Computational Methods and Function Theory
https://doi.org/10.1007/s40315-023-00512-5

Julia Sets, Jordan Curves and Quasi-circles

Norbert Steinmetz[1]

Received: 1 September 2023 / Revised: 16 September 2023 / Accepted: 26 September 2023

Abstract
In this paper, the classification of rational functions whose Julia sets are Jordan arcs or curves, which started in (Carleson and Gamelin in Complex dynamics, Springer, Berlin, 1993; Steinmetz in Math Ann 307:531–541, 1997), will be completed. The method of proof is based on two *quasi-conformal surgery procedures*, which enables shifting the critical points in simply connected (super-)attracting and parabolic basins into a single critical point of highest possible multiplicity.

Keywords Julia set · Jordan curve · Parabolic fixed point · Quasi-conformal surgery

Mathematics Subject Classification 37F10 · 30C62

1 Introduction

Any rational function f of degree $d > 1$ divides the Riemann sphere $\widehat{\mathbb{C}}$ into the compact and non-empty Julia set $\mathcal{J}_f$ and the open Fatou set $\mathcal{F}_f = \widehat{\mathbb{C}} \setminus \mathcal{J}_f$, which is the largest open set wherein the sequence (f^n) of iterates forms a normal family. For notation and basic facts in complex dynamics the reader is referred to [1, 2, 5]. The Fatou set is either empty or consists of one, two, or infinitely many connected components. If there are two Fatou components, they are simply connected and the Julia set is a Jordan curve separating these domains from each other. If $\mathcal{F}_f$ is simply connected, the Julia set is connected, and even locally connected if f is geometrically finite; this was proved independently by Tan Lei and Yin [3] and Mattler [4]. *Geometrically finite* means that the closure of the orbit $O^+(c) = \{f^n(c) : n \in \mathbb{N}\}$ of any critical point

In memoriam Larry Zalcman.

Communicated by Alexandre Eremenko and Walter Bergweiler.

✉ Norbert Steinmetz
stein@math.uni-dortmund.de

1 Institut für Mathematik, Universität Dortmund, 44221 Dortmund, Germany

Published online: 30 November 2023

intersects $\mathcal{J}_f$ only finitely often. The classification of rational functions having Jordan arc or Jordan curve Julia sets consists in determining suitable representatives with regard to the equivalence relation $\approx$ (qc-conjugacy): rational functions f and f_0 are called *qc-conjugate* to each other, written $f \approx f_0$, if there exists some quasi-conformal mapping $\phi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$, such that the conjugation

$$\phi \circ f = f_0 \circ \phi \tag{1}$$

holds on the Julia set $\mathcal{J}_f$ and also some open neighbourhood of $\mathcal{J}_f$ minus finitely many points. The image of any circle and closed interval under some qc-mapping of the sphere is called a *quasi-circle* and a *quasi-segment*, respectively. Ordinary conjugation $S \circ f = f_0 \circ S$, where S denotes any Möbius transformation, will be written $f \sim f_0$. In the present context, the quasi-conformal surgery process mentioned in the abstract will be applied to completely invariant and simply connected components U of the Fatou set $\mathcal{F}_f$. These domains have full boundary $\partial U = \mathcal{J}_f$ and are either (super-)attracting or parabolic basins, that is, $f : U \xrightarrow{d:1} U$ is a proper mapping of full degree $d = \deg f$, and the sequence (f^n) tends to some fixed point p that is either (super-)attracting ($p \in U$ and $|f'(p)| < 1$) or parabolic ($p \in \partial U$ with $f'(p) = 1$).

Proposition A ([2, Thm. 5.1, p. 106]) *If U is (super-)attracting, f is qc-conjugated to f_0 with (super-)attracting basin $U_0$and a single critical point of order $d-1$ in U_0,which coincides with the super-attracting fixed point.*

We note that in this case, f_0 is conjugated to some polynomial of degree d.

Proposition B ([6, Thm. 1]) *If U is parabolic and the parabolic fixed point is not a critical value, then f is qc-conjugated to f_0 with parabolic basin U_0 and a single critical point of order $d-1$ in U_0.*

2 Julia Sets and Jordan Arcs

In order that the Julia set $\mathcal{J}_f$ be a Jordan arc (with endpoints α and ω) it is necessary and sufficient that

(i) the Fatou set $\mathcal{F}_f$ consists of a simply connected (super-)attracting or parabolic basin U_f, and
(ii) the Julia set $\mathcal{J}_f$ contains $d-1$ simple critical points $c_1, \ldots, c_{d-1}$, such that

$$f^{-1}(\{\alpha, \omega\}) = \{\alpha, \omega, c_1, \ldots, c_{d-1}\}.$$

Examples

- The Fatou set of $f(z) = z^2 + i$ is simply connected, the orbit of the critical point $z = 0$, however, is $0 \mapsto i \mapsto -1 + i \mapsto i$; the Julia set is locally connected, but not a Jordan arc.

- The rational map

$$f(z) = a + \frac{1+a}{1-a}\left(z - 2 + \frac{1}{z}\right),$$

  see [6], has critical points $z = \pm 1$; $z = 1 \mapsto a \mapsto 1/a = f(1/a)$ with $f'(1/a) = (1+a)^2$ belongs to $\mathcal{J}_f$ if $\operatorname{Re} a > 0$ and also if $|a| > 2$. In the first case, f is qc-conjugated to the Chebychev polynomial T_2 and $\mathcal{J}_f$ is a quasi-segment with endpoints a and $1/a$. If $a + 1 \neq 0$, f has the fixed point $z_a = (1+a)/2$ with multiplier $\lambda = f'(z_a) = -(a+3)/(a+1)$; $|\lambda| < 1$ resp. $\lambda = 1$ holds if and only if $\operatorname{Re} a < -2$ resp. $a = -2$. In the first case, f is qc-conjugate to T_2 and $\mathcal{J}_f$ is a quasi-segment. This is also true if $a = -2$, in which case, however, f is qc-conjugated to f_0 with $4f_0 \sim T_2$ ([6, Thm. 3]).

Condition (ii) was proved in [5, Thm. 4, p. 141]. There are three different cases to realise it, being visualized below ($\star$: simple critical point; $\bullet$: fixed point and endpoint of $\mathcal{J}_f$; $\circ$: also end points of $\mathcal{J}_f$; $\rightarrow$: degree-one-map; $\rightrightarrows$: degree-two-map).

The dynamics of f on $\mathcal{J}_f$.

$\begin{array}{ccccc} & & \star & & \star \\ & & \downdownarrows & & \downdownarrows \\ \star & \rightrightarrows & \circ & \rightarrow & \bullet \\ & & \upuparrows & & \upuparrows \\ & & \star & & \star \end{array}$	$\begin{array}{cc} \star & \star \\ \downdownarrows & \downdownarrows \\ \bullet & \bullet \\ \upuparrows & \upuparrows \\ \star & \star \end{array}$	$\begin{array}{ccc} \star & & \star \\ \downdownarrows & & \downdownarrows \\ \circ & \leftrightarrows & \circ \\ \upuparrows & & \upuparrows \\ \star & & \star \end{array}$
Case A ($d = 6$)	Case B ($d = 5$)	Case C ($d = 5$)

Any polynomial f with connected Julia set satisfying (ii) is qc-conjugated to $\pm T_d$, where T_d denotes the dth Chebychev polynomial ([5, Thm. 5, p. 143]). For functions f satisfying (i) and (ii), the state of the art is as follows.

- If U is (super-)attracting, f is qc-conjugated to $\pm T_d$ and the Julia set $\mathcal{J}_f$ is a quasi-segment ([6, Thm. 2]).
- If U is a parabolic basin *with parabolic fixed point an endpoint of* $\mathcal{J}_f$, then f is qc-conjugated to f_0 with $d^2 f_0 \sim \pm T_d$; the Julia set $\mathcal{J}_f$ is a quasi-segment ([6, Thm. 3]).

We note that the proof of [6, Thm. 3] tacitly relied on the hypothesis emphasised in italic, which, however, was not explicitly stated. Hence it remains to discuss the case when f has a parabolic fixed point that is not an endpoint of $\mathcal{J}$. To this end we may assume that $\mathcal{J}_f$ has endpoints 0 and ∞ and parabolic fixed point $z_0 \neq 0, \infty$. Then $f(z) = zQ(z)^2$ (cases A and B in table above) or $f(z) = Q(z)^2/z$ (case C) holds for some rational function Q (whose zeros and poles are the critical points of f on $\mathcal{J}_f$). In any case, the semi-conjugate

$$F(z) = \sqrt{f(z^2)}$$

is an odd rational function with parabolic fixed points $\pm\sqrt{z_0}$. Each of the parabolic basins $U_\pm$ contains $d-1$ critical points, which, by Proposition B, may be merged to single critical points. Since the quasi-conformal procedure respects symmetries (the quasi-conformal mapping in (1) may be assumed odd), we obtain some odd rational function $F_a \approx F$ with parabolic fixed points ± 1, $(d-1)$-fold critical points $\pm a$ and critical values $\pm A = F_a(\pm a)$. This function may be written as

$$\frac{A - F_a(z)}{A + F_a(z)} = c\left(\frac{a-z}{a+z}\right)^d. \tag{2}$$

Theorem 1 *Suppose $\mathcal{J}_f$ is a Jordan arc and f has a parabolic fixed point that is not an endpoint of $\mathcal{J}$. Then f is qc-conjugated to some function*

$$f_a(z) = (F_a(\sqrt{z}))^2,$$

where F_a is defined in (2) and $a \neq 0$ is a root of the polynomial

$$P_c(a) = (a+1)^{2d} - (a-1)^{2d} - 4cad(a^2-1)^{d-1} \quad (c = \pm 1). \tag{3}$$

For any root $a \neq 0$, the Julia set $\mathcal{J}_{f_a}$ is a Jordan arc extending from 0 to ∞. The Fatou set of f_a consists of some parabolic basin with parabolic fixed point $z = 1$ and $(d-1)$-fold critical point $z = a^2$; neither $\mathcal{J}_{f_a}$ nor $\mathcal{J}_f$ is a quasi-segment.

Remark The polynomial $P_c(a) = aQ_c(a^2)$ in (3) has a simple resp. triple zero at $a = 0$, this depending on the sign of $(-1)^d c$. We also note that $a^{2d}P_c(1/a) = P_{(-1)^{d-1}c}(a)$, hence parameters a, $-a$, $1/a$ and $-1/a$ are admissible at the same time. Numerical experiments indicate that the non-zero zeros of P_c are simple:

d	P_{-1}	P_1
3	$8a(3+2a^2+3a^4)$	a^3
4	$32a^3(5+2a^2+a^4)$	$32\mathrm{a}(1+2a^2+5a^4)$
5	$8a(5+20a^2+78a^4+20a^6+5a^8)$	$64a^3(5+6a^2+a^4)$

Proof From $F_a(z) = 1 + (z-1) + c_2(z-1)^2 + \cdots$ we obtain

$$\begin{aligned}\frac{A - F_a(z)}{A + F_a(z)} &= \frac{A-1}{A+1}\left(1 - \frac{z-1}{A-1} + \cdots\right)\left(1 - \frac{z-1}{A+1} + \cdots\right)\\ &= \frac{A-1}{A+1}\left(1 - \frac{2A}{A^2-1}(z-1) + \cdots\right),\end{aligned}$$

and

$$c\left(\frac{a-z}{a+z}\right)^d = c\left(\frac{a-1}{a+1}\right)^d\left(1-\frac{z-1}{a-1}+\cdots\right)^d\left(1-\frac{z-1}{a+1}+\cdots\right)^d$$
$$= c\left(\frac{a-1}{a+1}\right)^d\left(1-\frac{2da}{a^2-1}+\cdots\right),$$

hence

$$\frac{A-1}{A+1} = c\left(\frac{a-1}{a+1}\right)^d \quad \text{and} \quad \frac{A}{A^2-1} = \frac{da}{a^2-1}. \tag{4}$$

In exactly the same way,

$$\frac{A+1}{A-1} = c\left(\frac{a+1}{a-1}\right)^d \quad \text{and} \quad \frac{A}{A^2-1} = \frac{da}{a^2-1} \tag{5}$$

is obtained. The first identity in each (4) and (5) then yields $c^2 = 1$ and

$$A = \frac{(a+1)^d - c(a-1)^d}{(a+1)^d + c(a-1)^d},$$

hence

$$(a^2-1)[(a+1)^{2d} - (a-1)^{2d} - 4cda(a^2-1)^{d-1}] = 0.$$

The term in square brackets is our polynomial P_c. Conversely, for $c = \pm 1$ and any root $a \neq 0$ of P_c, the corresponding rational map F_a is odd and has parabolic fixed points $z = \pm 1$ and $(d-1)$-fold critical values $z = \pm a$. The Julia set is an odd Jordan curve ($\gamma(-t) = -\gamma(t)$) extending from ∞ via $1, 0, -1$ to ∞. It is not a quasi-circle since it has cusps at the pre-images of $z = \pm 1$ under any iterate F_a^n. The Julia set of $f_a(z) = (F_a(\sqrt{z}))^2$ is a Jordan arc from $z = 0$ to $z = \infty$; it contains the parabolic fixed point $z = 1$ and is, of course, not a quasi-segment. □

3 Julia Sets and Jordan Curves

In order that $\mathcal{J}_f$ be a Jordan curve it is necessary and sufficient that the Fatou set consists of two simply connected domains U_1 and U_2, whose common boundary coincides with the Julia set $\mathcal{J}_f$. The Julia set is locally connected, since the rational function f is geometrically finite (Tan Lei [3] and Mattler [4]). The domains U_1 and U_2 are either fixed domains or form a cycle. In this case, one may consider the iterate f^2 in place of f. The following is known.

- If U_1 and U_2 are (super-)attracting basins (or form a (super-)attracting cycle), f is qc-conjugated to $z \mapsto z^{\pm d}$ and $\mathcal{J}_f$ is a quasi-circle (implicitly in [2, Thm. 5.3, p. 107]).

- If U_1 is a (super-)attracting and U_2 a parabolic basin, then f is conjugated to the polynomial $P(z) = (z^d + d - 1)/d$ with parabolic fixed point $z = 1$ and $(d-1)$-fold critical point $z = 0$; $\mathcal{J}_f$ is not a quasi-circle (implicitly in [2, Thm. 5.3, p. 107]).
- If U_1 and U_2 are parabolic basins with common parabolic fixed point, f is qc-conjugated to the Blaschke product

$$\frac{z^d + \frac{d-1}{d+1}}{1 + \frac{d-1}{d+1} z^d}$$

and the Julia set $\mathcal{J}_f$ is a quasi-circle ([6, Thm. 4]).

The case of different parabolic fixed points is implicitly dealt with in Theorem 1 in a special case. For this reason, Theorem 1 will be re-stated as follows.

Theorem 2 *Suppose f is self-conjugated, that is, $S \circ f = f \circ S$ holds for some non-trivial Möbius transformation S, and the Fatou set $\mathcal{F}_f$ consists of two parabolic basis with distinct parabolic fixed points. Then f is qc-conjugated to some function F_a in (2). The Julia set is not a quasi-circle.*

Example (W. Bergweiler and A. Eremenko, private communication) For $0 < \kappa < 2$, the rational map

$$f_\kappa(z) = z\frac{z^2 + \kappa z + 1}{z^2 - \kappa z + 1}$$

has two completely invariant parabolic basins U_0 and U_∞ with parabolic fixed points $z = 0$ and $z = \infty$, respectively; f_κ satisfies $f_\kappa(z) = -1/f_\kappa(-1/z)$ and is qc-conjugated to $f_{\sqrt{2}}$, which has twofold critical points $(\sqrt{2} \pm \sqrt{6})/2$ and is conjugated to F_a given by (2) with $d = 3$, $c = -1$, and $a = (-\sqrt{3} + i\sqrt{6})/3$.

Of course there might also exist rational functions f of degree d with parabolic fixed points p_1, p_2 and $(d-1)$-fold critical points c_1, c_2 that are not self-conjugated. In any case, f is conjugated to some function with parabolic fixed points $1, -1$ and $(d-1)$-fold critical points $a, -a$ if and only if the *cross-ratio equation*

$$(p_1, p_2, c_1, c_2) = (1, -1, a, -a)$$

is satisfied by a. For any such a some rational function F will be defined by

$$\frac{A - F(z)}{B - F(z)} = c\left(\frac{a - z}{a + z}\right)^d \quad (A = F(a),\ B = F(-a)). \tag{6}$$

As in the proof of Theorem 1 we obtain

$$\frac{A - 1}{B - 1} = c\left(\frac{a - 1}{a + 1}\right)^d, \quad \frac{A + 1}{B + 1} = c\left(\frac{a + 1}{a - 1}\right)^d, \quad \text{and}$$

$$\frac{B-A}{(A-1)(B-1)} = \frac{2da}{a^2-1} = \frac{B-A}{(A+1)(B+1)}.$$

The second equation gives $(A-1)(B-1) = (A+1)(B+1)$, hence $A+B = F(a) + F(-a) = 0$. This, however, was just the starting point for the proof of Theorem 1 and so of Theorem 2. We thus have proved

Theorem 3 *Let f be any rational map of degree d with two completely invariant parabolic basins and distinct parabolic fixed points. Then f is qc-conjugated to some function F_a given by (2); in particular, f is non-trivially self-conjugated.*

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Beardon, A.: Iteration of Rational Functions. Springer, Berlin (1991)
2. Carleson, L., Gamelin, T.W.: Complex Dynamics. Springer, Berlin (1993)
3. Lei, T., Yin, Y.: Local connectivity of the Julia set for geometrically finite rational maps. Sci. China **39**, 39–47 (1996)
4. Mattler, Ch.: Juliamengen und lokaler Zusammenhang, Dissertation (PhD thesis), Dortmund (1996)
5. Steinmetz, N.: Rational Iteration. Verlag Walter de Gruyter, Berlin (1993)
6. Steinmetz, N.: Jordan and Julia. Math. Ann. **307**, 531–541 (1997)