Groups of Rotations of Euclidean and Hyperbolic Spaces

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Abstract

In both the Euclidean plane $\mathbb{R}^2$ and the hyperbolic plane $\mathbb{H}^2$, a non-trivial group of rotations has a unique fixed point. We compare groups of rotations of the three-dimensional spaces $\mathbb{R}^3$ and $\mathbb{H}^3$, and in each case we discuss the existence of a (possibly non-unique) common fixed point of the elements in such a group.

Keywords Möbius maps · Hyperbolic rotations · Euclidean rotations · Unitary matrices

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1 Introduction

Our starting point is the following classical result about subgroups of the orthogonal group $SO(3)$.

Theorem 1 *Any finite subgroup G of the orthogonal group $SO(3)$ is either cyclic, dihedral, or isomorphic to one of the permutation groups A_4 , S_4 or A_5 .*

Theorem 1 has a strong geometric content for if we regard $SO(3)$ as the group of rotations of $\mathbb{R}^3$ about the origin, then the groups given there are precisely the groups of rotational symmetries of plane polygons, and of the five Platonic solids (see, for example [3, 7, 12, 14]). Nevertheless, by expressing the result in terms of the matrix

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group $\mathrm{SO}(3)$ rather than the group of isometries of $\mathbb{R}^3$, we are making the unnecessary assumption that each of the isometries in G fixes the origin in $\mathbb{R}^3$. Indeed, without any further assumptions, any finite group G of isometries of $\mathbb{R}^3$ is a group of rotations about some point in $\mathbb{R}^3$. To see this, take any orbit $\mathcal{O}$ under the action of G , and let $\mathcal{B}$ be the smallest closed ball that contains $\mathcal{O}$. Then $\mathcal{B}$ is uniquely determined by $\mathcal{O}$ and is G -invariant because $\mathcal{O}$ is. Thus the centre of $\mathcal{B}$ is also G -invariant. Better still, a common fixed point exists for any group of rotations, even when the group is infinite, and we have the next result.

Theorem 2 *Let G be a (finite or infinite) group of isometries of $\mathbb{R}^3$, in which every non-trivial element is a rotation. Then the non-trivial rotations in G have either a common axis or a unique common fixed point in $\mathbb{R}^3$.*

Next, let $\mathbb{B}^3$ be the open unit ball $\{x \in \mathbb{R}^3 : \|x\| < 1\}$ in $\mathbb{R}^3$; then $\mathbb{B}^3$ with the metric $2|dx|/(1 - \|x\|^2)$ is a model of three-dimensional hyperbolic space. As $\mathrm{SO}(3)$ acts as the group of hyperbolic rotations of $\mathbb{B}^3$ about the origin, it is perhaps not surprising that a version of Theorem 2 holds for the hyperbolic space $\mathbb{B}^3$.

Theorem 3 *Let G be a (finite or infinite) group of rotations of three-dimensional hyperbolic space $\mathbb{B}^3$. Then the non-trivial rotations in G either have a common axis or a unique common fixed point in $\mathbb{B}^3$.*

Of course, although Theorem 3 refers to the model $\mathbb{B}^3$, the analogous statement holds for any model of three-dimensional hyperbolic space. Theorem 3 is due to Fatou [5, p. 62], where it occurs as part of an extensive discussion of Möbius maps and hyperbolic spaces. An alternative proof of Theorem 3 is given in [10] which, in the words of its authors, “contains a simple and self-contained proof” of Theorem 3, where the argument presented “is given in a much fuller context” in (Fatou’s account) [5]. More precisely, the argument given in [10] yields the following algebraic, but equivalent, version of Theorem 3.

Theorem 4 *Every group of Möbius maps that contains only elliptic transformations (and the identity) is conjugate to a group of unitary Möbius maps, and therefore to a subgroup of $\mathrm{SO}(3)$.*

To see that this is equivalent to Theorem 3, let $\mathcal{M}$ be the group of Möbius maps $z \mapsto (az + b)/(cz + d)$, where $ad - bc \neq 0$, and let $\mathcal{U}$ be the subgroup of unitary Möbius maps, namely those of the form

$$z \mapsto \frac{az - \bar{c}}{cz + \bar{a}}, \quad |a|^2 + |c|^2 = 1,$$

where the matrix of coefficients is in the special unitary group $\mathrm{SU}(2, \mathbb{C})$. Next, a Möbius map is *elliptic* if and only if it is conjugate to a unitary Möbius map. Now $\mathcal{M}$ acts on the extended complex plane $\mathbb{C}_\infty$ and, under conjugation by the stereographic projection of $\partial\mathbb{B}^3$ onto $\mathbb{C}_\infty$, the group of Euclidean rotations of $\mathbb{B}^3$, equivalently $\mathrm{SO}(3)$, corresponds to the group $\mathcal{U}$ of unitary Möbius maps. These facts show that Theorems 3 and 4 are equivalent to each other.

A close inspection of [5] and [10] shows that Fatou's proof is largely geometric, and that the proof in [10] follows the same lines, but is stripped of all references to geometry. More precisely, the proof in [10] is about groups acting on the surfaces $\mathbb{C}_\infty$ and $\mathbb{S}^2$, and even though Theorem 3 is fundamentally about three-dimensional hyperbolic geometry, the proof of Theorem 4 in [10] does not even mention the existence of hyperbolic geometry. In this paper we present a new, and entirely geometric, proof of Theorem 3 which, we believe, is more direct, transparent, and simpler, than any of the earlier proofs, and in which Theorem 4 plays no role. In addition our proof explains the different roles played by the Euclidean and hyperbolic geometries in Theorems 2 and 3.

The proof in [10] is (essentially) reproduced in each of [1, p. 70–72], [9, p. 42] and [16, p. 236]. A similar proof (but from a slightly different perspective) is given in [4, pp. 22–24], and a geometric proof is suggested in [18]. Finally, and without going into details here, it has been shown more recently [17] that the analogous higher-dimensional version of Theorem 3 is true in four-dimensional hyperbolic space, but fails in hyperbolic spaces of dimension five or more.

2 The Hyperbolic Space $\mathbb{H}^3$

In this section we review the basic ideas about Möbius maps and hyperbolic spaces for these provide the geometric background of our discussion. We shall use the model $\mathbb{H}^3$, which is $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$, of hyperbolic space with the hyperbolic metric $|dx|/x_3$. For more details, see, for example, any of [1, 2, 8, 11, 13–15], and especially [6].

The extended complex plane $\mathbb{C}_\infty$ is the boundary of $\mathbb{H}^3$. A hyperbolic plane Π in $\mathbb{H}^3$ is either a Euclidean hemisphere, or a Euclidean half-plane, in $\mathbb{H}^3$ that is orthogonal to the co-ordinate plane $x_3 = 0$ which is identified with $\mathbb{C}$. The Euclidean closure of Π meets $\mathbb{C}_\infty$ in a circle C (which includes ∞ when C is a line), and we shall call C the *horizon* of Π . Every circle in $\mathbb{C}_\infty$ is the horizon of some hyperbolic plane. A *geodesic* (equivalently, a hyperbolic line) in $\mathbb{H}^3$ is the (non-empty) intersection of two distinct hyperbolic planes. Each geodesic has two endpoints, say u and v in $\mathbb{C}_\infty$, and we denote the geodesic by $\langle u, v \rangle$. Note that two geodesics in $\mathbb{H}^3$ lie in a plane Π if and only if their (four) endpoints lie on the horizon of Π : equivalently, *two geodesics are coplanar if and only if their endpoints are concyclic*. If two distinct geodesics γ_1 and γ_2 are coplanar, then exactly one of the following holds:

- (1) γ_1 and γ_2 are *concurrent* (they cross at some point in $\mathbb{H}^3$);
- (2) γ_1 and γ_2 are *parallel* (they have exactly one common end-point);
- (3) γ_1 and γ_2 are *ultra-parallel* (they are disjoint with no common end-point).

If two geodesics are not coplanar, we say they are *skew* geodesics.

The group $\mathcal{M}$ of Möbius maps is the group of conformal bijections of the extended complex plane $\mathbb{C}_\infty$ onto itself. Each Möbius map can be expressed as an even number of inversions in circles (including lines) in $\mathbb{C}_\infty$, and by using these inversions, the action of $\mathcal{M}$ extends so as to be the group of all conformal isometries of the hyperbolic metric on $\mathbb{H}^3$. Alternatively, each Möbius map can be extended algebraically (by using

quaternions) to act on $\mathbb{H}^3$, and this process is consistent with that obtained by using inversions (see, for example [1,6]).

It is well known that a Möbius map g is a (hyperbolic) rotation of $\mathbb{H}^3$ if and only if it is *elliptic*, and this is so if and only if it has at least one fixed point in $\mathbb{H}^3$. If g is elliptic then it has two distinct fixed points, say z_1 and z_2 in $\mathbb{C}_\infty$, and these are the endpoints of a geodesic $\mathcal{A}_g$ which is called the *axis* of g . In this case, $\mathcal{A}_g = \langle z_1, z_2 \rangle$, and g fixes each point of $\mathcal{A}_g$, but no other point in $\mathbb{H}^3$. The Möbius map $g(z) = (az+b)/(cz+d)$, where $ad - bc = 1$, determines its matrix of coefficients up to a factor -1 , and g is elliptic if and only if the trace $a + d$ of the matrix lies in the real interval $(-2, 2)$. As usual, we shall use the matrices and maps interchangeably.

3 A Proof of Theorem 3

Throughout this section G will be a group of Möbius maps such that every element of G apart from the identity I is an elliptic map (equivalently, a rotation of $\mathbb{H}^3$). Our proof of Theorem 3 follows from two lemmas.

Lemma 1 *Suppose that g and h are rotations in G with distinct axes $\mathcal{A}_g$ and $\mathcal{A}_h$. Then these axes are concurrent or ultraparallel; equivalently, they are coplanar but not parallel.*

Proof Let f be a rotation of $\mathbb{H}^3$, and let x be a point of $\mathbb{H}^3$ that is not on $\mathcal{A}_f$. Then $f(x) \neq x$ so the set of points equidistant from x and $f(x)$ is a hyperbolic plane Π . If $y \in \mathcal{A}_f$ then $\rho(y, x) = \rho(f(y), f(x)) = \rho(y, f(x))$, where ρ is the hyperbolic distance in $\mathbb{H}^3$. It follows that $y \in \Pi$; thus $\mathcal{A}_f \subset \Pi$.

Now consider different g and h in G with distinct axes $\mathcal{A}_g$ and $\mathcal{A}_h$. Then $g^{-1}h$ (in G) is elliptic, and so there is a point x in $\mathbb{H}^3$ so that $g^{-1}h(x) = x$. Thus $g(x) = h(x) = x'$, say, so that both $\mathcal{A}_g$ and $\mathcal{A}_h$ lie in the plane of points equidistant from x and x' . Thus $\mathcal{A}_g$ and $\mathcal{A}_h$ are coplanar. If they were parallel, we may take the common endpoint of $\mathcal{A}_g$ and $\mathcal{A}_h$ to be ∞ . Therefore, g and h act as Euclidean rotations of $\mathbb{C}$ about different centres. The commutator $ghg^{-1}h^{-1}$ of two rotations of $\mathbb{C}$ is the identity or a non-trivial translation of $\mathbb{C}$. Because g and h are rotations with distinct centres, they do not commute, so $ghg^{-1}h^{-1}$ is a non-trivial translation. As a translation has only one fixed point in $\mathbb{C}_\infty$, it is not elliptic, contrary to the fact that it lies in the purely elliptic group G . Thus $\mathcal{A}_g$ and $\mathcal{A}_h$ are coplanar but not parallel. $\square$

Lemma 2 *Suppose that g and h are in G , and have distinct axes $\mathcal{A}_g$ and $\mathcal{A}_h$. Then these axes are concurrent.*

Proof In view of Lemma 1 it is sufficient to show that if two elliptic elements, say g and h , have axes that are ultraparallel, then the group they generate, say Γ , contains a non-trivial, non-elliptic, Möbius map. By passing to conjugate elements we may suppose that

$$\mathcal{A}_g = \langle 0, \infty \rangle, \quad \mathcal{A}_h = \langle u, v \rangle, \quad 0 < u < v. \quad (3.1)$$

Thus g is a Euclidean rotation, say by an angle ψ , about the vertical co-ordinate axis in $\mathbb{R}^3$, and the axis $\mathcal{A}_h$ lies vertically ‘above’ the segment $[u, v]$ of the real axis. Now let $f = ghg^{-1}$. Then $\mathcal{A}_f = g(\mathcal{A}_h) = \langle e^{i\psi}u, e^{i\psi}v \rangle$ (so that $\mathcal{A}_f$ is a rotated image of $\mathcal{A}_h$) so we find that $\mathcal{A}_h$ and $\mathcal{A}_f$ are ultraparallel and, in addition (and this is important) $\text{trace}^2(f) = \text{trace}^2(h)$.

Now any two ultraparallel geodesics have a common normal (which lies in the plane which contains both geodesics). So, by considering the ultraparallel geodesics $\mathcal{A}_f$ and $\mathcal{A}_h$, and again passing to a conjugate group and relabelling, we may now suppose that

$$\mathcal{A}_h = \langle -1, 1 \rangle, \quad \mathcal{A}_f = \langle -k, k \rangle, \quad k > 1, \quad (3.2)$$

and that their common normal is $\langle 0, \infty \rangle$. Thus these axes are vertical semi-circles with diameters $[-1, 1]$ and $[-k, k]$.

It follows that the matrices for h and h^{-1} are, for some real θ ,

$$\begin{pmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix}, \begin{pmatrix} \cos \theta & -i \sin \theta \\ -i \sin \theta & \cos \theta \end{pmatrix}, \quad (3.3)$$

(for these maps fix 1 and -1 and are elliptic), while the matrices for f and f^{-1} are

$$\begin{pmatrix} \cos \theta & ik \sin \theta \\ ik^{-1} \sin \theta & \cos \theta \end{pmatrix}, \begin{pmatrix} \cos \theta & -ik \sin \theta \\ -ik^{-1} \sin \theta & \cos \theta \end{pmatrix}, \quad (3.4)$$

for these maps fix k and $-k$ and are elliptic with the same traces as h and h^{-1} . It follows that the group generated by g and h contains the map $hf^{-1} = hgh^{-1}g^{-1} = [h, g]$ with matrix

$$\begin{pmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -ik \sin \theta \\ -ik^{-1} \sin \theta & \cos \theta \end{pmatrix}, \quad (3.5)$$

and trace equal to $2 + (\sqrt{k} - 1/\sqrt{k})^2 \sin^2 \theta$. As this trace exceeds 2, the map $[h, g]$ is not elliptic. $\square$

Finally we establish Theorem 3. Lemmas 1 and 2 imply that if distinct non-trivial g and h are in G , then their axes $\mathcal{A}_g$ and $\mathcal{A}_h$ are either equal or concurrent. Next, it is easy to see that if g and h have distinct, but concurrent, axes, then the three axes of g , h and gh are concurrent but not coplanar. Indeed, this is true in Euclidean space, and by considering the case when these are rotations about the origin in $\mathbb{R}^3$, and interpreting the result in terms of hyperbolic rotations of the unit ball $\mathbb{B}^3$, it is seen to be true in hyperbolic space. Thus, except in the case when all elements of G have the same axes, G contains three rotations, say f , g and h , whose axes meet at a point ζ , and such that these three axes are not coplanar. Now take any rotation R in G and suppose that the axis of R is different from those of f , g and h . We shall suppose that ζ is not on $\mathcal{A}_R$ and reach a contradiction. Now $\mathcal{A}_R$ must meet both $\mathcal{A}_f$ and $\mathcal{A}_g$, in each case at a point other than ζ . It follows that $\mathcal{A}_R$ lies in the plane determined by $\mathcal{A}_f$ and $\mathcal{A}_g$. Thus $\mathcal{A}_R$

meets $\mathcal{A}_h$ at a point in this plane, and this must be at ζ , contrary to our assumption. Hence, $\zeta \in \mathcal{A}_R$ and our proof of Theorem 3 is complete.

4 Concluding Remarks

Our first remark is that this proof lays bare the use of the commutator in such arguments. The commutator $ghg^{-1}h^{-1}$ should be viewed as the composition of g with a displaced ‘copy’ $hg^{-1}h^{-1}$ of g^{-1} , and it is important to note that the two maps in this composition *have the same trace*.

Second, we are not aware that Theorem 2 is in the literature but a proof can easily be constructed following the ideas above, noting that in Euclidean geometry there are no ultraparallel geodesics. Finally, one might also ask whether or not the obvious analogue of Theorem 2 is true in all dimensions.

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