Correction to: Some Results on L -Functions Related to Sharing Two Finite Sets

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In what follows, by Theorem 1 (resp. Theorem 2) we mean Theorem 1 (resp. Theorem 2) of [4]. In the proof of Theorem 2 (see [4, p. 608, before (3.3)]) the argument that “Since f and $\mathcal{L}$ share the set $S_2 = \{b_1, b_2\}$ CM, it follows that P and Q share the set $\{\omega_1, \omega_2\}$ CM...” is not true in general. The proof is therefore incorrect as it depends on the said argument. A modification on the values of m related to the elements of S_1 is required to correct the entire proof, and so the same is needed in the statement of Theorem 2. Moreover, in the rectified statement, IM-sharing instead of CM-sharing of the set S_2 becomes sufficient to get the conclusion.

Before giving the corrected statement and the proof of Theorem 2, we introduce the following functions regarding the set $S_1 = \{a_1, a_2, \dots, a_m\} \subset \mathbb{C}$, in view of the fact that either $0 \notin S_1$ or $0 \in S_1$. Let

$$R_1(z) = \frac{z^m - \left(\sum a_i\right) z^{m-1} + \dots + (-1)^{m-2} \left(\sum a_{i_1} a_{i_2} \dots a_{i_{m-2}}\right) z^2}{(-1)^m \left(\sum a_{i_1} a_{i_2} \dots a_{i_{m-1}}\right) z + (-1)^{m+1} a_1 a_2 \dots a_m},$$

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where $a_i \neq 0$, $i = 1, 2, \dots, m$, and

$$\begin{aligned} & (z - a_1)(z - a_2) \cdots (z - a_m) \\ &= z^m - \left(\sum a_i \right) z^{m-1} + \cdots + (-1)^m a_1 a_2 \cdots a_m = P_1(z), \end{aligned}$$

say.

Also, let

$$R_2(z) = \frac{z^{m-1} - \left(\sum a_i \right) z^{m-2} + \cdots + (-1)^{m-3} \left(\sum a_{i_1} a_{i_2} \cdots a_{i_{m-3}} \right) z^2}{(-1)^{m-1} \left(\sum a_{i_1} a_{i_2} \cdots a_{i_{m-2}} \right) z + (-1)^m a_1 a_2 \cdots a_{m-1}},$$

where $a_i \neq 0$, $i = 1, 2, \dots, m-1$, and

$$\begin{aligned} & z(z - a_1)(z - a_2) \cdots (z - a_{m-1}) \\ &= z \left[z^{m-1} - \left(\sum a_i \right) z^{m-2} + \cdots + (-1)^{m-1} a_1 a_2 \cdots a_{m-1} \right] = P_2(z), \end{aligned}$$

say.

Suppose that the numerator and the denominator of each of $R_1(z)$ and $R_2(z)$ have no common zero. Otherwise, we redefine them by adding one more term in the numerator in the same way such that the denominator contains only the non-zero constant term.

Let the number of simple zeros and multiple zeros of the numerator of $R_1(z)$ be μ_1 and ν_1 respectively, where $\mu_1 + \nu_1 \geq 2$. Also, let the number of simple zeros and multiple zeros of the numerator of $zR_2(z) - z + 1$ be μ_2 and ν_2 respectively, where $\mu_2 + \nu_2 \geq 2$.

With the above defined μ_i and ν_i ($i = 1, 2$), the corrected statement of Theorem 2 is given as follows:

Theorem 2 *Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, m be a positive integer. Suppose that $S_1 = \{a_1, a_2, \dots, a_m\}$, $S_2 = \{b_1, b_2\}$ be two subsets of $\mathbb{C}$ such that $S_1 \cap S_2 = \emptyset$ and $(b_1 - a_1)^2(b_1 - a_2)^2 \cdots (b_1 - a_m)^2 \neq (b_2 - a_1)^2(b_2 - a_2)^2 \cdots (b_2 - a_m)^2$. If f and a non-constant L -function $\mathcal{L}$ share S_1 IM and S_2 IM, and one of these conditions is satisfied:*

- (i) $m > 2\mu_1 + 4\nu_1 + 5$ when S_1 has the form $S_1 = \{a_1, a_2, \dots, a_m\}$, $a_i \neq 0$, $i = 1, 2, \dots, m$,
- (ii) $m > 2\mu_2 + 4\nu_2 + 5$ when S_1 has the form $S_1 = \{a_1, a_2, \dots, a_{m-1}, 0\}$, then $\mathcal{L} = f$.

The examples (Example 2 and Example 3) as mentioned in [4, Rem. 1] should be applicable to Theorem 1 only. The following remark is related to Theorem 2.

Remark 2 Considering $S_1 = \{z : z[(z+1)^{11} + 1] = 0\} = \{a_1, a_2, \dots, a_{12}\}$, $S_2 = \{-1, 1\}$, we see that $S_1 \cap S_2 = \emptyset$ and $\prod_{i=1}^{12} (-1 - a_i)^2 \neq \prod_{i=1}^{12} (1 - a_i)^2$. This suggests that such sets S_1, S_2 do exist, and the conclusion of Theorem 2 holds if f and $\mathcal{L}$ share S_1, S_2 IM satisfying the value of m .

In order to prove the rectified version of Theorem 2, we use the symbol H to denote the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right),$$

where F and G are two non-constant meromorphic functions. By $N_E^{(1)}(r, 1; F)$ we shall mean the counting function of common simple 1-points of F and G , and by $\overline{N}(r, 1; F| \geq 2)$ we shall mean the reduced counting function of those 1-points of F whose multiplicities are not less than 2.

The additional lemmas required for the proof, and also the corrected proof of Theorem 2 are given below. The following lemma is obtained by combining [5, Lem. 2] and [2, Lem. 3].

Lemma 6 *If two non-constant meromorphic functions F and G share 1 IM and $H \not\equiv 0$, then*

$$\begin{aligned} N_E^{(1)}(r, 1; F) &\leq \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) + \overline{N}(r, F| \geq 2) + \overline{N}(r, G| \geq 2) \\ &\quad + \overline{N}_*(r, 1; F, G) + \overline{N}_0(r, 0; F') \\ &\quad + \overline{N}_0(r, 0; G') + S(r, F) + S(r, G), \end{aligned}$$

where $\overline{N}_0(r, 0; F')$ denotes the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$, and $\overline{N}_*(r, 1; F, G)$ denotes the reduced counting function of those common 1-points of F whose multiplicities differ from the multiplicities of the corresponding 1-points of G .

Taking $m = 0$ in [1, Lem. 2.2] we obtain the following lemma.

Lemma 7 *Let F and G be two non-constant meromorphic functions sharing 1 IM. Then*

$$\begin{aligned} \overline{N}(r, 1; F) + \overline{N}(r, 1; G) - N_E^{(1)}(r, 1; F) - \frac{1}{2} \overline{N}_*(r, 1; F, G) \\ \leq \frac{1}{2} [N(r, 1; F) + N(r, 1; G)] \end{aligned}$$

Lemma 8 *Let $F_1 = R_1(f)$, $G_1 = R_1(g)$, $F_2 = fR_2(f) - f + 1$ and $G_2 = gR_2(g) - g + 1$, where f and g are two non-constant meromorphic functions in $\mathbb{C}$. If $H \not\equiv 0$ (taking H as a function of F_j, G_j), and F_j, G_j share 1 IM for any $j \in \{1, 2\}$, then*

$$\begin{aligned} \frac{m}{2} [T(r, f) + T(r, g)] &\leq N_2(r, 0; F_j) + N_2(r, 0; G_j) + N_2(r, F_j) + N_2(r, G_j) \\ &\quad + \frac{3}{2} [N(r, 0; f) + N(r, 0; g) + \overline{N}(r, f) + \overline{N}(r, g)] \\ &\quad + S(r, f) + S(r, g), \end{aligned}$$

where $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2)$ for any $a \in \mathbb{C} \cup \{\infty\}$.

Proof We shall prove the lemma only for $j = 2$, because the other case can be proved in the same manner. Using Nevanlinna's second fundamental theorem, the Valiron-Mokhonko lemma (see [3]), and Lemmas 6 and 7, we have

$$\begin{aligned} m[T(r, f) + T(r, g)] &= T(r, F_2) + T(r, G_2) \\ &\leq \overline{N}(r, 1; F_2) + \overline{N}(r, 1; G_2) + \overline{N}(r, 0; F_2) + \overline{N}(r, 0; G_2) + \overline{N}(r, F_2) \\ &\quad + \overline{N}(r, G_2) - \overline{N}_0(r, 0; F'_2) - \overline{N}_0(r, 0; G'_2) + S(r, F_2) + S(r, G_2) \\ &\leq \frac{1}{2}[N(r, 1; F_2) + N(r, 1; G_2)] + \frac{3}{2}\overline{N}_*(r, 1; F_2, G_2) + N_2(r, 0; F_2) \\ &\quad + N_2(r, 0; G_2) + N_2(r, F_2) + N_2(r, G_2) + S(r, F_2) + S(r, G_2). \end{aligned}$$

Let $\overline{N}_L(r, 1; F_2)$ (resp. $\overline{N}_L(r, 1; G_2)$) denote the reduced counting function of those common 1-points of F_2, G_2 , where the multiplicity of the 1-point of F_2 (resp. G_2) is greater than that of G_2 (resp. F_2). Then $\overline{N}_*(r, 1; F_2, G_2) = \overline{N}_L(r, 1; F_2) + \overline{N}_L(r, 1; G_2)$.

We note that a 1-point of F_2 comes only from a zero of $P_2(f)$, and $P_2(z)$ has all simple zeros. Using Nevanlinna's first fundamental theorem we obtain

$$\begin{aligned} \overline{N}_L(r, 1; F_2) &\leq \overline{N}(r, 1; F_2 | \geq 2) \\ &\leq \overline{N}(r, 0; P_2(f) | \geq 2) \\ &\leq N(r, 0; P_2(f)) - \overline{N}(r, 0; P_2(f)) \\ &\leq \sum_{i=1}^{m-1} (N(r, a_i; f) - \overline{N}(r, a_i; f)) + N(r, 0; f) - \overline{N}(r, 0; f) \\ &\leq N\left(r, 0; \frac{f'}{f}\right) - N_{\otimes}(r, 0; f') + N(r, 0; f) - \overline{N}(r, 0; f) \\ &\leq \overline{N}(r, f) + N(r, 0; f) - N_{\otimes}(r, 0; f') + S(r, f), \end{aligned}$$

where $N_{\otimes}(r, 0; f') = \overline{N}(r, 0; f' | f \neq 0; a_1, a_2, \dots, a_{m-1})$. Similarly, we get

$$\overline{N}_L(r, 1; G_2) \leq \overline{N}(r, g) + N(r, 0; g) - N_{\otimes}(r, 0; g') + S(r, g).$$

This implies

$$\begin{aligned} \frac{m}{2}[T(r, f) + T(r, g)] &\leq \frac{3}{2}[N(r, 0; f) + N(r, 0; g) + \overline{N}(r, f) + \overline{N}(r, g)] + N_2(r, 0; F_2) \\ &\quad + N_2(r, 0; G_2) + N_2(r, F_2) + N_2(r, G_2) + S(r, f) + S(r, g). \end{aligned}$$

This completes the proof of the lemma. $\square$

Proof of Theorem 2 Let $F_1 = R_1(f)$, $G_1 = R_1(\mathcal{L})$, $F_2 = fR_2(f) - f + 1$ and $G_2 = \mathcal{L}R_2(\mathcal{L}) - \mathcal{L} + 1$. By [4, Lem. 3], it is obvious that $S(r, f) = S(r, \mathcal{L}) = O(\log r)$. Again, we have $\overline{N}(r, f) = \overline{N}(r, \mathcal{L}) = O(\log r)$ as f has finitely many poles, and $\mathcal{L}$ has at most one possible pole.

We first assume that S_1 is of the form: $S_1 = \{a_1, a_2, \dots, a_m\}$, $a_i \neq 0$, $i = 1, 2, \dots, m$. Since f and $\mathcal{L}$ share S_1 IM, therefore F_1 and G_1 share 1 IM. For F_1, G_1 , we suppose that $H \not\equiv 0$. If $\alpha_1, \alpha_2, \dots, \alpha_{\mu_1}$ and $\beta_1, \beta_2, \dots, \beta_{v_1}$ are respectively the distinct simple zeros and multiple zeros of the numerator of $R_1(z)$, and if γ is the zero of the denominator of $R_1(z)$ (when it exists), then by Lemma 8 we obtain

$$\begin{aligned} & \frac{m}{2} \{T(r, f) + T(r, \mathcal{L})\} \\ & \leq \frac{3}{2} \{N(r, 0; f) + N(r, 0; \mathcal{L})\} + \sum_{i=1}^{\mu_1} \{N(r, \alpha_i; f) + N(r, \alpha_i; \mathcal{L})\} \\ & \quad + 2 \sum_{j=1}^{v_1} \{\overline{N}(r, \beta_j; f) + \overline{N}(r, \beta_j; \mathcal{L})\} + N(r, \gamma; f) + N(r, \gamma; \mathcal{L}) \\ & \quad + O(\log r) \\ & \leq \left(\frac{5}{2} + \mu_1 + 2v_1 \right) \{T(r, f) + T(r, \mathcal{L})\} + O(\log r), \end{aligned}$$

which is clearly a contradiction as $m > 2\mu_1 + 4v_1 + 5$. Therefore our assumption was wrong, and so $H \equiv 0$, which on integration gives $1/(F_1 - 1) = c_1/(G_1 - 1) + c_2$, where $c_1 (\neq 0)$, c_2 are constants. Therefore, F_1 and G_1 share 1 CM.

Next, we assume that S_1 is of the form: $S_1 = \{a_1, a_2, \dots, a_{m-1}, 0\}$, $a_i \neq 0$, $i = 1, 2, \dots, m-1$. Clearly, F_2 and G_2 share 1 IM as f and $\mathcal{L}$ share S_1 IM. For F_2, G_2 , we suppose that $H \not\equiv 0$. Then, considering the zeros of the numerator and the denominator of $zR_2(z) - z + 1$ and applying Lemma 8 similarly as above, we get

$$\frac{m}{2} \{T(r, f) + T(r, \mathcal{L})\} \leq \left(\frac{5}{2} + \mu_2 + 2v_2 \right) \{T(r, f) + T(r, \mathcal{L})\} + O(\log r),$$

a contradiction as $m > 2\mu_2 + 4v_2 + 5$. Therefore the assumption that $H \not\equiv 0$ is also proved wrong for the functions F_2, G_2 , and so $H \equiv 0$, which implies, F_2 and G_2 share 1 CM.

Therefore, for $m > 2\mu_j + 4v_j + 5$, and for any $j \in \{1, 2\}$, we obtain that F_j and G_j share 1 CM.

For a particular $j \in \{1, 2\}$, since zeros of $F_j - 1$ and $G_j - 1$ are the zeros of $P_j(f)$ and $P_j(\mathcal{L})$ respectively, therefore $P_j(f)$ and $P_j(\mathcal{L})$ share 0 CM whenever $m > 2\mu_j + 4v_j + 5$. Combining this with the fact that f and $\mathcal{L}$ share S_1 IM, and that $P_j(z)$ has only simple zeros, we conclude that f and $\mathcal{L}$ share S_1 CM. Then, by Theorem 1 we obtain the conclusion of this theorem. This completes the proof. $\square$

In [4, Lem. 1], “ $\mathbb{C}$ ” should be replaced by “ $\mathbb{R}$ ”.

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