Computational Methods and Function Theory
https://doi.org/10.1007/s40315-024-00534-7

The Impact of the Limit q-Durrmeyer Operator on Continuous Functions

Övgü Gürel Yılmaz[1] · Sofiya Ostrovska[2] · Mehmet Turan[2]

Received: 28 May 2023 / Accepted: 26 September 2023

Abstract

The limit q-Durrmeyer operator, $D_{\infty,q}$, was introduced and its approximation properties were investigated by Gupta (Appl. Math. Comput. 197(1):172–178, 2008) during a study of q-analogues for the Bernstein–Durrmeyer operator. In the present work, this operator is investigated from a different perspective. More precisely, the growth estimates are derived for the entire functions comprising the range of $D_{\infty,q}$. The interrelation between the analytic properties of a function f and the rate of growth for $D_{\infty,q} f$ are established, and the sharpness of the obtained results are demonstrated.

Keywords q-Durrmeyer operator · Analytic function · Entire function · Growth estimates

Mathematics Subject Classification 30D15 · 30B40 · 47B38

1 Introduction

The significant influence of the Bernstein polynomials on modern mathematics—not only theoretical, but also applied and computational—brought about the emergence

Communicated by Feng Dai.

Övgü Gürel Yılmaz, Sofiya Ostrovska and Mehmet Turan contributed equally to this work.

✉ Övgü Gürel Yılmaz
ovgu.gurelyilmaz@erdogan.edu.tr

Sofiya Ostrovska
sofia.ostrovska@atilim.edu.tr

Mehmet Turan
mehmet.turan@atilim.edu.tr

1 Department of Mathematics, Recep Tayyip Erdogan University, 53100 Rize, Turkey

2 Department of Mathematics, Atilim University, Incek, 06830 Ankara, Turkey

Published online: 09 April 2024

of its numerous versions and modifications. See, for example, [2, 3, 12]. While the Bernstein polynomials serve to approximate the continuous functions on [0, 1], the Kantorovich polynomials constructed with respect to the Bernstein basis are applicable for the approximation of integrable functions. Kantorovich's breakthrough idea was further developed by Durrmeyer [7] and Derriennic [6]. The latter proved that the Bernstein–Durrmeyer polynomials approximate functions in $L_1[0,1]$, and also generate self-adjoint operators in $L_2[0,1]$.

With the increasing role of the q-Calculus (see, e.g. [1, 4, 5, 14]), the q-analogues of various Bernstein-type operators have come to the fore. The reader is referred to [3, 8, 15]. New versions of these operators, targeting a wide spectrum of various problems, are continuously coming out.

In 2008, Gupta [9] introduced a simple q-analogue of the Bernstein–Durrmeyer operators, denoted by $D_{n,q}$, and studied its approximation properties. One of the properties that he proved was that $\{D_{n,q}\}$ converges to the limit operator $D_{\infty,q}$ in the strong operator topology on $C[0,1]$. More results on the q-Durrmeyer operator have been obtained in [10, 13].

In the present work, further investigation is carried out concerning the limit q-Bernstein–Durrmeyer operator. Distinct from the preceding studies on the subject, this paper is focused on the analytic properties that the image of $f \in C[0,1]$ possesses under the operator $D_{\infty,q}$. Here, it is proved that, for each $f \in C[0,1]$, the function $D_{\infty,q} f$ admits an analytic continuation from [0, 1] to the whole complex plane $\mathbb{C}$. The growth estimates of the entire function $D_{\infty,q} f$ are provided, along with the interconnection between the growth of $D_{\infty,q} f$ and the behaviour of f. The sharpness of the obtained results is demonstrated.

To present the results, let us recall the necessary notation and definitions. The q-Pochhammer symbol denotes, for each $a \in \mathbb{C}$,

$$(a;q)_0 := 1, \qquad (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \qquad (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

The Euler Identities

$$(z;q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k} z^k, \quad |q|<1, \tag{1.1}$$

and

$$\frac{1}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k}, \quad |q|<1, \quad |z|<1, \tag{1.2}$$

will be used throughout. See [1, Ch. 10, Cor. 10.2.2].

The q-integral over an interval $[0, a]$, first introduced by Thomae [16] and later by Jackson [11], is defined as

$$\int_0^a f(t)\, d_q t := (1-q)a \sum_{j=0}^{\infty} q^j f(aq^j). \tag{1.3}$$

Definition 1.1 [9] Let $q \in (0, 1)$, $f \in C[0, 1]$. *The limit q-Durrmeyer operator* is defined by

$$(D_{\infty,q} f)(x) := D_{\infty,q}(f; x) = \begin{cases} \sum_{k=0}^{\infty} A_{\infty k}(f) p_{\infty k}(q; x), & x \in [0, 1), \\ f(1), & x = 1. \end{cases}$$

where

$$A_{\infty k}(f) := \frac{q^{-k}}{1-q} \int_0^1 f(t) p_{\infty k}(q; qt)\, d_q t, \quad k = 0, 1, \ldots, \tag{1.4}$$

and

$$p_{\infty k}(q; x) = \frac{(x; q)_\infty\, x^k}{(q; q)_k}, \quad k = 0, \ldots. \tag{1.5}$$

As coefficients (1.4) form a bounded sequence whenever $f \in C[0, 1]$, the function $D_{\infty,q} f$ admits an analytic continuation from $[0, 1]$ to the open disc $\{z : |z| < 1\}$. Taking into account (1.3), $A_{\infty k}(f)$ can also be expressed as

$$A_{\infty k}(f) = \frac{(q; q)_\infty}{(q; q)_k} \sum_{j=0}^{\infty} \frac{f(q^j) q^{(k+1)j}}{(q; q)_j}. \tag{1.6}$$

Throughout the paper, the letter C—with or without subscripts—denotes a positive constant whose specific value is of no importance. Subscripts, when used, indicate the dependence of C on certain parameters. It should be pointed out that the same letter may stand for different values. Moreover, if f is analytic in the closed disc $\Delta_r := \{z : |z| \le r\}$, the notation

$$M(r; f) := \max_{z \in \Delta_r} |f(z)|$$

will be employed.

The article is organized as follows: In Sect. 2, the main results are stated, while Sect. 3 contains the auxiliary technical lemmas. Finally, the proofs of the main results appear in Sect. 4.

2 Statement of Results

Theorem 2.1 *For each* $f \in C[0, 1]$, *the function* $(D_{\infty,q} f)(x)$ *admits an analytic continuation from* $[0, 1]$ *as an entire function given by*

$$(D_{\infty,q} f)(z) = \sum_{j=0}^{\infty} \frac{f(q^j)q^j}{(q;q)_j} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} (z;q)_{n+j}. \tag{2.1}$$

The proof of Theorem 2.1 presented in Sect. 4 yields, apart from (2.1), the following corollary:

Corollary 2.2 *The growth of* $D_{\infty,q} f$, *for each* $f \in C[0, 1]$, *enjoys the following estimate:*

$$M(r; D_{\infty,q} f) = O((-r;q)_\infty), \quad r \to \infty. \tag{2.2}$$

It is worth pointing out that coefficients (1.6) can be viewed as the values of the function $g(z) := (qz;q)_\infty \, \rho(z)$ at points $z = q^k$, $k = 0, 1, \ldots,$ where

$$\rho(z) = \sum_{j=0}^{\infty} \frac{f(q^j)q^j}{(q;q)_j} z^j. \tag{2.3}$$

Since $(qz;q)_\infty$ is entire and the series converges in the disc $\{z : |z| < 1/q\}$ for any $f \in C[0, 1]$, it follows that g is analytic in that disc. Clearly, the radius of convergence for ρ can be greater than $1/q$. The representation below of $D_{\infty,q}$ with the help of divided differences of g is important.

Theorem 2.3 *Given* $f \in C[0, 1]$, *let* $g(z) = (qz;q)_\infty \, \rho(z)$, *where* ρ *is defined by* (2.3). *Then,*

$$(D_{\infty,q} f)(z) = \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2} \, g[1; q; \ldots; q^k] z^k, \quad z \in \mathbb{C}.$$

Here, $g[x_0; \ldots; x_k]$ *stands for the divided difference of* g *at the distinct nodes* $x_0, \ldots, x_k$.

This representation allows us to not only refine the estimate of Corollary 2.2, but also establish a connection between the behaviour of f and the growth of its image under $D_{\infty,q}$.

Theorem 2.4 *Let* $R > 1$ *be such that* ρ *is analytic in* Δ_R. *Then,*

$$M(r; D_{\infty,q} f) = o\left(\frac{(-r;q)_\infty}{r^\lambda}\right), \quad r \to \infty,$$

for every $\lambda < (\ln R)/\ln(1/q)$.

As a consequence of Theorem 2.4, the crude estimate (2.2) can be improved. Since ρ is analytic in $\{z : |z| < 1/q\}$, it is possible to assume $\lambda = 0$ in Theorem 2.4 and obtain the following result.

Corollary 2.5 *For any $f \in C[0, 1]$,*

$$M(r; D_{\infty,q} f) = o((-r; q)_\infty), \quad r \to \infty.$$

Corollary 2.6 *If $f(q^j) = O(q^{\alpha j})$, $j \to \infty$, for some $\alpha > 0$, then*

$$M(r; D_{\infty,q} f) = o(r^{-\lambda}(-r; q)_\infty), \quad r \to \infty, \tag{2.4}$$

for all $\lambda < 1 + \alpha$.

Indeed, in this case, ρ is analytic in $\{z : |z| < q^{-1-\alpha}\}$.

Corollary 2.7 *If, for every $\alpha > 0$, the estimate $f(q^j) = o(q^{\alpha j})$, $j \to \infty$ holds, then, for every $\lambda \geq 0$, (2.4) is true.*

The estimate in Theorem 2.4 is sharp as demonstrated by the assertion below.

Theorem 2.8 *For every $\lambda > 1$, there exists $f \in C[0, 1]$ such that*

$$M(r; D_{\infty,q} f) \geq C r^{-\lambda}(-r; q)_\infty, \quad r \to \infty.$$

Theorem 2.4 and Corollaries 2.5–2.7 establish the connection between the radius of convergence for the series (2.3) and the rate of growth for $D_{\infty,q} f$. In a general sense, the greater the radius is, the slower the growth becomes. Approaching the problem from a different angle, the dependence of the growth on the differentiability of f at the origin is addressed in the next assertion. The statement makes it possible to obtain better estimates for $M(r; D_{\infty,q} f)$ than those guaranteed by Theorem 2.4 when f is differentiable at 0 even though the series (2.3) converges only in the smallest admissible disc.

Theorem 2.9 *Let f be m times differentiable at 0 from the right. Then,*

$$M(r; D_{\infty,q} f) = o(r^{-\lambda}(-r; q)_\infty), \quad r \to \infty, \tag{2.5}$$

for all $\lambda < 1 + m$.

Corollary 2.10 *If f is infinitely differentiable at 0 from the right, then (2.5) holds for all $\lambda > 0$. In particular, (2.5) is valid whenever f is analytic in a neighbourhood of 0.*

3 Auxiliary Results

In what comes next, the function τ given by

$$\tau(z) = (z;q)_\infty \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k^2}, \quad |z| < 1,$$

plays a key role.

Lemma 3.1 *The function τ admits an analytic continuation from the open unit disc as an entire function.*

Proof Consider

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k^2} = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} \frac{(q^{k+1};q)_\infty}{(q;q)_\infty}.$$

By (1.1), with $z = q^{k+1}$, one has

$$(q^{k+1};q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n} (q^{k+1})^n,$$

whence

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k^2} &= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} q^{(k+1)n}}{(q;q)_n} \\ &= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(q^n z)^k}{(q;q)_k}.\end{aligned}$$

By virtue of (1.2), it follows that

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k^2} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} \frac{1}{(q^n z;q)_\infty}, \quad |z| < 1.$$

Consequently, one obtains

$$\begin{aligned}\tau(z) &= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} \frac{(z;q)_\infty}{(q^n z;q)_\infty} \\ &= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} (z;q)_n, \quad |z| < 1. \qquad (3.1)\end{aligned}$$

Now, if $z \in \Delta_R$, then

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} (z;q)_n \right| \leq \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} (1+R)^n < \infty.$$

Hence, $\tau(z)$ is analytic in Δ_R for each $R > 0$ and (3.1) is valid for all $z \in \mathbb{C}$. Therefore, $\tau(z)$ is an entire function. □

Lemma 3.2 *Let $R > 1$ be such that ρ given by (2.3) is analytic in $\{z : |z| \leq R\}$. Then,*

$$\left| g[1; q; \ldots; q^k] \right| \leq C q^{\lambda k}$$

for every $\lambda < (\ln R)/\ln(1/q)$.

Proof It is known that (see for example, [12, Sect. 2.7., p.44, Eq. (4)])

$$g[a_0; \ldots; a_k] = \frac{1}{2\pi i} \oint_L \frac{g(\zeta) d\zeta}{(\zeta - a_0) \ldots (\zeta - a_k)},$$

where L is a positively-oriented, simple and closed curve encircling the distinct points $a_0, \ldots, a_k$ and g is analytic everywhere on and inside L.

Therefore,

$$g[1; q; \ldots; q^k] = \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{g(\zeta) d\zeta}{(\zeta - 1)(\zeta - q) \ldots (\zeta - q^k)}.$$

Now, assume that $0 < \lambda_0 < (\ln R)/\ln(1/q)$, that is, $1 < q^{-\lambda_0} < R$. Two cases will be considered:

Case 1. If $q^{-\lambda_0} \leq R - 1$, then $g[1; q; \ldots; q^k]$ can be estimated as

$$\begin{aligned}
\left| g[1; q; \ldots; q^k] \right| &\leq \frac{1}{2\pi} \cdot \frac{M(R; g)}{(R-1)(R-q) \ldots (R-q^k)} \cdot 2\pi R \\
&\leq \frac{M(R; g) R}{(R-1)^{k+1}} \leq 2M(R; g) q^{\lambda_0 k}.
\end{aligned}$$

Case 2. If $R - 1 < q^{-\lambda_0} \leq R$, then opt for $m_0 \in \mathbb{N}_0$ such that $R - q^m > q^{-\lambda_0}$ whenever $m \geq m_0$. Then, for $k \geq m_0$, one has

$$\begin{aligned}
\left| g[1; q; \ldots; q^k] \right| &\leq \frac{M(R; g) R}{(R-1) \cdots (R - q^{m_0 - 1})(R - q^{m_0}) \cdots (R - q^k)} \\
&\leq \frac{M(R; g) R}{(R-1) \cdots (R - q^{m_0 - 1})} \cdot \frac{1}{(R - q^{m_0})^{k - m_0 + 1}} \\
&\leq C_{R,q,g} \frac{1}{(R - q^{m_0})^k} < C q^{\lambda_0 k}, \quad k \geq m_0.
\end{aligned}$$

As a result, $\left|g[1; q; \ldots; q^k]\right| \le C q^{\lambda_0 k}$ for all k, possibly with a different C.

Combining the outcomes of the two cases yields $\left|g[1; q; \ldots; q^k]\right| \le C q^{\lambda_0 k}$, and, in turn, $\left|g[1; q; \ldots; q^k]\right| \le C q^{\lambda k}$ for all $\lambda \le \lambda_0$. Since λ_0 has been chosen arbitrarily, it follows that the latter inequality holds for all $\lambda < (\ln R)/\ln(1/q)$ as stated. □

4 Proofs of Main Results

Proof of Theorem 2.1 Using (1.6), one obtains

$$(D_{\infty,q} f)(z) = \sum_{k=0}^{\infty} \left(\frac{(q,q)_\infty}{(q,q)_k} \sum_{j=0}^{\infty} \frac{f(q^j) q^{(k+1)j}}{(q,q)_j} \right) p_{\infty k}(q; z), \quad |z| < 1.$$

Recalling (1.5) leads to

$$\begin{aligned}(D_{\infty,q} f)(z) &= \sum_{k=0}^{\infty} \frac{(q;q)_\infty}{(q;q)_k} \sum_{j=0}^{\infty} \frac{f(q^j) q^{(k+1)j}}{(q;q)_j} \frac{(z;q)_\infty z^k}{(q;q)_k} \\ &= (q;q)_\infty (z;q)_\infty \sum_{j=0}^{\infty} \frac{f(q^j) q^j}{(q;q)_j} \sum_{k=0}^{\infty} \frac{(q^j z)^k}{(q;q)_k^2} \\ &= (q;q)_\infty (z;q)_\infty \sum_{j=0}^{\infty} \frac{f(q^j) q^j}{(q;q)_j} \frac{\tau(q^j z)}{(q^j z;q)_\infty}, \\ &= (q;q)_\infty \sum_{j=0}^{\infty} \frac{f(q^j) q^j}{(q;q)_j} (z;q)_j \tau(q^j z), \quad |z| < 1.\end{aligned}$$

By (3.1),

$$\tau(q^j z) = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} (q^j z; q)_n,$$

and, hence,

$$\begin{aligned}(D_{\infty,q} f)(z) &= \sum_{j=0}^{\infty} \frac{f(q^j) q^j}{(q;q)_j} (z;q)_j \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} (q^j z; q)_n \\ &= \sum_{j=0}^{\infty} \frac{f(q^j) q^j}{(q;q)_j} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} (z;q)_{j+n}, \quad |z| < 1. \qquad (4.1)\end{aligned}$$

Since, for $R > 0$ and $z \in \Delta_R$, one has $|(z;q)_{j+n}| \le (-R;q)_\infty$ for all $j, n \in \mathbb{N}_0$, the series in (4.1) converges uniformly in any closed disc Δ_R. Therefore,

$$\left|\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n}(z;q)_{j+n}\right| \le (-R;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = (-R;q)_\infty(-q;q)_\infty$$

which implies that, when $z \in \Delta_R$,

$$\begin{aligned}\left|(D_{\infty,q}f)(z)\right| &\le (-R;q)_\infty(-q;q)_\infty \sum_{j=0}^{\infty} \frac{\left|f(q^j)\right| q^j}{(q;q)_j}\\ &\le \|f\|_{C[0,1]}(-R;q)_\infty \frac{(-q;q)_\infty}{(q;q)_\infty}\\ &=: C_{f,q}(-R;q)_\infty.\end{aligned}$$

Consequently, $(D_{\infty,q}f)(z)$ is analytic in any disc of radius $R > 0$. Thus, $(D_{\infty,q}f)(z)$ is entire. This completes the proof. □

Proof of Theorem 2.3 Starting from (1.6), one arrives at

$$A_{\infty k}(f) = (q^{k+1};q)_\infty \sum_{j=0}^{\infty} \frac{f(q^j)q^{(k+1)j}}{(q;q)_j} = \left[(qz;q)_\infty\, \rho(z)\right]\Big|_{z=q^k} = g(q^k).$$

Therefore,

$$(D_{\infty,q}f)(z) = (z;q)_\infty \sum_{k=0}^{\infty} g(q^k)\frac{z^k}{(q;q)_k}, \quad |z| < 1/q.$$

Application of Euler's identity (1.1) leads to

$$\begin{aligned}(D_{\infty,q}f)(z) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} g(q^j) z^{k+j}}{(q;q)_k (q;q)_j}\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{k-j} q^{(k-j)(k-j-1)/2} g(q^j) z^k}{(q;q)_{k-j}(q;q)_j}\\ &= \sum_{k=0}^{\infty}(-1)^k q^{k(k-1)/2}\left(\sum_{j=0}^{k} \frac{(-1)^{-j} g(q^j)}{q^{j(j-1)/2}(q;q)_j q^{j(k-j)}(q;q)_{k-j}}\right) z^k,\\ &\quad |z| < \frac{1}{q}.\end{aligned}$$

Employing [12, p. 44, Eq. (3)] with $x_j = q^j$, one arrives at

$$g[1; q; \ldots; q^k] = \sum_{j=0}^{k} \frac{(-1)^{-j} g(q^j)}{q^{j(j-1)/2}(q; q)_j q^{j(k-j)}(q; q)_{k-j}}.$$

Therefore, formula

$$(D_{\infty,q} f)(z) = \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2}\, g[1; q; \ldots; q^k] z^k$$

holds for $|z| < 1/q$ and also in every disc where $D_{\infty,q} f$ possess an analytic continuation. Applying Theorem 2.1, one completes the proof. □

Proof of Theorem 2.4 By Theorem 2.3,

$$(D_{\infty,q} f)(z) = \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2}\, g[1; q; \ldots; q^k] z^k, \quad z \in \mathbb{C}.$$

Select $\lambda < (\ln R)/\ln(1/q)$ and take μ such that $\lambda < \mu < (\ln R)/\ln(1/q)$. Now, the growth of $D_{\infty,q} f$ may be estimated with the help of Lemma 3.2, which implies $|g[1; q; \ldots; q^k]| \leq C q^{\mu k}$. Therefore,

$$\left|(D_{\infty,q} f)(z)\right| \leq C \sum_{k=0}^{\infty} q^{k(k-1)/2} \left(q^{\mu}|z|\right)^k \leq C \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q; q)_k} \left(q^{\mu}|z|\right)^k,$$

and, hence,

$$M(r; D_{\infty,q} f) \leq C(-q^{\mu} r; q)_{\infty}.$$

Recall [17, Eq. (2.6)] that, for r large enough,

$$C_1 \exp\left\{\frac{\ln^2 r}{2\ln\frac{1}{q}} + \frac{\ln r}{2}\right\} \leq (-r; q)_{\infty} \leq C_2 \exp\left\{\frac{\ln^2 r}{2\ln\frac{1}{q}} + \frac{\ln r}{2}\right\}.$$

Consequently,

$$C_1 \frac{(-r; q)_{\infty}}{r^{\mu}} \leq (-q^{\mu} r; q)_{\infty} \leq C_2 \frac{(-r; q)_{\infty}}{r^{\mu}} \tag{4.2}$$

for r large enough.

As a result,

$$M(r; D_{\infty,q} f) = O\left(\frac{(-r; q)_{\infty}}{r^{\mu}}\right), \quad r \to \infty,$$

$$= o\left(\frac{(-r;q)_\infty}{r^\lambda}\right), \quad r \to \infty,$$

as stated. □

Proof of Theorem 2.8 For $\lambda > 1$, set $\alpha = q^{\lambda-1} \in (0, 1)$ and

$$s_j = \sum_{k=0}^{j} \frac{\alpha^k}{(q;q)_{j-k}}, \quad j \in \mathbb{N}_0.$$

Obviously, the sequence $\{s_j\}$ is bounded. In addition, it is increasing because, for $j \in \mathbb{N}_0$,

$$s_{j+1} - s_j = \sum_{k=0}^{j} \alpha^k \left(\frac{1}{(q;q)_{j+1-k}} - \frac{1}{(q;q)_{j-k}}\right) + \alpha^{j+1} > 0.$$

Consequently, $\{s_j\}$ converges. Now, let $f \in C[0, 1]$ be such that $f(q^j) = (q;q)_j s_j$. This is possible due to the fact that $\{(q;q)_j s_j\}$ is convergent as a product of two convergent sequences. For this f, one has

$$\rho(z) = \sum_{j=0}^{\infty} s_j (qz)^j.$$

Evidently, ρ is analytic in $\{z : |z| < 1/q\}$ and

$$\begin{aligned}\rho(z) &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{\alpha^k}{(q;q)_{j-k}}\right)(qz)^j \\ &= \sum_{j=0}^{\infty} \frac{(qz)^j}{(q;q)_j} \sum_{k=0}^{\infty} (\alpha q z)^k \\ &= \frac{1}{(qz;q)_\infty} \cdot \frac{1}{1-\alpha q z}, \quad |z| < \frac{1}{q}.\end{aligned}$$

Hence, $g(z) = \rho(z)(qz;q)_\infty = 1/(1-\alpha q z)$, whence g is analytic in $\{z : |z| < 1/(\alpha q)\}$. Simple calculations reveal:

$$g^{(k)}(z) = \frac{(\alpha q)^k k!}{(1-\alpha q z)^{k+1}}, \quad k \in \mathbb{N}_0.$$

By the Intermediate Value Theorem,

$$g[1; q; \ldots; q^k] = \frac{g^{(k)}(\xi)}{k!}, \quad \xi \in (q^k, 1).$$

Since all $g^{(k)}(x)$ are increasing on [0, 1], there holds

$$g[1; q; \ldots; q^k] \geq \frac{g^{(k)}(q^k)}{k!} = \frac{(\alpha q)^k}{(1-\alpha q^{k+1})^{k+1}} \geq (\alpha q)^k, \quad k \in \mathbb{N}_0.$$

As a result,

$$\begin{aligned} M(r; D_{\infty,q} f) &= \sum_{k=0}^{\infty} q^{k(k-1)/2} g[1; q; \ldots; q^k] r^k \\ &\geq (q; q)_\infty \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q; q)_k} (\alpha q r)^k \\ &= (q; q)_\infty (-\alpha q r; q)_\infty. \end{aligned}$$

Writing $\alpha = q^{\lambda-1}$ and using (4.2), one obtains

$$M(r; D_{\infty,q} f) \geq C r^{-\lambda} (-r; q)_\infty, \quad r \to \infty,$$

which completes the proof. □

Proof of Theorem 2.9 By Taylor's Theorem, one can write

$$f(x) = T_m(x) + S_m(x)$$

where $T_m(x)$ is a polynomial of degree at most m and $S_m(x) = o(x^m)$ as $x \to 0^+$. Since $D_{\infty,q}$ maps a polynomial to a polynomial of the same degree (see [9, Rem. 3]), there holds

$$(D_{\infty,q} f)(z) = P_m(z) + (D_{\infty,q} S_m)(z),$$

where $P_m(z)$ is a polynomial of degree at most m and, as such,

$$M(r; P_m) = o(r^{-\lambda}(-r; q)_\infty), \quad r \to \infty,$$

for all $\lambda > 0$. As for $M(r; D_{\infty,q} S_m)$, it can be estimated by means of Corollary 2.6 with $\alpha = m$. □

Funding Open access funding provided by the Scientific and Technological Research Council of TÜrkiye (TÜBİTAK).

References

1. Andrews, G.E., Askey, R., Roy, R.: Special Functions, Encyclopedia of Mathematics and Its Applications. The University Press, Cambridge (1999)
2. Boehm, W., Müller, A.: On de Casteljau's algorithm. Comput. Aided Geom. Design **16**, 587–605 (1999)
3. Bustamante, J.: Bernstein Operators and Their Properties. Birkhäuser/Springer, Cham (2017)
4. Castellani, L., Wess, J. (eds.): Quantum Groups and Their Applications in Physics. IOS Press, Amsterdam (1996)
5. Chung, W.S., Hassanabadi, H.: The q-boson algebra and $su_q(2)$ algebra based on q-deformed binary operations. Int. J. Theor. Phys. **60**(6), 2102–2114 (2021)
6. Derriennic, M.M.: Sur l'approximation de fonctions integrables sur [0,1] par des polynomes de Bernstein modifies. J. Approx. Theory **31**, 325–343 (1981)
7. Durrmeyer, J.L.: Une formule d'inversion de la transformee de Laplace: applications e la theorie des moments. These de 3e cycle, Paris (1967)
8. Gal, S.G.: Approximation by Complex Bernstein and Convolution Type Operators. World Scientific Publishing Company, Singapore (2009)
9. Gupta, V.: Some approximation properties of q-Durrmeyer operators. Appl. Math. Comput. **197**(1), 172–178 (2008)
10. Gupta, V., Wang, H.: The rate of convergence of q-Durrmeyer operators for $0<q<1$. Math. Methods Appl. Sci. **31**(16), 1946–1955 (2008)
11. Jackson, F.H.: On q-definite integrals. Q. J. Pure Appl. Math. **41**, 193–203 (1910)
12. Lorentz, G.G.: Bernstein Polynomials. Chelsea, New York (1986)
13. Mahmudov, N.I.: Approximation by q-Durrmeyer type polynomials in compact disks in the case $q>1$. Appl. Math. Comput. **237**, 293–303 (2014)
14. Ostrovska, S.: The q-versions of the Bernstein operator: from mere analogies to further developments. Results Math. **69**(3–4), 275–295 (2016)
15. Phillips, G.M.: Interpolation and Approximation by Polynomials. CMS Books Math., Springer, New York (2003)
16. Thomae, J.: Beiträge zur Theorie der durch die Heinsche Reihe darstellbaren Functionen. J. Reine Angew. Math. **70**, 258–281 (1869)
17. Zeng, J., Zhang, C.: A q-analog of Newton's series, Stirling functions and Eulerian functions. Results Math. **25**, 370–391 (1994)