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The Intrinsic Geometry of Simply and Rectifiably Connected Plane Sets

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Abstract
We prove that the metric completion of the intrinsic length space associated with a simply and rectifiably connected plane set is a Hadamard space. We also characterize when such a space is Gromov hyperbolic.

Keywords Intrinsic length distance · Non-positive curvature · CAT(0) · Hadamard spaces

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1 Introduction

Throughout this section X is a simply and rectifiably connected plane set; we make no assumption about X being open or closed. Then $\overline{X_l}$ is the metric completion of $X_l := (X, l)$, the metric space where l is the intrinsic (Euclidean) length distance on X.

Theorem *Suppose X is a simply and rectifiably connected plane set. Then $\overline{X_l}$ is a Hadamard space. Moreover, $\overline{X_l}$ is Gromov hyperbolic if and only if X does not contain Euclidean disks of arbitrarily large radii, i.e., if and only if*

$$R := \sup\left\{r > 0 \;\middle|\; \exists\, \mathsf{D}(x; r) \subset X\right\} < +\infty;$$

when this holds, X is $2R$-hyperbolic and this is best possible.

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Recall that a *Hadamard* metric space is a complete CAT(0) space[1]; see Sect. 2.2.2. Our result provides a bountiful supply of easily constructed Hadamard spaces.

The special case where X is a closed bounded Jordan plane region was established in [1]; there the work of [3] is an essential ingredient. We present an elementary argument for the case where X is a simply connected plane domain; see Sect. 3.1.[2]

It is easy to construct compact rectifiably connected plane sets X for which X_l is geodesic but fails to have non-positive curvature. It would be useful to have a characterization of the plane sets X for which $\overline{X_l}$ has non-positive curvature. As Bishop mentions, in [5, Ex. 9.1.6, p. 310] the authors assert that for "any locally simply connected plane set X", X_l has non-positive curvature. However, as Bishop comments, their discussion fails to mention certain essential details including, in particular, the existence of geodesics.

2 Preliminaries

For real numbers r and s,

$$r \wedge s := \min\{r, s\} \quad \text{and} \quad r \vee s := \max\{r, s\}.$$

2.1 Metric Space Notation and Terminology

Throughout this section X is an arbitrary metric space with distance denoted $|x - y|$; this is not meant to imply that X possesses any sort of linear or group structure. In this setting, all topological notions refer to the metric topology; here $\mathsf{cl}(A)$, $\mathsf{bd}(A)$, $\mathsf{int}(A)$ are the topological closure, boundary, interior (respectively) of $A \subset X$.

Every metric space can be isometrically embedded into a complete metric space. We let $\bar{X}$ denote the metric completion of the metric space X; thus $\bar{X}$ is the closure of the image of X under such an isometric embedding. We call $\partial X := \bar{X} \setminus X$ the metric boundary of X.

When $A \subset X$, there is a natural embedding $\bar{A} \hookrightarrow \bar{X}$ and $\mathsf{bd}(A) \subset \partial A$. Here if $A \subset X$ is open and X complete, then $\partial A = \mathsf{bd}(A)$, but in general $\bar{A} = \overline{\mathsf{cl}}(A)$ and $\partial A = \overline{\mathsf{bd}}(A) \setminus A$ where $\overline{\mathsf{cl}}$ and $\overline{\mathsf{bd}}$ denote topological closure and boundary in $\bar{X}$

2.1.1 Paths, Arcs, Geodesics, and Length

A *path in* X is a continuous map $\mathbb{R} \supset I \xrightarrow{\gamma} X$ where $I = I_\gamma$ is the *parameter interval for* γ and may be closed or open or neither and finite or infinite. The *trajectory* of such a path γ is $|\gamma| := \gamma(I)$ which we call a *curve* and often–when easily understood in context–we abuse notation and just write γ in place of $|\gamma|$.

[1] By definition, our CAT(0) spaces are geodesic.

[2] After this manuscript was completed, the author learned of the work [9]; the case when X is a simply connected plane domain follows at once from their Proposition 12.1.

A path $I \xrightarrow{\gamma} X$ is a *geodesic* if it is an isometry:

$$\forall\, s, t \in I, \quad |\gamma(s) - \gamma(t)| = |s - t|$$

and X is a *geodesic metric space* if each pair of points can be joined by a geodesic.

When I is closed and $I \neq \mathbb{R}$, $\partial\gamma := \gamma(\partial I)$ denotes the set of *endpoints of* γ and consists of one or two points depending on whether or not I is compact. For example, if $I_\gamma = [u, v] \subset \mathbb{R}$, then $\partial\gamma = \{\gamma(u), \gamma(v)\}$. When $\partial\gamma = \{a, b\}$, we write $\gamma : a \curvearrowright b$ (*in* X) to indicate that γ is a path (in X) with *initial point* a and *terminal point* b; this implies an orientation—a precedes b on γ.

We call γ a *compact path* if its parameter interval is compact. A compact path γ is a *loop* if $\partial\gamma$ is a single point, and then $|\gamma|$ is often dubbed a *closed curve*. A loop $\gamma : [u, v] \to X$ is a *Jordan loop* (aka, a *simple closed curve*) if $\gamma|_{[u,v)}$ is injective.

An *arc* α is an injective compact path; here $|\alpha|$ is often called a *simple* curve; again, we sometimes abuse notation and call $|\alpha|$ an arc. The *interior of* α is $\overset{\circ}{\alpha} := \alpha \setminus \partial\alpha$.

Given points $a, b \in |\alpha|$, there is a unique subarc $\alpha[a, b]$ of α with endpoints a, b; precisely, there are unique $u, v \in I$ with $\alpha(u) = a$, $\alpha(v) = b$ and $\alpha[a, b] := \alpha|_{[u,v]}$. (Again, sometimes $\alpha[a, b]$ is this map and sometimes it denotes its trajectory.) We also use this notation for a general path γ, but here $\gamma[a, b]$ denotes the unique subpath of γ that joins a, b obtained by using the last time γ is at a up to the first time γ is at b.

When $\alpha : a \curvearrowright b$ and $\beta : b \curvearrowright c$ are paths that join a to b and b to c respectively, $\alpha\star\beta$ denotes the concatenation[3] of α and β; so $\alpha\star\beta : a \curvearrowright c$. The *reverse of* γ is the path $\tilde{\gamma}$ defined by $\tilde{\gamma}(t) := \gamma(1 - t)$ (when $I_\gamma = [0, 1]$) and going from $\gamma(1)$ to $\gamma(0)$. Of course, $|\alpha\star\beta| = |\alpha| \cup |\beta|$ and $|\tilde{\gamma}| = |\gamma|$.

Every compact path contains an arc with the same endpoints; see [12].

The length of a compact path $[0, 1] \xrightarrow{\gamma} X$ is defined in the usual way by

$$\ell(\gamma) := \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \,\middle|\, 0 = t_0 < t_1 < \cdots < t_n = 1 \right\},$$

γ is *rectifiable* when $\ell(\gamma) < \infty$, and X is *rectifiably connected* provided each pair of points in X can be joined by a rectifiable path. An arbitrary path γ is *locally rectifiable* if each compact subpath of γ is rectifiable, and such a γ is *rectifiable* if

$$\ell(\gamma) := \sup\big\{\ell(\alpha) \,\big|\, \alpha \text{ a compact subpath of } \gamma\big\} < +\infty.$$

Rectifiable paths always have endpoints, and so have unique extensions to compact paths with the same length. Here is a precise statement; cf. [11, Thm. 3.2, p.7].

Fact 2.1 Let $\mathbb{R} \supset I \xrightarrow{\gamma} X$ be a rectifiable path with I a finite interval. Then there is a unique extension $\bar{I} \xrightarrow{\bar{\gamma}} \bar{X}$ of γ to a compact rectifiable path $\bar{\gamma}$ and $\ell(\bar{\gamma}) = \ell(\gamma)$.

3 We are ignoring how to parametrize the concatenation as this is not needed for our work.

Every rectifiable path can be parametrized with respect to its arclength [11, p. 5]. When γ is a rectifiable path, we tacitly assume its parameter interval is $I_\gamma = [0, \ell(\gamma)]$ unless specifically stated otherwise.

2.1.2 Intrinsic Length Distance

Every rectifiably connected metric space X admits a natural *intrinsic* distance, its so-called *(inner) length distance* given by

$$l(a, b) := \inf\big\{\ell(\gamma) \,\big|\, \gamma : a \curvearrowright b \text{ a rectifiable path in } X\big\}.$$

A metric space $(X, |\cdot|)$ is a *length* space provided for all points $a, b \in X$, $|a - b| = l(a, b)$, and we call such a $|\cdot|$ a *length (or intrinsic)* distance function. An l-geodesic $[a, b]_l$ is a shortest path joining a and b, and any shortest path can be parametrized to be an l-geodesic.

The notation $X_l := (X, l)$ is convenient, and then $\partial_l X := \overline{X_l} \backslash X_l$. We note that $(\overline{X_l})_l = \overline{X_l}$, which is a consequence of the facts that the length distance $l = l_d$ associated with a length distance d is just d, and the completion of a length distance is also a length distance.

More generally, a continuous function $X \xrightarrow{\rho} (0, \infty)$ on a rectifiably connected metric space X induces a length distance d_ρ on X defined by

$$d_\rho(a, b) := \inf_{\gamma : a \curvearrowright b} \ell_\rho(\gamma) \quad \text{where} \quad \ell_\rho(\gamma) := \int_\gamma \rho \, ds$$

and where the infimum is taken over all rectifiable paths $\gamma : a \curvearrowright b$ in X. We describe this by calling $\rho\, ds = \rho(x)|dx|$ a *conformal metric* on X.

There are two useful properties of length spaces that we use repeatedly. First, for any open set U in a length space X, we always have $\mathsf{dist}(x, \mathsf{bd}\, U) = \mathsf{dist}(x, X \backslash U)$ for all points $x \in U$. Second, $\bar{X}$ is also a length space. In fact, for all $x \in X$, $\xi \in \partial X$, $\varepsilon > 0$ there is a path $\gamma : x \curvearrowright \xi$ in $X \cup \{\xi\}$ with $\ell(\gamma) < |x - \xi| + \varepsilon$.

We utilize the fact that rectifiable arcs in $\overline{X_l}$ can be approximated by arcs in X. Here is a precise statement.

Lemma 2.2 *Let X be rectifiably connected. Suppose $\tilde{\gamma} : \tilde{p} \curvearrowright \tilde{q}$ is a rectifiable arc in $\overline{X_l}$. Then for each $\varepsilon > 0$, there is a rectifiable path $\gamma : p \curvearrowright q$ in X with $l\big(\gamma(t), \tilde{\gamma}(t)\big) < \varepsilon$ for all $t \in I := [0, \ell(\tilde{\gamma})]$ (where I is also the parameter interval for γ); thus there are rectifiable arcs $\alpha : p \curvearrowright q$ in X and $\tilde{\alpha} : \tilde{p} \curvearrowright \tilde{q}$ in $X \cup \{\tilde{p}, \tilde{q}\}$ with $\alpha, \tilde{\alpha} \subset \mathsf{N}_l(\tilde{\gamma}; \varepsilon)$.*

Proof Sketch Given $\varepsilon \in (0, \ell(\tilde{\gamma}))$, let n be the smallest positive integer with $\ell(\tilde{\gamma})/n \le \varepsilon/10$. Put $t_i := (i/n)\ell(\tilde{\gamma})$ for $0 \le i \le n$. Define $x_i := \tilde{\gamma}(t_i)$ if $\tilde{\gamma}(t_i) \in X$; otherwise, if $\tilde{\gamma}(t_i) \in \partial_l X$, choose any $x_i \in X$ with $l\big(x_i, \tilde{\gamma}(t_i)\big) < \varepsilon/10$. Then $l(x_{i-1}, x_i) < 3\varepsilon/10$ so there are rectifiable arcs $\gamma_i : x_{i-1} \curvearrowright x_i$ in X with $\ell(\gamma_i) < 3\varepsilon/10$. Then $\gamma := \gamma_1 \star \cdots \star \gamma_n$ has the asserted properties, where $\gamma_i : [t_{i-1}, t_i] \to X$ is parametrized proportional to arc length. □

Here is information that we employ to construct Jordan loops inside X.

Lemma 2.3 *Let X be rectifiably connected. Suppose $\gamma_i : \tilde{p} \curvearrowright q_i$ $(i = 1, 2)$ are rectifiable arcs in $X \cup \{\tilde{p}\}$ with $\gamma_1 \cap \gamma_2 = \{\tilde{p}\} \subset \overline{X_l}$. Then for each $\varepsilon > 0$, there are points $p_i \in \gamma_i$ and a rectifiable arc $\alpha : p_1 \curvearrowright p_2$ in X with $l(p_1, \tilde{p}) < \varepsilon, l(p_2, \tilde{p}) < \varepsilon, \ell(\alpha) < \varepsilon$ and such that $\gamma_1^{-1}[q_1, p_1] \star \alpha \star \gamma_2[p_2, q_2]$ is a rectifiable arc $q_1 \curvearrowright q_2$ in X.*

Proof Let $\varepsilon > 0$ be given. Choose points $a_i \in \gamma_i$ and a rectifiable arc $\beta : a_1 \curvearrowright a_2$ in X with each of $l(a_1, \tilde{p}), l(a_2, \tilde{p}), \ell(\beta)$ less than $\varepsilon/10$. Let p_1 be the last point of β in γ_1 and let p_2 be the first point of $\beta[p_1, a_2]$ in γ_2. Then $\alpha := \beta[p_1, p_2]$ has the asserted properties. □

Let $[0, 1) \xrightarrow{\gamma} X$ be a path in X. If there is a point $\xi \in \partial X$ such that $\lim_{t \to 1^-} |\gamma(t) - \xi| = 0$, then ξ is called a *path accessible* (metric) boundary point of X. In this situation, we define $\gamma(1) := \xi$ and obtain a path $\gamma : [0, 1] \to X \cup \{\xi\} \subset \bar{X}$. We describe this by saying that γ is *a path in X with terminal endpoint $\xi \in \partial X$.*

We write $\partial^{\mathrm{pa}} X$ for the set of all path accessible boundary points of X. Restricting attention to rectifiable paths γ yields *rectifiably accessible* (metric) boundary points of X, denoted by $\partial^{\mathrm{ra}} X$. Clearly, $\partial^{\mathrm{ra}} X \subset \partial^{\mathrm{pa}} X \subset \partial X$ and each containment may be strict. We define $X^{\mathrm{ra}} := X \cup \partial^{\mathrm{ra}} X$.

A path in X need not be a path in X_l; see [6, Ex. 3.6]. However, a rectifiable path in X is also continuous as a map into X_l and therefore a path in X_l. Two rectifiable arcs in X with a common endpoint in $\partial_l X$, say

$$[0, \ell(\alpha)] \xrightarrow{\alpha} X \cup \{\xi\}, [0, \ell(\beta)] \xrightarrow{\beta} X \cup \{\xi\} \quad \text{with} \;\; \alpha(0) = \xi = \beta(0) \in \partial_l X,$$

are *l-equivalent* if and only if

$$\lim_{s \to 0^+} l\big(\alpha(s), \beta(s)\big) = 0.$$

There is a natural one-to-one correspondence between $\partial_l X$ and the l-equivalence classes of such rectifiable arcs; see [6, Prop. 3.29].

The identity map $X_l \xrightarrow{\mathsf{id}} X$ is 1-Lipschitz and so has a 1-Lipschitz extension $\overline{X_l} \xrightarrow{\iota} \bar{X}$. In general, $\iota = \iota_X$ need not be surjective nor injective. However, we always have $\iota(\partial_l X) = \partial^{\mathrm{ra}} X$.

We make repeated appeals to the following elementary fact; see [6, Lem. 3.17] and also Fact 2.1.

Lemma 2.4 *Let $X = (X, |\cdot|)$ be a rectifiably connected metric space with associated length distance space $X_l = (X, l)$. Suppose $[0, 1) \xrightarrow{\gamma} X$ is a rectifiable path in X. Then*

$$\lim_{s,t \to 1^-} \ell\big(\gamma|_{[s,t]}\big) = 0 = \lim_{s,t \to 1^-} l\big(\gamma(s), \gamma(t)\big)$$

so there exist points $z \in \partial X$ and $\zeta \in \partial_l X$ such that

$$\lim_{t \to 1^-} |\gamma(t) - z| = 0 = \lim_{t \to 1^-} l\big(\gamma(t), \zeta\big);$$

therefore there are rectifiable paths

$$[0,1] \xrightarrow{\bar{\gamma}} X \cup \{z\} \subset \bar{X} \quad \textit{and} \quad [0,1] \xrightarrow{\tilde{\gamma}} X \cup \{\zeta\} \subset \overline{X_l} \quad \textit{with} \;\; \bar{\gamma} = \iota \circ \tilde{\gamma}$$

that are obtained by defining

$$\bar{\gamma}(t) := \begin{cases} \gamma(t) & \textit{for } t \in [0,1), \\ z & \textit{for } t = 1; \end{cases} \quad \textit{and} \quad \tilde{\gamma}(t) := \begin{cases} \gamma(t) & \textit{for } t \in [0,1), \\ \zeta & \textit{for } t = 1; \end{cases}$$

Moreover, $z \in \partial X$ *if and only if* $\zeta \in \partial_l X$*. Also,* $\ell(\tilde{\gamma}) = \ell(\bar{\gamma}) = \ell(\gamma)$*.*

Corollary 2.5 *Suppose* X *is rectifiably connected and* $a, b \in X^{\mathrm{ra}} := X \cup \partial^{\mathrm{ra}} X$*. Then* a, b *can be joined by a rectifiable path in* $X \cup \{a, b\}$*. Moreover, if* $\gamma : a \curvearrowright b$ *in* $X \cup \{a, b\}$*, then there are unique points* $\tilde{a}, \tilde{b} \in \overline{X_l}$ *and a rectifiable* $\tilde{\gamma} : \tilde{a} \curvearrowright \tilde{b}$ *in* $X_l \cup \left\{\tilde{a}, \tilde{b}\right\}$ *with* $\gamma = \iota \circ \tilde{\gamma}$ *and* $\ell(\gamma) = \ell(\tilde{\gamma})$*.*

We define $X^{\mathrm{ra}} \times X^{\mathrm{ra}} \xrightarrow{l^{\mathrm{ra}}} [0, +\infty)$ by

$$l^{\mathrm{ra}}(a, b) := \inf\left\{\ell(\gamma) \,\middle|\, \gamma : a \curvearrowright b \text{ a rectifiable path in } X \cup \{a, b\}\right\}.$$

In general, l^{ra} need not be a distance on X^{ra} because the triangle inequality may fail. However, its restriction l_1^{ra} to $X_1^{\mathrm{ra}} \times X_1^{\mathrm{ra}}$, where

$$X_1^{\mathrm{ra}} := X \cup \partial_1^{\mathrm{ra}} X \quad \text{with} \quad \partial_1^{\mathrm{ra}} X := \left\{z \in \partial^{\mathrm{ra}} X \mid \operatorname{card} \iota^{-1}(z) = 1\right\},$$

is a distance on X_1^{ra}. The triangle inequality is easy to check if the intermediate point lies in X and not difficult to verify when this point lies in $\partial_1^{\mathrm{ra}} X$.

Setting

$$\partial_l^1 X := \left\{\xi \in \partial_l X \mid \iota^{-1}\big(\iota(\xi)\big) = \{\xi\}\right\}, \quad X_l^1 := X_l \cup \partial_l^1 X, \quad X_l^{\mathrm{ra}} := (X_1^{\mathrm{ra}}, l_1^{\mathrm{ra}})$$

and $\iota^1 := \iota|_{X_l^1}$, we easily obtain the following.

Lemma 2.6 *When* X *is rectifiably connected,* $X_l^1 \xrightarrow{\iota^1} X_l^{\mathrm{ra}}$ *is an isometry.*

Proof Let $\tilde{a}, \tilde{b} \in X_l^1$. Then $a := \iota(\tilde{a}), b := \iota(\tilde{b}) \in X_1^{\mathrm{ra}}$. Let $\gamma : a \curvearrowright b$ be an arc in $X \cup \{a, b\}$. The ends of γ determine $\tilde{a}, \tilde{b}$, so by Corollary 2.5 there is a rectifiable $\tilde{\gamma} : \tilde{a} \curvearrowright \tilde{b}$ in $X_l \cup \left\{\tilde{a}, \tilde{b}\right\}$ with $\gamma = \iota \circ \tilde{\gamma}$ and $\ell(\tilde{\gamma}) = \ell(\gamma)$. Thus

$$l(\tilde{a}, \tilde{b}) \le \ell(\tilde{\gamma}) = \ell(\gamma).$$

Taking an infimum over all such γ gives

$$l(\tilde{a}, \tilde{b}) \le l(a, b)$$

and the opposite inequality holds because ι is 1-Lipschitz. □

2.2 CAT(0) Metric Spaces

Here our terminology and notation conforms with that in [4]; also, see [5]. We recall a few fundamental concepts, mostly copied directly from [4].

2.2.1 Geodesic and Comparison Triangles

A *geodesic triangle* Δ in X consists of three points in X, say $a, b, c \in X$, called the *vertices of* Δ and three geodesics, say $\alpha : a \curvearrowright b$, $\beta : b \curvearrowright c$, $\gamma : c \curvearrowright a$ (that we may write as $[a, b]$, $[b, c]$, $[c, a]$) called the *sides of* Δ. We use the notation

$$\Delta = \Delta(\alpha, \beta, \gamma) \quad \text{or} \quad \Delta = [a, b, c] := [a, b]\star[b, c]\star[c, a] \quad \text{or} \quad \Delta = \Delta(a, b, c)$$

depending on the context and the need for accuracy.

A Euclidean triangle $\bar{\Delta} = \Delta(\bar{a}, \bar{b}, \bar{c})$ in $\mathbb{C}$ is a *comparison triangle* for $\Delta = \Delta(a, b, c)$ provided $|a - b| = |\bar{a} - \bar{b}|$, $|b - c| = |\bar{b} - \bar{c}|$, $|c - a| = |\bar{c} - \bar{a}|$. We also write $\bar{\Delta} = \bar{\Delta}(a, b, c)$ when a specific choice of $\bar{a}, \bar{b}, \bar{c}$ is not required. A point $\bar{x} \in [\bar{a}, \bar{b}]$ is a *comparison point* for $x \in [a, b]$ when $|x - a| = |\bar{x} - \bar{a}|$. Assuming that $b \neq a \neq c$ (so $\bar{b} \neq \bar{a} \neq \bar{c}$), the *comparison angle of* Δ *at* a is defined to be the interior Euclidean angle of $\bar{\Delta}$ at $\bar{a}$ and denoted by

$$\overline{\measuredangle}_a(b, c) := \measuredangle^{\mathsf{euc}}_{\bar{a}}(\bar{b}, \bar{c}).$$

Assume $a \neq p \neq b$ and let $\alpha : p \curvearrowright a$, $\beta : p \curvearrowright b$ be rectifiable arcs in X parameterized by arc length. The (upper) *Alexandrov angle between* α *and* β is defined by

$$\measuredangle_p(\alpha, \beta) := \limsup_{s,t \to 0^+} \overline{\measuredangle}_p(\alpha(s), \beta(t));$$

see [4, 1.12, p.9]. When $[p, a]$, $[p, b]$ are geodesics, $\measuredangle_p(a, b) := \measuredangle_p([p, a], [p, b])$.

2.2.2 CAT(0) Definition

A geodesic triangle Δ in X satisfies the *CAT(0) distance inequality* if and only if the distance between any two points of Δ is not larger than the Euclidean distance between the corresponding comparison points; that is,

$$\forall\, x, y \in \Delta \text{ and corresponding comparison points } \bar{x}, \bar{y} \in \bar{\Delta}, \quad |x - y| \leq |\bar{x} - \bar{y}|.$$

We also say that Δ is *CAT(0)-thin* when it satisfies the CAT(0) distance inequality.

A geodesic metric space is *CAT(0)* if and only if each of its geodesic triangles is CAT(0)-thin. A complete CAT(0) metric space is called a *Hadamard space*. A geodesic metric space X has *non-positive curvature* if and only if it is locally CAT(0), meaning that for each point $a \in X$ there is an $r > 0$ (that can depend on a) such that the metric ball $\mathsf{B}(a; r)$ (endowed with the distance inherited from X) is CAT(0).

Of the many conditions which guarantee that a space is CAT(0), for instance, see [4, Prop. 1.7, p. 161] or [5, Thm. 4.3.5, p.116], we mention only that a geodesic metric space X is CAT(0) if and only if each of its geodesic triangles satisfies the CAT(0) vertex angle criterion. Here Δ satisfies the *CAT(0) vertex angle criterion* if and only if Δ has distinct vertices and the Alexandrov angle between any two sides of Δ is not greater than the interior Euclidean angle between the corresponding sides of a comparison triangle for Δ; equivalently, if and only if the (Alexandrov) vertex angles of Δ are not greater than the corresponding (Euclidean) vertex angles of a comparison triangle for Δ.

2.2.3 Triangle Tails

Let $\Delta = [a, b, c] = [a, b]\star[b, c]\star[c, a]$ be a geodesic triangle. Suppose there are points $b_o \in [a, b]$ and $c_o \in [a, c]$ such that the subgeodesics $[a, b_o] \subset [a, b]$ and $[a, c_o] \subset [a, c]$ coincide: i.e., $[a, b_o] = [a, c_o]$. This common geodesic segment is a *tail* of Δ, and Δ is *tail-less* if there are no such tails.[4]

It is not difficult to verify the following. (If the lengths of two sides of an Euclidean triangle are increased by the same amount, then certain angles also increase.)

Fact 2.7 Let X be a geodesic metric space. Suppose every tail-less geodesic triangle in X satisfies the CAT(0) vertex angle criterion. Then X is CAT(0).

2.2.4 Gromov Hyperbolicity Definition

A geodesic metric space X is δ-*hyperbolic* if and only if for all geodesic triangles Δ in X, each edge of Δ lies in the δ-neighborhood of the union of the other two edges, and X is *Gromov hyperbolic* if and only if it is δ-hyperbolic for some $\delta \in [0, +\infty)$.

2.3 General Plane Information

We view the Euclidean plane as the complex number field $\mathbb{C}$. Everywhere Ω is a *plane domain* (i.e., an open connected set), $\Omega^c := \mathbb{C}\setminus\Omega$ and $\partial\Omega$ denote the complement and boundary (respectively) of Ω.

The open disk of radius r centered at the point $a \in \mathbb{C}$ is

$$\mathsf{D}(a; r) := \{z : |z - a| < r\},$$

$\mathbb{D} := \mathsf{D}(0; 1)$ is the open unit disk, and the open r-neighborhood of a set $A \subset \mathbb{C}$ is

$$\mathsf{N}(A; r) := \bigcup_{a \in A} \mathsf{D}(a; r) = \{z : \mathsf{dist}(z, A) < r\}.$$

[4] Caution: this does *not* mean that the sides of Δ do not overlap somewhere away from the vertices.

2.3.1 Complex Analysis

The well known Riemann and Carathéodory mapping theorems assert that when Ω is a simply connected plane domain, there is a conformal map (i.e., a holomorphic homeomorphism) $f : \mathbb{D} \to \Omega$, and if Ω is a Jordan domain, f extends to a homeomorphism $\bar{\mathbb{D}} \to \bar{\Omega}$. So, each boundary point of a Jordan domain is path accessible from the domain.

We repeatedly use the less known fact that when Ω is a simply connected plane domain with rectifiable boundary (e.g., if $\partial\Omega$ is a rectifiable Jordan loop), then each point of $\partial\Omega$ is rectifiably accessible from Ω; that is, $\partial^{\mathrm{ra}}\Omega = \partial\Omega$ and $\Omega^{\mathrm{ra}} = \bar{\Omega}$. I am indebted to Distinguished Professor Chris Bishop for explaining this to me. It is a consequence of the fact that any Riemann map onto such a domain belongs to the Hardy class H^1; see the "easy half" of Chris' result in [2].

A Riemann map $\mathbb{D} \xrightarrow{f} \Omega$ provides a *conformal model* for the length space $\overline{\Omega_l}$. Indeed, the conformal metric $|f'(z)|\,|dz|$ on $\mathbb{D}$ induces the length distance

$$d_f(a,b) := \inf_{\gamma: a \frown b} \int_\gamma |f'(z)|\,|dz|$$

where the infimum is over all rectifiable arcs $\gamma : a \frown b$ in $\mathbb{D}$ and $\mathbb{D}_f := (\mathbb{D}, d_f) \xrightarrow{f} (\Omega, l) =: \Omega_l$ is an isometry. One can demonstrate that $\overline{\mathbb{D}_f} = \mathbb{D} \cup \partial_f\mathbb{D}$, where $\partial_f\mathbb{D} := \big\{\zeta \in \partial\mathbb{D} \;\big|\; f\big([0,\zeta)\big) \text{ is rectifiable}\big\}$; evidently, $\partial_f\mathbb{D} \subset \partial\mathbb{D}_f$, and the opposite containment can be established with the help of [10, Prop. 2.14-p.29, Cor. 2.17-p.35, Thm. 4.20-p.88]. Thus $\overline{\Omega_l}$ is isometrically equivalent to $\overline{\mathbb{D}_f}$. With this model, the map $\iota : \overline{\Omega_l} \to \Omega^{\mathrm{ra}}$ can be realized as the radial limit extension $f : \overline{\mathbb{D}_f} \to \Omega^{\mathrm{ra}} \subset \bar{\Omega}$.

For example, we now see that a Jordan loop Λ in $\overline{\Omega_l}$ corresponds to a Jordan loop in $\overline{\mathbb{D}_f} \subset \bar{\mathbb{D}}$ whose interior is a simply connected domain in $\mathbb{D}$ with f image a simply connected $D \subset \Omega$ satisfying $\partial D = \iota(\Lambda)$, which is a rectifiably connected loop (perhaps not Jordan) in Ω^{ra}.

3 Proofs

Here we establish the Theorem stated in the Introduction. Now X is a given simply and rectifiably connected plane set with $\overline{X_l}$ the metric completion of the intrinsic (Euclidean) length space X_l associated with X. Also, $\overline{X_l} \xrightarrow{\iota} \bar{X}$ is the 1-Lip extension of the identity map $X_l \to X$.

First, we consider simply connected plane domains, then arbitrary simply and rectifiably connected plane sets.

3.1 CAT(0) Proof for *X* a Simply Connected Plane Domain

Assume $X = \Omega$ is a simply connected plane domain. Evidently, $\overline{\Omega_l}$ is a complete length metric space. We demonstrate that it is a 4-point limit of CAT(0) spaces, so by [4, Thm. 3.9, p.196] it is also CAT(0) and hence a Hadamard space.

The Riemann Mapping Theorem provides a conformal map $\mathbb{D} \xrightarrow{f} \Omega$ (i.e., a holomorphic homeomorphism). Let (r_ν) be a strictly increasing sequence in $(0, 1)$ with $r_\nu \nearrow 1$. For each $\nu \in \mathbb{N}$, define

$$f_\nu(\zeta) := f(r_\nu\zeta), \quad \Omega_\nu := f_\nu(\mathbb{D}) = f(r_\nu\mathbb{D}),$$

and let $U_\nu := f_\nu(r_\nu\mathbb{D}) = f(r_\nu^2\mathbb{D})$, $\lambda_\nu := \lambda_{\Omega_\nu}$, $M_\nu := \max_{\bar{U}_\nu} \lambda_\nu$, and $\varepsilon_\nu := \big(10\,\mathsf{diam}_l(\Omega_\nu)M_\nu^2\big)^{-1}$.

Note that U_ν is compactly contained in Ω_ν which in turn is compactly contained in Ω, and

$$\mathsf{diam}_l(\Omega_\nu) = \mathsf{diam}(\Omega_\nu, l) \quad \text{with } l = l_\Omega.$$

Also, for any plane domain D, $\lambda_D\,ds$ denotes the Poincaré hyperbolic metric in D, i.e., $\lambda_D\,ds$ is the maximal complete metric in D with constant Gaussian curvature -1; see [8]. Note that $\{U_\nu \mid \nu \in \mathbb{N}\}$ is an increasing open cover of Ω and (M_ν), (ε_ν) are increasing, decreasing positive sequences with $M_\nu \nearrow +\infty$, $\varepsilon_\nu \searrow 0$ respectively.

It is easy to check that $\overline{\Omega_l}$ is a 4-point limit of the spaces $(\bar{\Omega}_\nu, l_\nu)$ where l_ν is Euclidean length distance in Ω_ν. Since Ω_ν is a Jordan domain, we could appeal to Bishop's result now, but it is easy to provide a simple alternative argument.

Consider the conformal metric $\rho_\nu\,ds$ in Ω_ν where

$$\rho_\nu := 1 + \varepsilon_\nu\lambda_\nu \quad (\text{and note that} \rho_\nu \leq 1 + \big(10\,\mathsf{diam}_l(\Omega_\nu)M_\nu\big)^{-1} \text{in } \bar{U}_\nu).$$

Let d_ν be the length distance obtained from the metric $\rho_\nu\,ds$ in Ω_ν. Since $\log\lambda_\nu$ is subharmonic and $\mathscr{C}^\infty$ smooth in Ω_ν, so is $\log\rho_\nu$ (see [7, 2.1, 2.2]) and therefore by classical results (e.g., see [4, Thm.1A.6, Thm. 4.1, pp. 173,193]) each space (Ω_ν, d_ν) is CAT(0).

We check that $\overline{\Omega_l}$ is a 4-point limit of the CAT(0) spaces (Ω_ν, d_ν).

Let $x_1, x_2, x_3, x_4 \in \overline{\Omega_l}$ and $\varepsilon \in (0, \mathsf{diam}_l\,\Omega_1)$ be given. Define $z_1, z_2, z_3, z_4 \in \Omega$ as follows: if $x_i \in \Omega$, let $z_i := x_i$; otherwise, $x_i \in \partial_l\Omega$, and we pick any $z_i \in \Omega$ with $l(z_i, x_i) < \varepsilon/10$.

Next, for all $1 \leq i < j \leq 4$, choose arcs $\sigma_n^{ij} : z_i \curvearrowright z_j$ in Ω with $\ell(\sigma_n^{ij})$ decreasing to $l(z_i, z_j)$. Fix N so that for all $1 \leq i < j \leq 4$, $n \geq N \implies \ell(\sigma_n^{ij}) < l(z_i, z_j) + \varepsilon/10$; so $\ell(\sigma_n^{ij}) < \frac{11}{10}\,\mathsf{diam}_l(\Omega_\nu)$.

For each n, $K_n := \bigcup_{1\leq i<j\leq 4} \sigma_n^{ij}$ is a compact subset of Ω, so there is an increasing sequence $(\nu_n)_{n\geq N}$ such that for each $n \geq N$, $M_{\nu_n} > 2\varepsilon^{-1}$ and $K_n \subset U_{\nu_n} \subset \bar{U}_{\nu_n} \subset \Omega_{\nu_n}$. Then for each $n \geq N$ and all $1 \leq i < j \leq 4$,

$$
\begin{aligned}
l(z_i, z_j) &\le l_{\nu_n}(z_i, z_j) \le d_{\nu_n}(z_i, z_j) \le \ell_{\rho_{\nu_n}}(\sigma_n^{ij}) = \int_{\sigma_n^{ij}} \rho_{\nu_n}\, ds \\
&\le \Big(1 + \big(10\,\mathsf{diam}_l(\Omega_\nu) M_{\nu_n}\big)^{-1}\Big)\ell(\sigma_n^{ij}) \quad (\text{because } \sigma_{ij}^n \subset K_n \subset U_{\nu_n}) \\
&\le \ell(\sigma_n^{ij}) + \frac{11}{100 M_{\nu_n}} < l(z_i, z_j) + \frac{\varepsilon}{10} + \frac{11\varepsilon}{200} < l(z_i, z_j) + \frac{\varepsilon}{5}
\end{aligned}
$$

and so

$$
\begin{aligned}
l(x_i, x_j) &\le l(x_i, z_i) + l(z_i, z_j) + l(z_j, x_j) \le d_{\nu_n}(z_i, z_j) + \frac{\varepsilon}{5} \le l(z_i, z_j) + \frac{2\varepsilon}{5} \\
&\le l(z_i, x_i) + l(x_i, x_j) + l(x_j, z_j) + \frac{2\varepsilon}{5} \le l(x_i, x_j) + \frac{3\varepsilon}{5}.
\end{aligned}
$$

Thus, for all $n \ge N$ and $1 \le i < j \le 4$: $z_i, z_j \in \Omega_{\nu_n}$ and $|l(x_i, x_j) - d_{\nu_n}(z_i, z_j)| < \varepsilon$. □

3.2 CAT(0) Proof for General Case

Let X be a simply and rectifiably connected plane set. Our primary goal here is to demonstrate that $\overline{X_l}$ is uniquely geodesic; the CAT(0) property follows.

Since X is simply connected, whenever Λ is a Jordan loop in X, $\mathscr{D}(\Lambda) := \Lambda \cup \mathsf{Int}(\Lambda) \subset X$. As we employ this observation again and again, it is worthwhile to review methods for constructing Jordan loops.

Given distinct points p, q in X,

$$\Gamma(p, q) := \{\text{all rectifiable arcs } \gamma : p \curvearrowright q \text{ in} X\} \ne \varnothing.$$

Suppose $\beta, \gamma \in \Gamma(p, q)$ and there is a point $c \in \gamma \setminus \beta$. There are several ways to construct a Jordan loop Λ in X that contains an open subarc of γ which in turn contains c. Most simply, we move backwards, forwards along γ from c (towards p, q respectively) and let a, b be (respectively) the first points of $\beta \cap \gamma$. Here $\Lambda := \gamma[a, b] \cup \beta[a, b]$ has the asserted properties with $c \in \gamma(a, b)$.

A minor possible problem is that we do not know the order of a, b along β. To remedy this, set $b_1 := b$ and then, move backwards along γ (from c to p), and let a_1 be the first point of $\beta[p, b_1] \cap \gamma$. Now $\Lambda_1 := \gamma[a_1, b_1] \star \beta^{-1}[b_1, a_1]$ has the asserted properties and $p \le a < b \le q$ along both β and γ. Yet another alternative is to set $a_2 := a$, move forwards along γ (from c to q), let b_2 be the first point of $\beta[a_2, q] \cap \gamma$, and use $\Lambda_2 := \gamma[a_2, b_2] \star \beta^{-1}[b_2, a_2]$. Note that the three Jordan loops $\Lambda, \Lambda_1, \Lambda_2$ could all be different.

For definitiveness, we always use the first alternative construction.

We assume $\mathsf{int}(X) \ne \varnothing$, so

$$\mathscr{O} := \{\text{all components } \Omega \text{ of } \mathsf{int}(X)\} \ne \varnothing.$$

Note that even if some $\Omega \in \mathscr{O}$ has non-rectifiably accessible boundary points, $\bar{\Omega} \subset X$ is still possible. For each $\Omega \in \mathscr{O}$, $\overline{\Omega_l} \xrightarrow{\iota_\Omega} \bar{\Omega}$ is the 1-Lip extension of the identity map $\Omega_l \to \Omega$.

The following facts are useful.

(3.1a) *Rectifiable Jordan loops.* Suppose Λ is a rectifiable Jordan loop in X. Then there is a unique $\Omega \in \mathscr{O}$ with $\mathscr{D}(\Lambda) \subset \Omega^{\text{ra}} \cap X$, and if $\Lambda \cap \Omega = \varnothing$, then $\Omega = \mathsf{Int}(\Lambda)$.

(3.1b) *Components of* $\mathsf{int}(X)$. For distinct $\Omega_1, \Omega_2 \in \mathscr{O}$, $\mathsf{card}(\Omega_1^{\text{ra}} \cap \Omega_2^{\text{ra}} \cap X) \leq 1$.

(3.1c) *Unique length boundary points.* For each $\Omega \in \mathscr{O}$, $z \in \partial^{\text{ra}}\Omega \cap X \implies \mathsf{card}\, \iota_\Omega^{-1}(z) = 1$.[5]

Proof of (3.1a) Since X is simply connected, $\mathscr{D}(\Lambda) \subset X$, so $D := \mathsf{Int}(\Lambda) \subset X$ and there is an $\Omega \in \mathscr{O}$ with $D \subset \Omega$. Evidently, $\mathscr{D}(\Lambda) = \bar{D} = D^{\text{ra}} \subset \Omega^{\text{ra}} \cap X$; see Sect. 2.3.1. Fix a point $o \in \mathsf{D}$. Given $p \in \Omega$, let $\alpha : o \curvearrowright p$ in Ω. If $\Lambda \cap \Omega = \varnothing$, then $\alpha \cap \Lambda = \varnothing$, so $\alpha \subset D$ whence $p \in D$ and $\Omega = D$. $\square$

Proof of (3.1b) Let a, b be distinct points in $\Omega_1^{\text{ra}} \cap \Omega_2^{\text{ra}} \cap X$ for some $\Omega_1, \Omega_2 \in \mathscr{O}$. For $j \in \{1, 2\}$, pick rectifiable arcs $\alpha_j : a \curvearrowright b$ in $\Omega_j \cup \{a, b\}$. Since $\overset{\circ}{\alpha}_j \subset \Omega_j$, $\overset{\circ}{\alpha}_1 \cap \overset{\circ}{\alpha}_2 \neq \varnothing \implies \Omega_1 = \Omega_2$, so we may assume that $\alpha_1 \cap \alpha_2 = \{a, b\}$. Then $\Lambda := \alpha_1 \star \alpha_2^{-1}$ is a rectifiable Jordan loop in X, so by (3.1a) there is a unique $\Omega \in \mathscr{O}$ with $\mathscr{D}(\Lambda) \subset \Omega^{\text{ra}} \cap X$.

Fix a point $o \in D := \mathsf{Int}(\Lambda)$ and points $z_j \in \overset{\circ}{\alpha}_j \subset \Omega_j$. Then for each $j \in \{1, 2\}$, any $z \in \Omega_j$ can be joined to z_j (by a rectifiable path in Ω_j) and then to o (by a rectifiable path in $\Omega \cup \{z_j\}$), so there is a rectifiable path $z \curvearrowright o$ in $\mathsf{int}(X)$. It follows that $\Omega_1 = \Omega = \Omega_2$. $\square$

Proof of (3.1c) Suppose $z \in \partial^{\text{ra}}\Omega \cap X$ for some $\Omega \in \mathscr{O}$. Let α, β be rectifiable arcs in $\Omega \cup \{z\}$ both having z an endpoint. We show that α and β determine the same point in $\partial_l \Omega$.

Assume $[0, \ell(\alpha)] \xrightarrow{\alpha} \Omega \cup \{z\}$ and $[0, \ell(\beta)] \xrightarrow{\beta} \Omega \cup \{z\}$ with $\alpha(0) = z = \beta(0)$. We verify that $\lim_{s \to 0+} l\big(\alpha(s), \beta(s)\big) = 0$.

First, suppose that for all $\upsilon \in \big(0, \ell(\alpha) \wedge \ell(\beta)\big)$, $\alpha\big((0, \upsilon)\big) \cap \beta\big((0, \upsilon)\big) \neq \varnothing$. Given such an υ, pick $\sigma, \tau \in (0, \upsilon)$ with $\alpha(\sigma) = \beta(\tau)$. Then for any $s \in (0, \upsilon)$,

$$l\big(\alpha(s), \beta(s)\big) \leq l\big(\alpha(s), \alpha(\sigma)\big) + l\big(\alpha(\sigma), \beta(\tau)\big) + l\big(\beta(\tau), \beta(s)\big) \leq 2\upsilon < \varepsilon$$

provided $\upsilon < 2\varepsilon$.

Otherwise, we may assume $\alpha \cap \beta = \{z\}$. Let γ be a rectifiable arc in Ω from the terminal point a of α to the terminal point b of β. Then $\alpha \star \beta^{-1}$ and γ are arcs in $\Gamma(a, b)$ and $z \in (\alpha \star \beta^{-1}) \backslash \gamma$, so there is a rectifiable Jordan loop Λ in X that contains an open subarc of $\alpha \star \beta^{-1}$ which in turn contains z. Evidently, $\mathsf{Int}(\Lambda) \subset \Omega$ and $\mathscr{D}(\Lambda) \subset \Omega^{\text{ra}} \cap X$, and thus $\lim_{s \to 0+} l\big(\alpha(s), \beta(s)\big) = 0$. $\square$

[5] Thus $z \in \partial_1^{\text{ra}}\Omega$; see Lemma 2.6.

3.2.1 Entry and Exit Points

Let p, q be distinct points in X and $\Omega \in \mathscr{O}$. We say that $\gamma \in \Gamma(p,q)$ *enters* Ω if $\mathsf{card}(\gamma \cap \Omega^{\mathrm{ra}}) \geq 2$. We employ the following crucial facts.

(3.2a) *Both points in* Ω^{ra}. $\mathsf{card}\big(\{p,q\} \cap \Omega^{\mathrm{ra}}\big) = 2 \implies \forall\, \gamma \in \Gamma(p,q),\ \gamma \subset \Omega^{\mathrm{ra}}$.

(3.2b) *One point in* Ω^{ra}. $\mathsf{card}\big(\{p,q\} \cap \Omega^{\mathrm{ra}}\big) = 1 \implies \exists\, e := e_\Omega \in \partial^{\mathrm{ra}}\Omega \cap X$ such that $\forall\, \gamma \in \Gamma(p,q)$

$$\{p,q\} \cap \Omega^{\mathrm{ra}} = \{p\} \implies e \in \gamma,\ \gamma[p,e] \subset \Omega^{\mathrm{ra}},\ \text{and } \gamma(e,q] \cap \Omega^{\mathrm{ra}} = \varnothing,$$
$$\{p,q\} \cap \Omega^{\mathrm{ra}} = \{q\} \implies e \in \gamma,\ \gamma[e,q] \subset \Omega^{\mathrm{ra}},\ \text{and } \gamma[p,e) \cap \Omega^{\mathrm{ra}} = \varnothing.$$

(3.2c) *Neither point in* Ω^{ra}. $\mathsf{card}\big(\{p,q\} \cap \Omega^{\mathrm{ra}}\big) = 0 \implies$ if some arc in $\Gamma(p,q)$ enters Ω, then $\exists\, a := a_\Omega, b := b_\Omega \in \partial^{\mathrm{ra}}\Omega \cap X$ such that $\forall\, \gamma \in \Gamma(p,q)$

$$a, b \in \gamma,\ \gamma[a,b] \subset \Omega^{\mathrm{ra}},\ \text{ and } \big(\gamma[p,a) \cup \gamma(b,q]\big) \cap \Omega^{\mathrm{ra}} = \varnothing.$$

The points a, b in (3.2c) (and e in (3.2b)) are called *entry, exit points* (respectively) for Ω relative to p, q. These entry, exit points depend only on p, q, and Ω.

Proof of (3.2a) Assume $p, q \in \Omega^{\mathrm{ra}} \cap X$ and let $\gamma \in \Gamma(p,q)$. We show that $\gamma \subset \Omega^{\mathrm{ra}}$.

Let $\alpha : p \curvearrowright q$ be a rectifiable arc in $\Omega \cup \{p,q\} \subset \Omega^{\mathrm{ra}} \cap X$. Suppose there is a point $o \in \gamma \setminus \alpha$. As discussed in the third paragraph at the beginning of this subsection, there are points $a, b \in \alpha \cap \gamma$ such that $p \leq a < b \leq q$ along both α and γ with $\Lambda := \gamma[a,b] \star \alpha^{-1}[b,a]$ a rectifiable Jordan loop in X and with $o \in \gamma(a,b)$.

By (3.1a) there is a unique $\Omega_o \in \mathscr{O}$ with $\mathscr{D}(\Lambda) \subset \Omega_o^{\mathrm{ra}} \cap X$. We claim that $D := \mathsf{Int}(\Lambda) \subset \Omega$, so $\Omega = \Omega_o$ and $o \in \Omega^{\mathrm{ra}}$, and as o is an arbitrary point of $\gamma \setminus \alpha$, $\gamma \subset \Omega^{\mathrm{ra}}$ as asserted.

Let $z \in D$ and fix any point $c \in \alpha(a,b)$. Since $\Lambda = \partial D$ is rectifiable, there is a rectifiable arc $\beta : z \curvearrowright c$ in $D \cup \{c\}$. As $c \in \Omega \subset \mathsf{int}(X)$ and $D = \mathsf{Int}(\Lambda) \subset \Omega_o \subset \mathsf{int}(X)$, $\beta \subset \mathsf{int}(X)$ and so $z \in \beta \subset \Omega$. □

The proof of (3.2b) is similar to, but easier than, the proof of (3.2c) and so left to the reader.

Proof of (3.2c) Assume $\gamma \in \Gamma(p,q)$ enters Ω. There are distinct points $p_o, q_o \in \gamma \cap \Omega^{\mathrm{ra}}$ and we label these so that $p < p_o < q_o < q$ along γ. According to (3.2a), $\gamma[p_o, q_o] \subset \Omega^{\mathrm{ra}}$. Roughly speaking, a, b are the endpoints of the maximal subarc of γ that contains $\gamma[p_o, q_o]$ and lies in Ω^{ra}. Some care is required because Ω^{ra} need not be closed in $\mathbb{C}$ nor in X.

The set $A := \big\{z \in \gamma[p, p_o] \,\big|\, z \in \Omega^{\mathrm{ra}}\big\}$ is non-empty and bounded below, so it has a greatest lower bound a. Similarly, there is a least upper bound b for $B := \big\{z \in \gamma[q_o, q] \,\big|\, z \in \Omega^{\mathrm{ra}}\big\}$. Clearly $a, b \in \gamma \cap \partial\Omega$, $\big(\gamma[p,a) \cup \gamma(b,q]\big) \cap \Omega^{\mathrm{ra}} = \varnothing$, and it is not difficult to check that $\gamma(a,b) \subset \Omega^{\mathrm{ra}}$.

To corroborate that $a, b \in \partial^{\mathrm{ra}}\Omega$, we employ (3.1c) in conjunction with Lemma 2.6 as follows. Since $\gamma(a,b) \subset \Omega^{\mathrm{ra}} \cap X$, it lies in the image of the isometry $\iota^1_\Omega : \Omega^1_l \to \Omega^{\mathrm{ra}}_l$.

Thus $\tilde{\gamma} := (\iota^1_\Omega)^{-1} \circ \gamma(a, b)$ is a rectifiable arc in Ω^1_l and so has endpoints $\tilde{a}, \tilde{b}$ that we label to have ι_Ω images a, b. Thus $a = \iota_\Omega(\tilde{a}), b = \iota_\Omega(\tilde{b}) \in \iota_\Omega(\partial_l \Omega) = \partial^{\mathrm{ra}}\Omega$.

Let $\beta \in \Gamma(p, q)$. The path $\gamma^{-1}[a, p]\star\beta\star\gamma^{-1}[q, b]$ contains a rectifiable arc $a \curvearrowright b$ that must lie in Ω^{ra}. Since $\big(\gamma[p, a) \cup \gamma(b, q]\big) \cap \Omega^{\mathrm{ra}} = \varnothing$, it must be that $a, b \in \beta$, so $\beta[a, b] \subset \Omega^{\mathrm{ra}}$. If, e.g., there were a point $c \in \beta(b, q] \cap \Omega^{\mathrm{ra}}$, then letting d be the first point of $\beta[c, q]$ in $\gamma[b, q]$ would give an arc $\gamma[b, d]\star\beta^{-1}[d, c]$, but $b, c \in \Omega^{\mathrm{ra}}$ would imply $\gamma[b, d] \subset \Omega^{\mathrm{ra}}$ violating our choice of b. Similarly $\beta[p, a) \cap \Omega^{\mathrm{ra}} = \varnothing$. □

Here is a noteworthy consequence of (3.1c) and (3.2a):

$$\forall\ \Omega \in \mathscr{O}, \quad \text{there is an isometric embedding } \overline{\Omega_l} \xrightarrow{h_\Omega} \overline{X_l} \text{ with } \iota_X \circ h_\Omega = \iota_\Omega. \tag{3.3}$$

Proof of (3.3) The identity map $\Omega \xhookrightarrow{\mathsf{id}_\Omega} X$ induces a 1-Lipschitz embedding $\Omega_l \xhookrightarrow{\mathsf{id}_{\Omega_l}} X_l$ which then has a 1-Lipschitz extension $h_\Omega : \overline{\Omega_l} \to \overline{X_l}$. We explain why id_{Ω_l} is an isometric embedding.

Fix $a, b \in \Omega$ and let $\gamma \in \Gamma(a, b)$. By (3.2a), $\gamma \subset \Omega^{\mathrm{ra}} \cap X$, so by (3.1c) γ lies in the image of the isometry $\iota^1_\Omega : \Omega^1_l \to \Omega^{\mathrm{ra}}_l$; see Lemma 2.6. Thus $\tilde{\gamma} := (\iota^1_\Omega)^{-1} \circ \gamma$ is a rectifiable arc in Ω^1_l with $\ell(\gamma) = \ell(\tilde{\gamma}) \ge l_\Omega(a, b)$. Taking an infimum over all such arcs γ, and using the fact that id_{Ω_l} is 1-Lipschitz, we now obtain $l_X(a, b) = l_\Omega(a, b)$. It now follows that h_Ω is an isometric embedding.

Evidently, $h_\Omega(z) = z$ for $z \in \Omega$. Suppose $\zeta \in \partial_l \Omega$. Let $[0, 1) \xrightarrow{\alpha} \Omega$ be a rectifiable path that represents ζ. According to Lemma 2.4, α extends to rectifiable arcs $\tilde{\alpha}$ in $\overline{\Omega_l}$ and $\bar{\alpha} = \iota_\Omega \circ \tilde{\alpha}$ in $\Omega^{\mathrm{ra}} \subset \bar{\Omega}$ with $\zeta = \tilde{\alpha}(1)$ and $z = \bar{\alpha}(1) = \iota_\Omega(\zeta) \in \partial^{\mathrm{ra}}\Omega$.

But, $\alpha = \mathsf{id}_{\Omega_l} \circ \alpha$ is also a rectifiable arc in X and so has extensions $\bar{\alpha}$ in X^{ra} (i.e., $z \in X^{\mathrm{ra}}$) and α_X in $\overline{X_l}$. If $z \in X$, then $z \in \Omega^{\mathrm{ra}} \cap X$, so $\{\zeta\} = \iota^{-1}_\Omega(z)$, $h_\Omega(\zeta) = z$, and $\iota_X\big(h_\Omega(\zeta)\big) = \iota_X(z) = z = \iota_\Omega(\zeta)$. If $z \in \partial^{\mathrm{ra}}X$, then $\xi := \alpha_X(1) \in \partial_l X$, $z = \iota_X(\xi)$, and $h_\Omega(\zeta) = \xi$, so $\iota_X\big(h_\Omega(\zeta)\big) = \iota_X(\xi) = z = \iota_\Omega(\zeta)$. □

To simplify notation, often we identify $\overline{\Omega_l}$ with its image $h_\Omega(\overline{\Omega_l}) \subset \overline{X_l}$, but we must remember that some points[6] in $\partial_l \Omega$ may lie in X (and so not in $\partial_l X$).

We require similar information to deal with points in $\partial_l X$. Given distinct points p, q in $\overline{X_l}$, define

$$\Gamma_l(p, q) := \{\text{all rectifiable arcs } \gamma : p \curvearrowright q \text{ in} X \cup \{p, q\}\} \neq \varnothing$$

and

$$\bar{\Gamma}_l(p, q) := \Big\{\text{all rectifiable arcs } \gamma : p \curvearrowright q \text{ in } \overline{X_l}\Big\} \neq \varnothing.$$

The arcs in $\Gamma_l(p, q)$ are easier to work with (and all we need to compute $l(p, q)$), but facts about $\bar{\Gamma}_l(p, q)$ will help us establish uniqueness of l-geodesics.

Let p, q be distinct points in $\overline{X_l}$ and $\Omega \in \mathscr{O}$. The arcs in $\bar{\Gamma}(p, q)$ also have unique entry, exit points as in (3.2c); here γ *enters* $\overline{\Omega_l}$ if $\mathsf{card}(\gamma \cap \overline{\Omega_l}) \ge 2$. We identify $\overline{\Omega_l}$ with its image $h_\Omega(\overline{\Omega_l}) \subset \overline{X_l}$.

[6] These are precisely the points in $\iota^{-1}_\Omega(\partial^{\mathrm{ra}}\Omega \cap X)$.

(3.4a) *Both points in* $\overline{\Omega_l}$. $\mathsf{card}\big(\{p,q\}\cap\overline{\Omega_l}\big)=2 \implies \forall\,\gamma\in\bar{\Gamma}_l(p,q),\ \gamma\subset\overline{\Omega_l}$.

(3.4b) *One point in* $\overline{\Omega_l}$. $\mathsf{card}\big(\{p,q\}\cap\overline{\Omega_l}\big)=1 \implies \exists\ e:=e_\Omega\in\partial^{\rm ra}\Omega\cap X$ such that $\forall\,\gamma\in\bar{\Gamma}_l(p,q)$

$$\{p,q\}\cap\overline{\Omega_l}=\{p\} \implies e\in\gamma,\ \gamma[p,e]\subset\overline{\Omega_l},\ \text{and}\ \gamma(e,q]\cap\overline{\Omega_l}=\varnothing,$$
$$\{p,q\}\cap\overline{\Omega_l}=\{q\} \implies e\in\gamma,\ \gamma[e,q]\subset\overline{\Omega_l},\ \text{and}\ \gamma[p,e)\cap\overline{\Omega_l}=\varnothing.$$

(3.4c) *Neither point in* $\overline{\Omega_l}$. $\mathsf{card}\big(\{p,q\}\cap\overline{\Omega_l}\big)=0 \implies$ if some arc in $\bar{\Gamma}_l(p,q)$ enters $\overline{\Omega_l}$, then $\exists\ a:=a_\Omega,\ b:=b_\Omega\in\partial^{\rm ra}\Omega\cap X$ such that $\forall\,\gamma\in\bar{\Gamma}_l(p,q)$

$$a,b\in\gamma,\ \gamma[a,b]\subset\overline{\Omega_l},\ \text{and}\ \big(\gamma[p,a)\cup\gamma(b,q]\big)\cap\overline{\Omega_l}=\varnothing.$$

Again, we call the points a, b in (3.4c) (and e in (3.4b)) *entry, exit points* (respectively) for $\overline{\Omega_l}$ relative to p, q. These entry, exit points depend only on p, q, and Ω.

Proof of (3.4a) Suppose $p,q\in\overline{\Omega_l}$ and $\gamma\in\bar{\Gamma}_l(p,q)$, but $\gamma\not\subset\overline{\Omega_l}$. By replacing γ with an appropriate subarc, we may assume $\gamma\cap\overline{\Omega_l}=\{p,q\}$. Let $p_1,q_1\in\gamma$ be the first points[7] at distance $d:=\frac{1}{10}l(p,q)$ from p,q respectively. Put $\varepsilon:=d\wedge\mathsf{dist}_l(\gamma[p_1,q_1],\overline{\Omega_l})$.

Pick any points $p_0,q_0\in\Omega$ with $l(p_0,p)<\varepsilon$, $l(q_0,q)<\varepsilon$. Mimicking the proof of Lemma 2.2 gives us a rectifiable arc $\alpha : p_0\curvearrowright q_0$ in X with $\alpha\subset\mathsf{N}_l(\gamma;\varepsilon)$. According to (3.2a), $\alpha\subset\Omega^{\rm ra}\cap X\subset\overline{\Omega_l}$. Since $l(p,q)<l(p_0,q_0)+2\varepsilon$,

$$l(p_0,q_0)\ge 10d-2\varepsilon.$$

Fix a point $a\in\alpha$ with $l(a,p_0)=l(a,q_0)\ge\frac{1}{2}l(p_0,q_0)\ge 5d-\varepsilon$. Evidently, for all $b\in\gamma[p_1,q_1]$,

$$a\in\alpha\subset\overline{\Omega_l} \implies l(b,a)\ge\mathsf{dist}_l(b,\overline{\Omega_l})\ge\mathsf{dist}_l(\gamma[p_1,q_1],\overline{\Omega_l})\ge\varepsilon.$$

Also, if $b\in\gamma[p,p_1]$, then

$$l(b,a)\ge l(p,a)-l(p,b)\ge l(p_0,a)-l(p_0,p)-l(p,b)\ge 4d-2\varepsilon\ge 2\varepsilon.$$

Similarly, $b\in\gamma[q_1,q] \implies l(b,a)>\varepsilon$. This contradicts $\alpha\subset\mathsf{N}_l(\gamma;\varepsilon)$. □

Items (3.4b) and (3.4c) now readily follow. To see that the entry and exit points lie in X, note that we can use arcs in $X\cup\{p,q\}$ to determine these points.

3.2.2 Stable Points

Given distinct $p,q\in\overline{X_l}$, we call $x\in\overline{X_l}$ a *(p,q)-stable point* if x lies in every $\gamma\in\bar{\Gamma}_l(p,q)$. Let $\Sigma(p,q)$ be the set of all (p,q)-stable points. Evidently, p, q, and

[7] As we move along γ away from its endpoints p, q, respectively.

all entry and exit points associated with p, q belong to $\Sigma(p,q)$. It is not difficult to see that $\Sigma(p,q)$ is closed in $X \cup \{p,q\}$, ordered via any arc in $\bar{\Gamma}(p,q)$, and $l(p,q) = l(p,x) + l(x,q)$ for any $x \in \Sigma(p,q)$. To see that all arcs induce the same ordering on any $x, y \in \Sigma(p,q)$: note that if $\beta, \gamma \in \bar{\Gamma}_l(p,q)$ with $x < y$, $y < x$ along γ, β, respectively, then $\gamma[p,x]\star\beta[x,q]$ is a path (which contains an arc) $p \curvearrowright q$ but avoids y contradicting $y \in \Sigma(p,q)$.

By Lemma 2.2, $x \in \Sigma(p,q)$ provided $x \in \gamma$ for all $\gamma \in \Gamma_l(p,q)$. Also

$$\Sigma(p,q) = \{p,q\} \cup \bigcup_{x<y} \Sigma(x,y)$$

where the union is over all $x, y \in X \cap \Sigma(p,q)$, and,

$$x, y \in \Sigma(p,q) \text{ with } p \le x < y \le q \implies x \in \Sigma(p,y).$$

Indeed, $x, y \in \gamma \in \Gamma_l(p,q)$ and $\exists\ x \notin \beta \in \Gamma_l(p,y) \implies x \notin \beta\star\gamma[y,q] \in \Gamma_l(p,q)$.

Here are two especially useful facts.

$$\forall\, \gamma \in \Gamma_l(p,q), \quad z \in \gamma\setminus\Sigma(p,q) \implies \quad \begin{array}{l}\exists\, \Omega \in \mathscr{O} \text{ and a subarc} \\ \alpha \subset \gamma \cap \Omega^{\mathrm{ra}} \text{ with } z \in \overset{\circ}{\alpha}.\end{array} \tag{3.5a}$$

Consequently,

$$\Sigma(p,q) = \{p,q\} \implies \exists\, \Omega \in \mathscr{O} \text{ such that } p, q \in \overline{\Omega_l} \subset \overline{X_l}. \tag{3.5b}$$

Proof Sketch for (3.5) Suppose $\gamma \in \Gamma_l(p,q)$ and $z \in \gamma\setminus\Sigma(p,q) \subset X$. Pick an arc $\beta \in \Gamma_l(p,q)$ with $z \notin \beta$. Again, we can construct a Jordan loop Λ in X that contains an open subarc of γ which in turn contains z, but a wee bit of care is required. Since the ends of β, γ both determine the same points $p, q \in \overline{X_l}$, there are points p_1, p_2 and q_1, q_2 on β, γ respectively, close in X_l, and as close to p, q as desired. Then $\gamma[p_1,q_1] \cup \beta[p_2,q_2]$, together with short arcs $p_1 \curvearrowright p_2, q_1 \curvearrowright q_2$ in X, forms a loop in X which contains the asserted Jordan loop Λ; see Lemma 2.3 for details. Now (3.5b) follows from (3.1a).

To corroborate (3.5b), start with any $\gamma \in \Gamma_l(p,q)$. By (3.5b), for each $z \in \overset{\circ}{\gamma} = \gamma(p,q)$, there is an $\Omega_z \in \mathscr{O}$ and an arc $\alpha_z \subset \gamma \cap \Omega_z^{\mathrm{ra}}$ with $z \in \overset{\circ}{\alpha}_z$. If $w \in \alpha_z$, then $\varnothing \ne \overset{\circ}{\alpha}_w \cap \overset{\circ}{\alpha}_z \subset \Omega_w^{\mathrm{ra}} \cap \Omega_z^{\mathrm{ra}} \cap X$, so by (3.1b) $\Omega_w = \Omega_z$. It now follows that there is a single $\Omega \in \mathscr{O}$ with $\overset{\circ}{\gamma} \subset \Omega^{\mathrm{ra}}$, but then $\gamma \subset \overline{\Omega_l}$. □

3.2.3 Contructing Geodesics

Let $p, q \in \overline{X_l}$. We exhibit an l-geodesic $p \curvearrowright q$ in $\overline{X_l}$.

Assume $p, q \in X$. Suppose there exists an $\Omega \in \mathscr{O}$ with $p, q \in \Omega^{\mathrm{ra}}$. By (3.1c), there are unique points $\tilde{p}, \tilde{q} \in \overline{\Omega_l}$ with $p = \iota_\Omega(\tilde{p}), q = \iota_\Omega(\tilde{q})$. By Sect. 3.1, there is a unique l-geodesic $\sigma_\Omega : \tilde{p} \curvearrowright \tilde{q}$ in $\overline{\Omega_l}$. Then by (3.3) and its proof, $\sigma_X := h_\Omega \circ \sigma_\Omega$ is an l-geodesic in $\overline{X_l}$ with endpoints $h_\Omega(\tilde{p}) = p$ and $h_\Omega(\tilde{q}) = q$.

Now suppose that for all $\Omega \in \mathscr{O}$, $\{p, q\} \not\subset \Omega^{\mathrm{ra}}$. We construct a path $\sigma : p \curvearrowright q$ in $\overline{X_l}$ that has $\ell(\sigma) \le \ell(\gamma)$ for all $\gamma \in \Gamma(p, q)$. Thus σ is a shortest path and hence an l-geodesic

If $p \in \Omega_p^{\mathrm{ra}}$ for some $\Omega_p \in \mathscr{O}$, let $e_p \in \partial^{\mathrm{ra}}\Omega_p \cap X$ be the exit point associated with q, Ω_p as given in (3.2b) and let σ_p be the h_{Ω_p} image of the l-geodesic $\tilde{p} \curvearrowright \tilde{e}_p$ in $\overline{(\Omega_p)_l}$ where $\tilde{p}, \tilde{e}_p$ are the unique points in $\overline{(\Omega_p)_l}$ with $p = \iota_{\Omega_p}(\tilde{p})$, $e_p = \iota_{\Omega_p}(\tilde{e}_p)$. If no such Ω_p exists, put $\Omega_p := \varnothing$, $e_p := p$, $\sigma_p := \{p\}$. Define Ω_q, e_q, σ_q in a similar manner.

Let $\gamma \in \Gamma(p, q)$. Note that $\{e_p, e_q\} \subset \gamma$. Suppose $z \in \gamma[e_p, e_q] \setminus \Sigma(p, q)$. By (3.5b), there is an $\Omega \in \mathscr{O}$ and an arc $\alpha \subset \gamma[e_p, e_q] \cap \Omega^{\mathrm{ra}}$ with $z \in \overset{\circ}{\alpha}$. Thus γ enters Ω and so $z \in \gamma(a_\Omega, b_\Omega)$ where $a_\Omega, b_\Omega \in \partial^{\mathrm{ra}}\Omega \cap X$ are the entry, exit points associated with Ω as given in (3.2c).

It now follows that $\gamma[e_p, e_q] \setminus \Sigma(p, q)$ is a union of countably many $\gamma_n := \gamma(a_n, b_n)$ where $a_n := a_{\Omega_n}, b_n := b_{\Omega_n}$ are the entry, exit points (given by (3.2c)) associated with the countably many Ω_n that satisfy $\mathsf{card}(\gamma \cap \Omega_n^{\mathrm{ra}}) \ge 2$ with $\Omega_p \ne \Omega_n \ne \Omega_q$. Note that $a_n, b_n \in \partial^{\mathrm{ra}}\Omega_n \cap X$ and these entry, exit points correspond to unique points $\tilde{a}_n, \tilde{b}_n \in \partial_l \Omega_n \subset \overline{(\Omega_n)_l}$.

For each n, let $\sigma_n : a_n \curvearrowright b_n$ in $\overline{X_l}$ be the h_{Ω_n} image of the l-geodesic $\tilde{a}_n \curvearrowright \tilde{b}_n$ in $\overline{(\Omega_n)_l}$. Replacing each of the subarcs $\gamma[p, e_p], \gamma[e_q, q], \gamma_n$ of γ with $\sigma_p, \sigma_q, \sigma_n$, respectively, we obtain an arc

$$\sigma := \sigma_p \cup \Sigma(p, q) \cup \sigma_q \cup \bigcup_n \sigma_n : p \curvearrowright q \quad \text{in } \overline{X_l}.$$

Since the new subarcs have lengths no larger than the replaced subarcs, $\ell(\sigma) \le \ell(\gamma)$. Since the entry, exit points relative to p, q do not depend on γ, the construction of σ is independent of γ and σ is indeed an arc $p \curvearrowright q$ in $\overline{X_l}$ with shortest length.

Assume $p \in X$ and $q \in \partial_l X$. Suppose $q \in \overline{\Omega_l}$ for some $\Omega \in \mathscr{O}$. Assume $p \notin \Omega^{\mathrm{ra}}$. By (3.4b), there is a unique exit point $e \in \partial^{\mathrm{ra}}\Omega \cap X$ (that depends only on p), and by earlier work, there are unique l-geodesics $\sigma_q : e \curvearrowright q$ in $\overline{\Omega_l} \subset \overline{X_l}$ and $\sigma_p : p \curvearrowright e$ in $\overline{X_l}$, and we see that $\sigma := \sigma_p \star \sigma_q$ is a shortest arc $p \curvearrowright q$ in $\overline{X_l}$.

Suppose that for all $\Omega \in \mathscr{O}$, $q \notin \overline{\Omega_l}$. Start with any $\gamma \in \Gamma_l(p, q)$ and let (z_n) be an increasing sequence along γ with $\ell(\gamma[z_n, q]) \to 0$. If $z_n \in \Sigma(p, q)$, set $x_n := z_n$. If $z_n \notin \Sigma(p, q)$, then (3.5b) and (3.4c) provide entry, exit points $a_n, b_n \in \gamma \cap \partial^{\mathrm{ra}}\Omega_n$ with $z_n \in \gamma(a_n, b_n)$; here we set $x_n := b_n$. Thus (x_n) is an increasing sequence in $\Sigma(p, q)$ with $l(x_n, q) \to 0$ as $n \to +\infty$.

As $p, x_n \in X$, there are l-geodesics $\sigma_n : p \curvearrowright x_n$ in $\overline{X_l}$. However, $x_n \in \Sigma(p, x_{n+1})$, so $\sigma_n \subset \sigma_{n+1}$.[8] Therefore, it follows that $\sigma := \bigcup_{n \ge 1} \sigma_n$ is a rectifiable arc in $\overline{X_l}$ with terminal endpoint q and with $\ell(\sigma) = l(p, q)$. Thus σ is a shortest arc, hence an l-geodesic in $\overline{X_l}$.

[8] See the second paragraph of Sect. 3.2.2.

Assume $p, q \in \partial_l X$. If $\Sigma(p,q) = \{p,q\}$, then by (3.5b) $p, q \in \overline{\Omega_l} \subset \overline{X_l}$ for some $\Omega \in \mathscr{O}$ and hence there is an l-geodesic joining these points. Suppose there exists an $x \in X \cap \Sigma(p,q)$. Then by a previous case there are l-geodesics $p \curvearrowright x$ and $x \curvearrowright q$ which paste together to give a path $\sigma : p \curvearrowright q$ in $\overline{X_l}$ with $\ell(\sigma) = l(p,x) + l(x,q) = l(p,q)$.

3.2.4 Uniqueness and CAT(0)

Our penultimate task is to verify uniqueness of l-geodesics in $\overline{X_l}$. Let p, q be distinct points in $\overline{X_l}$, let $\sigma : p \curvearrowright q$ be the l-geodesic in $\overline{X_l}$ constructed above, and suppose $\psi : p \curvearrowright q$ is also an l-geodesic in $\overline{X_l}$. Then

$$\sigma \cap \Sigma(p,q) = \Sigma(p,q) = \psi \cap \Sigma(p,q).$$

Let $z \in \sigma \setminus \Sigma(p,q)$.

Appealing to (3.5b) we obtain an $\Omega \in \mathscr{O}$ and an arc $\alpha \subset \sigma \cap \Omega^{\mathrm{ra}}$ with $z \in \overset{\circ}{\alpha}$. This means that $\mathsf{card}(\sigma \cap \overline{\Omega_l}) \geq 2$, so by (3.4c) $z \in \sigma[a,b] \subset \overline{\Omega_l}$ where $a, b \in \partial^{\mathrm{ra}}\Omega \subset \overline{X_l}$ are the entry, exit points associated with p, q, Ω. Also by (3.4c), $a, b \in \psi$. Since $\psi[a,b] \subset \overline{\Omega_l}$ is a shortest arc and $\overline{\Omega_l}$ is CAT(0) (by Sect. 3.1), it must be that $\psi[a,b] = \sigma[a,b]$.

By symmetry it now follows that $\sigma = \psi$.

Finally, we confirm the CAT(0) property for $\overline{X_l}$. Let $\Delta := [a,b,c]_l = [a,b]_l \cup [b,c]_l \cup [c,a]_l$ be a geodesic triangle in $\overline{X_l}$. By Fact 2.7 we may assume that Δ is tail-less. Since l-geodesics in $\overline{X_l}$ are unique, this means that Δ is a rectifiable Jordan loop in $\overline{X_l}$.

Since $[a,b]_l \cap \big([b,c]_l \star [c,a]_l\big) = \{a,b\}$, $\Sigma(a,b) = \{a,b\}$. Now (3.5b) produces an $\Omega \in \mathscr{O}$ with $a, b \in \overline{\Omega_l} \subset \overline{X_l}$ and so $[a,b]_l \subset \overline{\Omega_l}$. Similarly, $\Sigma(b,c) = \{b,c\}$ and $\Sigma(c,a) = \{c,a\}$, so $[b,c]_l \cup [c,a]_l \subset \overline{\Omega_l}$. Thus $\Delta \subset \overline{\Omega_l}$ and therefore Δ satisfies the CAT(0) vertex angle criterion.

3.3 Gromov Hyperbolicity Proof

Clearly, if X contains Euclidean disks of arbitrarily large radius, then $\overline{X_l}$ is not Gromov hyperbolic. Suppose

$$R := \sup\big\{r > 0 \;\big|\; \exists\, \mathsf{D}(x;r) \subset X\big\} < +\infty.$$

We show that X is $2R$-hyperbolic, and that this is best possible.

Let $\Delta = [a,b,c]_l$ be a geodesic triangle in $\overline{X_l}$. We may assume that Δ is tail-less; therefore, as explained immediately above, $\Delta \subset \overline{\Omega_l}$ for some $\Omega \in \mathscr{O}$.

There is a simply connected $\Omega_o \subset \Omega$ with $\partial\Omega_o = \iota(\Delta)$; see the last paragraph of Sect. 2.3.1. Let $D_o := \mathsf{D}(o;r)$ be a maximal open disk in Ω_o. Then $\partial D_o \cap \partial\Omega_o$ either consists of two antipodal points or has cardinality at least three. Below we verify that $\mathsf{card}(\partial D_o \cap \partial\Omega_o) = 3$.

Notice that $\bar{D}_o = D_o \cup \partial D_o$ isometrically embeds into $\overline{\Omega_l} \hookrightarrow \overline{X_l}$ so $S_l := \partial_l D_o$ is a Euclidean circle in $\overline{X_l}$ that bounds a Euclidean disk D_l in $\overline{X_l}$. Note too that $S_l \cap \Delta \subset \overline{\Omega_l}$ is isometrically equivalent to $\partial D_o \cap \partial \Omega_o$; one way to see this is to use the conformal model for $\overline{\Omega_l}$ (see the last paragraph of Sect. 2.3.1) in conjunction with [10, Prop. 2.14, Cor.2.17, pp. 29,35].

So, $\mathsf{card}(S_l \cap \Delta) \geq 2$; we show that $S_l \cap \Delta = \{a_o, b_o, c_o\}$ with $a_o \in (b,c)_l, b_o \in (c,a)_l, c_o \in (a,b)_l$.

First, let $E \in \{[a,b]_l, [b,c]_l, [c,a]_l\}$ be an edge of Δ. If $E \cap S_l$ contained two distinct points x, y, then the Euclidean segment $[x,y] \neq E[x,y]$ would be an l-geodesic $x \curvearrowright y$ in $\overline{\Omega_l}$ which would violate unique geodesicity for $\overline{\Omega_l}$; thus

$$\mathsf{card}(E \cap S_l) \leq 1. \tag{3.6a}$$

Also,

$$\exists\, p \in E \cap S_l \quad \Longrightarrow \quad \forall\, q \in E \setminus \{p\}, \;\; \measuredangle_p(q,o) \geq \frac{\pi}{2}. \tag{3.6b}$$

Indeed, E is a complete convex subspace of $\overline{\Omega_l}$, so any $p \in E \cap S_l$ is the unique point of E nearest to o, and so (3.6b) follows from [4, Prop. II.2.4(3), p. 177]. Here $\measuredangle_p(q,o) = \measuredangle_p(E[p,q],[p,o])$.

Thus $2 \leq \mathsf{card}(S_l \cap \Delta) \leq 3$. Suppose $S_l \cap \Delta = \{p, q\}$. Then $\iota(p), \iota(q)$ are antipodal points of ∂D_o, so the Euclidean segment $[p,q]$ is the l-geodesic $p \curvearrowright q$ in $\overline{\Omega_l}$. Now p, q are not both vertices of Δ (otherwise (3.6a) would be violated) so we can select a vertex, say a, of Δ so that

$$p \in [a,b]_l, \;\; q \in [a,c]_l, \;\; p \neq a \neq q \neq c \text{ but maybe } p = b.$$

According to (3.6b), $\measuredangle_p(a,o) \geq \pi/2$ and $\measuredangle_q(a,o) \geq \pi/2$. However, the comparison triangle $\bar{\Delta}(a,p,q)$ (in $\mathbb{C}$) cannot have two vertex angles that are both of size $\pi/2$ or larger.

Thus $\mathsf{card}(S_l \cap \Delta) = 3$. Employing (3.6a) again we see that $S_l \cap \Delta \cap \{a,b,c\} = \varnothing$. It now follows that there are points $a_o \in (b,c)_l, b_o \in (c,a)_l, c_o \in (a,b)_l$ with $S_l \cap \Delta = \{a_o, b_o, c_o\}$.

Take any point $z \in \Delta$, say $z \in [a,b]_l$, or even $z \in [a, c_o]_l$. Look at a comparison triangle $\Delta(\bar{a}, \bar{c}_o, \bar{b}_0)$ for $[a, c_o, b_o]_l$ and pick a point $w \in [a, b_o]_l$ so that the Euclidean segment $[\bar{z}, \bar{w}]$ is parallel to $[\bar{c}_o, \bar{b}_o]$. Now we see that

$$l(z,w) \leq |\bar{z} - \bar{w}| \leq |\bar{c}_o - \bar{b}_o| = l(c_o, b_o) = |\iota(c_o) - \iota(b_o)| \leq 2r \leq 2R$$

and therefore $\overline{X_l}$ is $2R$-hyperbolic.

To see that this is best possible, fix $R > 0$ and consider the set

$$X := \left\{ z \in \mathbb{C} \;\middle|\; |\mathrm{Im}(z)| \leq R \right\};$$

the points $a := t, b := Ri, c := -Ri$ where $t > 2R$; and $\Delta := [a, b, c] = [a, b] \cup [b, c] \cup [c, a]$. Let 2φ be the vertex angle for Δ at a; i.e., the angle between the edges $[a, b]$ and $[a, c]$. Pick $y \in (0, R)$ so that $z := 2R + iy \in [a, b]$. Then

$$\frac{t}{H} = \cos\varphi = \frac{2R}{h} \quad \text{and} \quad \frac{R}{H} = \sin\varphi = \frac{R-y}{h}$$

where

$$H := |a-b| = \sqrt{t^2 + R^2} \quad \text{and} \quad h := |z-b| = \frac{2RH}{t}.$$

Now $\dfrac{\mathsf{dist}(z, [a, c])}{H-h} = \sin 2\varphi = 2\dfrac{tR}{H^2}$, so

$$\mathsf{dist}(z, [a, c]) = 2R\frac{t}{H}\frac{H-h}{H} = 2R\frac{t}{\sqrt{t^2+R^2}}\left(1 - \frac{2R}{t}\right) \to 2R \text{ as } t \to +\infty.$$

Since $\mathsf{dist}(z, [b, c]) = 2R$, we see that X is δ-hyperbolic for $\delta := 2R$ but no smaller δ works. □

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