\ell})^{k_j+\frac{\varepsilon}{\ell}} \geq \prod_{j=1}^{\ell} (u_j v_j)^{k_j+\frac{\varepsilon}{\ell}},$$

and hence

$$\begin{aligned} \sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^{\ell} \times \mathbb{N}^{\ell}} |a(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-|\mathbf{k}|-\varepsilon}| &\ll \sum_{\substack{u_j=1 \\ 1 \leq j \leq \ell}}^{\infty} \sum_{\substack{v_j=1 \\ 1 \leq j \leq \ell}}^{\infty} \frac{u_1^{k_1-1} \cdots u_{\ell}^{k_{\ell}-1}}{(u_1 v_1 + \cdots u_{\ell} v_{\ell})^{|\mathbf{k}|+\varepsilon}} \\ &\leq \sum_{u_j=1}^{\infty} \sum_{v_j=1}^{\infty} \frac{u_1^{k_1-1} \cdots u_{\ell}^{k_{\ell}-1}}{\prod_{j=1}^{\ell} (u_j v_j)^{k_j+\frac{\varepsilon}{\ell}}} = \zeta\left(1 + \frac{\varepsilon}{\ell}\right)^{\ell} \prod_{j=1}^{\ell} \zeta\left(k_j + \frac{\varepsilon}{\ell}\right) < \infty. \end{aligned}$$

Since $t_j(0) = h_j(0) = 0$ for all $1 \leq j \leq \ell$, all involved weak functions have a removable singularity in $z = 0$ and so have their product. We have for all $s \in \mathbb{C}$

$$\left(\frac{2\pi}{N}\right)^{-s} \Gamma(s) L(f; s) = \int_0^{\infty} \sum_{\alpha=1}^R \mu_{\alpha} (\vartheta_{k_1}(\omega_{h_{\alpha,1}} \otimes \omega_{t_{\alpha,1}}) \cdots \vartheta_{k_{\ell}}(\omega_{h_{\alpha,\ell}} \otimes \omega_{t_{\alpha,\ell}}))(ix) x^{s-1} dx.$$

Hence, due to absolute convergence, we obtain for all s with $\sigma > |\mathbf{k}|$:

$$L(f; s) = \lim_{T \rightarrow \infty} \frac{1}{\Gamma(s)} \left(\frac{2\pi}{N}\right)^s 2^{\ell} N^{\ell-|\mathbf{k}|} \sum_{\substack{u_j=1 \\ 1 \leq j \leq \ell}}^T \sum_{\alpha=1}^R \mu_{\alpha} u_1^{k_{\alpha,1}-1} \cdots u_{\ell}^{k_{\alpha,\ell}-1} t_{\alpha,1}(u_1) \cdots t_{\alpha,\ell}(u_{\ell})$$

$$\times \int_0^\infty \omega_{h_{\alpha,1}} \left(\frac{u_1 x i}{N} \right) \cdots \omega_{h_{\alpha,\ell}} \left(\frac{u_\ell x i}{N} \right) x^{s-1} dx.$$

Together with Lemma 3.10 we obtain, that this equals

$$\begin{aligned} & 2^\ell N^{\ell-|k|} \sum_{\substack{u_j, v_j=1 \\ 1 \leq j \leq \ell}}^\infty \sum_{\alpha=1}^R \mu_\alpha \left(\prod_{j=1}^\ell u_j^{k_j-1} t_{\alpha,j}(u_j) (\mathcal{F}_M h_{\alpha,j})(v_j) \right) (u_1 v_1 + \cdots + u_\ell v_\ell)^{-s} \\ &= 2^\ell N^{\ell-|k|} \sum_{\substack{u_j, v_j=1 \\ 1 \leq j \leq \ell}}^\infty \prod_{j=1}^\ell u_j^{k_j-1} \sum_{\alpha=1}^R \mu_\alpha \left(\prod_{j=1}^\ell t_{\alpha,j}(u_j) (\mathcal{F}_M h_{\alpha,j})(v_j) \right) (u_1 v_1 + \cdots + u_\ell v_\ell)^{-s}. \end{aligned}$$

This proves the proposition. $\square$

Proposition 3.11 provides us coefficients $a(\mathbf{u}, \mathbf{v})$ that belong to splittings of $L(f; s)$ over $\mathcal{Q} = \mathbb{N}^\ell \times \mathbb{N}^\ell$. We may use this to define a linear map from “splitting coefficients” to modular forms. Firstly, consider the vector space

$$A_k := \left\{ a : \mathbb{N}^\ell \times \mathbb{N}^\ell \longrightarrow \mathbb{C} \mid s \in \mathbb{C}, \operatorname{Re}(s) > |k| : \sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell} |a(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s}| < \infty \right\}.$$

Secondly, look at the subspace $B_{M,N,k} \subset A_k$ of functions that generate L -functions of higher weak modular forms in $U_k(\Gamma_1(M, N))$. The linear map $B_{M,N,k} \rightarrow \mathcal{O}(\{\operatorname{Re}(s) > |k|\})$ into holomorphic functions with $a(\mathbf{u}, \mathbf{v}) \mapsto \sum a(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s}$ induces a linear map $\psi_{M,N,k} : B_{M,N,k} \rightarrow U_k(\Gamma_1(M, N))$. Note that this map is well-defined, since the L -function of a modular form is uniquely determined, and of course surjective (Proposition 3.11 provides proper pre-images). However, this map does not have to be injective since there is obviously no Identity theorem for Dirichlet series of the form $\sum a(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s}$. This lack of uniqueness is measured by the kernel $\Lambda_{M,N,k} := \ker(\psi_{M,N,k})$. So, when considering the coefficients $a(\mathbf{u}, \mathbf{v})$ in (3.3), all coefficients $b(\mathbf{u}, \mathbf{v})$ generating $L(f; s) = \sum b(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s}$ (assuming absolute convergence in $\operatorname{Re}(s) > |k|$), are contained in the translated set $a + \Lambda_{M,N,k}$. All of this can be summarized in the following proposition.

Proposition 3.12 *Let $f \in U_k(\Gamma_1(M, N))$. Then $a + \Lambda_{M,N,k}$ consists of all coefficient functions $b(\mathbf{u}, \mathbf{v})$, such that*

$$L(f; s) = \sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell} b(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s}$$

and the series converges absolutely for $\operatorname{Re}(s) > |k|$. In particular, all of them define splittings of $L(f; s)$ ($S = \langle \mathbb{N}^\ell, \mathbb{N}^\ell \rangle \subset \mathbb{N}$) over $\mathcal{Q} = \mathbb{N}^\ell \times \mathbb{N}^\ell$ equipped with the integer map $|\langle \mathbf{u}, \mathbf{v} \rangle|_{\mathcal{Q}} := \langle \mathbf{u}, \mathbf{v} \rangle$.

Of course, when using Proposition 3.4, one could reconstruct the original ordinary Dirichlet series with a standard rearrangement. However, in the following we study a completely different rearrangement $(U_{M,N,m})_{m \in \mathbb{N}}$ that arises from the results in the previous section. With this we want to extend the region of convergence of the series $L(f; s) = \sum a(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s}$ naturally. Fix an integer N . We define for $p, q \in \mathbb{N}$

$$T_{M,N,p,q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell \mid N(p-1) < \max(\mathbf{u}) \leq Np, M(q-1) < \max(\mathbf{v}) \leq Mq\}.$$

Note that the $T_{N,p,q}$ define a disjoint covering of $\mathbb{N}^\ell \times \mathbb{N}^\ell$. We then define the sub-rearrangement

$$\begin{aligned} U_{M,N,1} &:= T_{M,N,1,1}, \\ U_{M,N,2} &:= T_{M,N,1,2} \cup T_{M,N,2,1} \cup T_{M,N,2,2}, \\ U_{M,N,3} &:= T_{M,N,1,3} \cup T_{M,N,2,3} \cup T_{M,N,3,1} \cup T_{M,N,3,2} \cup T_{M,N,3,3}, \end{aligned}$$

and so on. After Proposition 3.12 provided us some natural splittings (in fact, all Dirichlet series of $L(f; s)$ arising from products of weak functions for $\mathbf{k}$ and not from the usual Fourier series), we show that we can improve the region of convergence by rearranging the splittings by $U_{M,N,m}$.

Theorem 3.13 *Let $N > 1$ and $\ell \geq 1$ be integers and $h_j \in (\mathbb{F}_{M-1})_0^{\mathbb{C}_0}$ with $\mathcal{F}_M h_j \in [M, c_j]$ and the $t_j \in [N, d_j]$ be even or odd N -periodic functions for $1 \leq j \leq \ell$ and some non-negative integers c_j and d_j . We further assume that we have $(h_j t_j) = (-1)^{k_j}$ for every $1 \leq j \leq \ell$. Consider the modular form*

$$f(\tau) = \prod_{j=1}^{\ell} \vartheta_{k_j}(\omega_{h_j} \otimes \omega_{t_j}; \tau) \in U_{\mathbf{k}}(\Gamma_1(M, N)).$$

For all values $s \in \mathbb{C}$ with $\operatorname{Re}(s) > \max(|\mathbf{k}| - \ell - d, -c)$, where $c = \sum_{j=1}^{\ell} c_j$ and $d = \sum_{j=1}^{\ell} d_j$, we have the series representation

$$L(f; s) = 2^\ell N^{\ell-|\mathbf{k}|} \sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell} \Pi_{\mathbf{k}}(\mathbf{u}) t(\mathbf{u}) (\mathcal{F}_N^{(\ell)} h)(\mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s},$$

where $t(\mathbf{u}) := t_1(u_1) \cdots t_\ell(u_\ell)$ and $(\mathcal{F}_N^{(\ell)} h)(\mathbf{v}) := (\mathcal{F}_N h_1)(v_1) \cdots (\mathcal{F}_N h_\ell)(v_\ell)$ is the multi-dimensional Fourier transform. The summation respects the rearrangement $(U_{M,N,m})_{m \in \mathbb{N}}$. In particular, we have

$$\inf_{b \in a + \Lambda_{M,N,k}} \tilde{\sigma} \left(\sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell} b(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s} \right) \leq \max(|\mathbf{k}| - \ell - d, -c). \quad (3.4)$$

Here, $a(\mathbf{u}, \mathbf{v})$ are the standard coefficients obtained in Proposition 3.11, for the set $a + \Lambda_{M,N,k}$ see also Proposition 3.12.

Proof The series on the right of (3.2) converges absolutely for all s with $\operatorname{Re}(s) > |\mathbf{k}|$. Since $t_j(0) = h_j(0) = 0$ for all $1 \leq j \leq \ell$, all involved weak functions have a removable singularity in $z = 0$ and so have their product. We have for all $s \in \mathbb{C}$

$$\left(\frac{2\pi}{N} \right)^{-s} \Gamma(s) L(f; s) = \int_0^\infty f(ix) x^{s-1} dx = \int_0^\infty \prod_{j=1}^{\ell} \vartheta_{k_j}(\omega_{h_j} \otimes \omega_{t_j}; xi) x^{s-1} dx.$$

The functions $t_1, \dots, t_\ell$ have heights $d_1, \dots, d_\ell$ which means by Theorem 2.14 that there is a constant $C > 0$ such that for all $T \in \mathbb{N}$ and $0 \leq x \leq 1$:

$$\left| \sum_{\substack{u_j=1 \\ 1 \leq j \leq \ell}}^{NT} u_1^{k_1-1} \cdots u_\ell^{k_\ell-1} t_1(u_1) \cdots t_\ell(u_\ell) \omega_{h_1} \left(\frac{u_1 xi}{N} \right) \cdots \omega_{h_\ell} \left(\frac{u_\ell xi}{N} \right) x^{s-1} \right|$$

$$= x^{\sigma-1} \prod_{j=1}^{\ell} \left| \sum_{u_j=1}^{NT} u_j^{k_j-1} t_j(u_j) \omega_{h_j} \left(\frac{u_j x i}{N} \right) \right| \leq C x^{\sigma+d-1-(|k|-\ell)}$$

and the right hand side is an integrable majorant for $\sigma > |k| - \ell - d$. For these values we therefore have dominated convergence on the interval $[0, 1]$ and uniform convergence on the interval $[1, \infty)$, hence we obtain for $\operatorname{Re}(s) > |k| - \ell - d$

$$\begin{aligned} L(f; s) &= \frac{1}{\Gamma(s)} \left(\frac{2\pi}{N} \right)^s \int_0^{\infty} \lim_{T \rightarrow \infty} 2^{\ell} N^{\ell-|k|} \sum_{\substack{u_j=1 \\ 1 \leq j \leq \ell}}^{NT} u_1^{k_1-1} \cdots u_{\ell}^{k_{\ell}-1} t_1(u_1) \cdots t_{\ell}(u_{\ell}) \\ &\quad \times \omega_{h_1} \left(\frac{i x u_1}{N} \right) \cdots \omega_{h_{\ell}} \left(\frac{i x u_{\ell}}{N} \right) x^{s-1} dx \\ &= \lim_{T \rightarrow \infty} \frac{1}{\Gamma(s)} \left(\frac{2\pi}{N} \right)^s 2^{\ell} N^{\ell-|k|} \sum_{\substack{u_j=1 \\ 1 \leq j \leq \ell}}^{NT} u_1^{k_1-1} \cdots u_{\ell}^{k_{\ell}-1} t_1(u_1) \cdots t_{\ell}(u_{\ell}) \\ &\quad \times \int_0^{\infty} \omega_{h_1} \left(\frac{i x u_1}{N} \right) \cdots \omega_{h_{\ell}} \left(\frac{i x u_{\ell}}{N} \right) x^{s-1} dx. \end{aligned}$$

In the proof of the Dominated convergence theorem the upper bound was independent of the choice of the partial sums for the series of ω . Hence, together with Lemma 3.10 we obtain for $\operatorname{Re}(s) > -c$:

$$L(f; s) = \lim_{T \rightarrow \infty} 2^{\ell} N^{\ell-|k|} \sum_{\substack{u_j=1 \\ 1 \leq j \leq \ell}}^{NT} u_1^{k_1-1} \cdots u_{\ell}^{k_{\ell}-1} t_1(u_1) \cdots t_{\ell}(u_{\ell}) \sum_{\substack{v_j=1 \\ 1 \leq j \leq \ell}}^{MT} \frac{\mathcal{F}_N^{(\ell)} h(\mathbf{v})}{(u_1 v_1 + \cdots + u_{\ell} v_{\ell})^s}.$$

Since the order of summation in the partial sums respects the rearrangement $(U_{M,N,m})_{m \in \mathbb{N}}$, note that

$$\sum_{\substack{u_j=1 \\ 1 \leq j \leq \ell}}^{NT} \sum_{\substack{v_j=1 \\ 1 \leq j \leq \ell}}^{MT} - \sum_{\substack{u_j=1 \\ 1 \leq j \leq \ell}}^{N(T-1)} \sum_{\substack{v_j=1 \\ 1 \leq j \leq \ell}}^{M(T-1)} = \sum_{U_{M,N,T}}.$$

Since $\{\operatorname{Re}(s) > |k| - \ell - d\} \cap \{\operatorname{Re}(s) > -c\} = \{\operatorname{Re}(s) > \max(|k| - \ell - d, -c)\}$ and (3.4) follows with

$$\tilde{\sigma} \left(\sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^{\ell} \times \mathbb{N}^{\ell}} a(\mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s} \right) \leq \max(|k| - \ell - d, -c),$$

the theorem is proved. $\square$

From this we obtain a much more general result as (ii) presented in the above examples.

Corollary 3.14 *Let $t \neq 0$ be N -periodic and be an element of $[N, d]$. Then the series*

$$\lim_{T \rightarrow \infty} \sum_{n=1}^{NT} t(n) n^{-s} = \sum_{r=0}^{\infty} \sum_{\ell=1}^N t(\ell) (Nr + \ell)^{-s}$$

converges for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > -d$ to a holomorphic function $L(t, s)$. In particular, $L(t, -\alpha) = 0$ for all $0 \leq \alpha < d$.

Proof Put $k = d + 1$. Choose $h \neq 0$ such that $\operatorname{sgn}(t \cdot h) = (-1)^k$. Then we obtain with Theorem 3.13 that the series

$$\lim_{T \rightarrow \infty} N^{1-k} \sum_{u=1}^{NT} \sum_{v=1}^{\infty} \frac{u^{k-1} t(u) (\mathcal{F}_N h)(v)}{(uv)^s} = \lim_{T \rightarrow \infty} N^{1-k} \left(\sum_{u=1}^{NT} u^{d-s} t(u) \right) \left(\sum_{v=1}^{\infty} \frac{(\mathcal{F}_N h)(v)}{v^s} \right)$$

converges for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$ to a holomorphic function. Since

$$\lim_{T \rightarrow \infty} \sum_{u=1}^{NT} u^{d-s} t(u) = \sum_{r=0}^{\infty} \sum_{\ell=1}^N t(\ell) (Nr + \ell)^{-s+d},$$

the claim follows. $\square$

One consequence of this observation is an application to infinite products.

Example 3.15 Consider the function

$$a_4(n) = \begin{cases} -1, & \text{if } n \equiv \pm 1 \pmod{4}, \\ 2, & \text{if } n \equiv 2 \pmod{4}, \\ 0, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Then a_4 has height 1, since obviously $\sum_{j=1}^4 a_4(j) = \sum_{j=1}^4 a_4(j)j = -1 + 4 - 3 = 0$. One sees quickly that

$$f(s) = \sum_{n=1}^{\infty} a_4(n) n^{-s} = (3 \cdot 2^{-s} - 2 \cdot 4^{-s} - 1) \zeta(s).$$

Together with Corollary 3.14 we conclude that

$$\sum_{n=0}^{\infty} \sum_{j=1}^4 a_4(4n+j) (4n+j)^{-s}$$

converges to a holomorphic function for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > -1$ and we find

$$\sum_{n=0}^{\infty} (\log(4n+1) - 2 \log(4n+2) + \log(4n+3)) = f'(0).$$

Since $\zeta(0) = -\frac{1}{2}$, we obtain

$$\prod_{n=0}^{\infty} \frac{(4n+1)(4n+3)}{(4n+2)^2} = \frac{1}{\sqrt{2}}.$$

Remark 3.16 With a rearranged splitting

$$\sum_{n=1}^{\infty} ((2n-1)^{-s} - 2(2n)^{-s} + (2n+1)^{-s}) = 2(1 - 2^{1-s})\zeta(s) - 1,$$

that converges for $\operatorname{Re}(s) > -1$, we similarly conclude (when using $\zeta'(0) = -\frac{1}{2} \log(2\pi)$) the Wallis product

$$\prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2} = \frac{2}{\pi}.$$

Remark 3.17 Let χ be a non-principal even character modulo N . Then, using the well-known Weierstraß product expansion

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}},$$

we find

$$\prod_{n=0}^{\infty} (Nn+1)(Nn+2)^{\chi(2)}(Nn+3)^{\chi(3)} \cdots (Nn+N-1)^{\chi(N-1)} = \prod_{m=1}^N \Gamma\left(\frac{m}{N}\right)^{\chi(m)}.$$

As a consequence, we obtain the following well-known identity

$$e^{L'(\chi,0)} = \prod_{m=1}^N \Gamma\left(\frac{m}{N}\right)^{\chi(m)}.$$

The next final corollary provides natural generalized Dirichlet series representations for L -functions associated to products of Eisenstein series for non-principal primitive Dirichlet characters.

Corollary 3.18 Let $\chi, \psi : \mathbb{Z}^{\ell} \rightarrow \mathbb{C}^{\times}$ be non-principal, primitive characters modulo M and N , respectively, such that $\chi_j(-1)\psi_j(-1) = (-1)^{k_j}$ for all $j = 1, \dots, \ell$. For all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > \max\left(|k| - \ell - \frac{1}{2} \sum_{j=1}^{\ell} (\psi_j(-1) + 1), -\frac{1}{2} \sum_{j=1}^{\ell} (\chi_j(-1) + 1)\right)$ we have

$$L\left(\prod_{j=1}^{\ell} E_{k_j}(\chi_j, \psi_j; \tau), s\right) = \left(-\frac{2\pi i}{N}\right)^{|k|} \prod_{j=1}^{\ell} \frac{2\mathcal{G}(\psi_j)}{(k_j - 1)!} \sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^{\ell} \times \mathbb{N}^{\ell}} \Pi_{\mathbf{k}}(\mathbf{u}) \overline{\psi}(\mathbf{u}) \chi(\mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s},$$

where the summation respects the rearrangement $(U_{M,N,m})_{m \in \mathbb{N}}$.

Proof Since all characters are primitive, we have

$$E_{k_j}(\chi_j, \psi_j; \tau) = \frac{\chi_j(-1)(-2\pi i)^{k_j} \mathcal{G}(\psi_j)}{N(k_j - 1)! \mathcal{G}(\overline{\chi_j})} \vartheta_{k_j}(\omega_{\overline{\chi_j}} \otimes \omega_{\overline{\psi_j}}; \tau).$$

Hence we obtain with Theorem 3.13

$$L\left(\prod_{j=1}^{\ell} E_{k_j}(\chi_j, \psi_j; \tau), s\right) = \lambda_1 \cdots \lambda_{\ell} 2^{\ell} N^{\ell - |k|} \sum_{(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^{\ell} \times \mathbb{N}^{\ell}} \Pi_{\mathbf{k}}(\mathbf{u}) \overline{\psi}(\mathbf{u}) (\mathcal{F}_N^{(\ell)} \overline{\chi})(\mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle^{-s},$$

where

$$\lambda_j = \frac{\chi_j(-1)(-2\pi i)^{k_j} \mathcal{G}(\psi_j)}{N(k_j - 1)! \mathcal{G}(\overline{\chi_j})}.$$

We can simplify the expression $(\mathcal{F}_N^{(\ell)})(\overline{\chi})$ by

$$\left(\mathcal{F}_N^{(\ell)}\right)(\overline{\chi})(\mathbf{v}) = \chi(\mathbf{v}) \left(\mathcal{F}_N^{(\ell)} \overline{\chi}\right)(\mathbf{1}) = \chi(\mathbf{v}) \prod_{j=1}^{\ell} \chi_j(-1) \mathcal{G}(\overline{\chi_j}),$$

so we obtain

$$\lambda_1 \cdots \lambda_{\ell} N^{\ell-|\mathbf{k}|} \left(\mathcal{F}_N^{(\ell)} \overline{\chi}\right)(\mathbf{v}) = \left(-\frac{2\pi i}{N}\right)^{|\mathbf{k}|} \prod_{j=1}^{\ell} \frac{\mathcal{G}(\psi_j)}{(k_j-1)!} \chi(\mathbf{v}).$$

The extended domain of convergence follows, because of the rearrangement, with Theorem 3.13 and the fact that the height of ψ_j is given by $\frac{1}{2}(\psi_j(-1) + 1)$. $\square$

Note that this representation of the L -function of the considered product is more natural since it is a direct generalization of the formula for $L(E_k(\chi, \psi; \tau), s)$ in the case $\ell = 1$, where the series directly splits into a product of two Dirichlet L -functions:

$$\frac{2(-2\pi i)^k \mathcal{G}(\psi)}{N^k(k-1)!} \sum_{(u,v) \in \mathbb{N} \times \mathbb{N}} u^{k-1} \overline{\psi}(u) \chi(v) (uv)^{-s} = \frac{2(-2\pi i)^k \mathcal{G}(\psi)}{N^k(k-1)!} L(\overline{\psi}; s-k+1) L(\chi; s).$$

The region of convergence may be improved when summing with respect to the rearrangement $(U_{M,N,m})_{m \in \mathbb{N}}$. In this case we end up with

$$\sum_{(u,v) \in \mathbb{N} \times \mathbb{N}} u^{k-1} \overline{\psi}(u) \chi(v) (uv)^{-s} = \lim_{T \rightarrow \infty} \sum_{u=1}^{NT} \sum_{v=1}^{MT} u^{k-1} \overline{\psi}(u) \chi(v) (uv)^{-s}.$$

By Corollary 3.14 this converges, if $k \geq 2$, for $\operatorname{Re}(s) > k-1$ if ψ is odd and for $\operatorname{Re}(s) > k-2$ if ψ is even (and of course, non-principal). In the case $k = 1$ we have convergence in the region $\operatorname{Re}(s) > -1$ if and only if ψ and χ are both even and for $\operatorname{Re}(s) > 0$ else. Finally, we give an example.

Remark 3.19 Let χ be a primitive even Dirichlet character modulo $N > 1$. We then look on the Eisenstein series $E_2(\chi, \chi; \tau)$ of weight $k = 2$ and define $f(\tau) := E_2(\chi, \chi; \tau)^2$. Then f is a modular form of weight 4 for the group $\Gamma(N^2)$ and vanishes in the cusps $z = 0$ and $z = i\infty$, hence its L -function $L(f; s)$ is entire. We are especially interested in the critical value $L(f; 1)$. With Corollary 3.18 we obtain

$$L(f; 1) = \frac{64\pi^4 \mathcal{G}(\chi)^2}{N^4} \lim_{T \rightarrow \infty} \sum_{u_1, u_2, v_1, v_2=1}^{NT} \frac{u_1 u_2 \overline{\chi}(u_1) \overline{\chi}(u_2) \chi(v_1) \chi(v_2)}{u_1 v_1 + u_2 v_2}.$$

Note that this converges, since $|\mathbf{k}| = 4$, $\ell = 2$ and $\frac{1}{2} \sum_{j=1}^2 (\chi(-1) + 1) = 2$, and $4 - 2 - 2 = 0 < 1$.

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