Correction to: Boundary Integral Formula for Harmonic Functions on Riemann Surfaces

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Abstract

We present a corrected proof of Proposition 2.3 in the article “Boundary integral formula for harmonic functions on Riemann surfaces” [2].

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Let $\tilde{V}$ be a compact Riemann surface and let

$$V = \tilde{V} \setminus \left(\bigcup_{r=1}^m V_r \right) \quad (1)$$

be a subdomain in $\tilde{V}$, where $\{V_r\}_{r=1}^m$ is a collection of disjoint neighborhoods with smooth boundaries. In [2, Prop. 2.3], the author claimed that for a specially embedded $V \subset \mathbb{CP}^2$ and an arbitrary harmonic function $u \in C^1(\overline{V})$ there exists a harmonic function $h \in C^1(\overline{V})$ such that

$$\int_{\partial V_r} \partial(u - h) = 0 \quad \text{for } r = 1, \dots, m, \quad (2)$$

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where ∂ is the operator of holomorphic differentiation. As substantiating evidence the author suggested the existence of harmonic functions with log-type singularities at the points $z^{(r)} \in \tilde{V} \setminus V$, $r = 1, \dots, m$. However, this argument on its own is not sufficient, and in the present note we provide additional arguments to complete the proof. In the proposition below, we prove the existence of a representation satisfying (2) for a general Riemann surface of type (1).

Proposition 1 *Let V be a smoothly bounded Riemann surface as in (1). Then there exists a set of real-valued harmonic functions $\{h_j\}_{j=1}^{m-1} \in C^1(\bar{V})$ such that for any harmonic function $u \in C^1(\bar{V})$ there exist coefficients $\{a_j\}_{j=1}^{m-1} \in \mathbb{C}$ satisfying*

$$\int_{bV_r} \partial \left(u - \sum_{j=1}^{m-1} a_j h_j \right) = 0 \quad \text{for } r = 1, \dots, m. \quad (3)$$

The formulas in the original article [2], which refer to Proposition 2.13 need also to be corrected. In particular, the final formula (1.12) and formula (3.6) should be replaced with the formula

$$u(w^{(k)}) = \frac{1}{d+1} \cdot \operatorname{Re} \left\{ \frac{w_0^{(k)} \cdot \det A_k[g](w, w^{(1)}, \dots, w^{(d-1)})}{\det A(w)} \right\} + \sum_{j=1}^{m-1} a_j h_j(w^{(k)}), \quad (4)$$

where $\{h_j\}_1^{m-1}$ are fixed harmonic functions with log-type singularities at selected fixed points in $\tilde{V} \setminus V$, and the coefficients $\{a_j\}_1^{m-1}$ depending on the form $\partial u|_{bV}$ are constructed in the proof of Proposition 1.

The proof of Proposition 1 is based on the application of three lemmas below. In those lemmas we use O. Forster's idea from his book [1] to consider two special cases of construction of harmonic functions on open Riemann surfaces.

Lemma 1 *Let $z^{(1)}, z^{(2)} \in U \Subset \tilde{V}$, where U is a coordinate neighborhood in $\tilde{V}$ with coordinate function ζ such that $\zeta(z^{(1)}) = 0$, $\zeta(z^{(2)}) = 1/2$. Then there exist neighborhoods $U_j \ni z^{(j)}$ and a holomorphic function f on $\tilde{V} \setminus (U_2 \cup z^{(1)})$ such that*

$$f|_{U_1} = \zeta^{-1} \cdot \phi, \quad (5)$$

where ϕ is a holomorphic function in U_1 with $\phi(z^{(1)}) \neq 0$, and

$$\begin{cases} \frac{1}{2\pi i} \int_{bU_1} \partial \log |f(\zeta)|^2 = -1, \\ \frac{1}{2\pi i} \int_{bU_2} \partial \log |f(\zeta)|^2 = 1. \end{cases} \quad (6)$$

Proof We consider a smooth function ψ such that

$$\psi(\zeta) = \begin{cases} 1 & \text{if } |\zeta| < r_1 \text{ for some } r_1 \in (\frac{3}{4}, 1), \\ 0 & \text{if } |\zeta| > r_2 \text{ for some } r_2 \in (r_1, 1), \end{cases} \quad (7)$$

and define the function f_0 on $U \setminus z^{(1)}$ by the formulas

$$f_0(\zeta) = \begin{cases} \exp\left(\psi(\zeta) \cdot \log\left(\frac{\zeta-1/2}{\zeta}\right)\right) & \text{if } r_1 < |\zeta| < 1, \\ \frac{\zeta-1/2}{\zeta} & \text{if } |\zeta| < r_1, \end{cases} \quad (8)$$

where $\log((\zeta - 1/2)/\zeta)$ is a univalent branch of the log, which is well defined for ζ satisfying $|\zeta| \in (r_1, 1)$. From the definition of the function f_0 it follows that it can be extended to $\tilde{V} \setminus U$ as $f_0(\zeta) = 1$.

We consider the neighborhoods

$$\begin{aligned} U_1 &= \left\{ \zeta \in U : |\zeta| < \frac{1}{4} \right\}, \\ U_2 &= \left\{ \zeta \in U : \left| \zeta - \frac{1}{2} \right| < \frac{1}{8} \right\}, \\ U_2(\epsilon) &= \left\{ \zeta \in U : \left| \zeta - \frac{1}{2} \right| < \frac{1}{8} - \epsilon \right\} \Subset U_2, \\ W &= \tilde{V} \setminus \left(\overline{U}_2(\epsilon) \cup \left\{ |\zeta| \leq \frac{1}{8} \right\} \right), \end{aligned} \quad (9)$$

satisfying the following equalities

$$\begin{aligned} U_1 \cup W &= \tilde{V} \setminus \overline{U}_2(\epsilon), \\ U_1 \cap W &= \left\{ \zeta \in U : \frac{1}{8} < |\zeta| < \frac{1}{4} \right\}. \end{aligned} \quad (10)$$

To construct a meromorphic function on $\tilde{V} \setminus \overline{U}_2$ satisfying (5) we consider the smooth differential form $\alpha^{(0,1)}$

$$\begin{aligned} \alpha^{(0,1)} &= \frac{\bar{\partial} f_0}{f_0} && \text{on } W, \\ \alpha^{(0,1)} &= 0 && \text{on } U_1. \end{aligned}$$

Using the solvability of the $\bar{\partial}$ -equation on the open Riemann surface $\tilde{V} \setminus \overline{U}_2(\epsilon)$ (see [1]) we obtain a smooth function g such that $\bar{\partial} g = \alpha^{(0,1)}$ and the meromorphic function $f(z) = e^{-g} \cdot f_0$ on $\tilde{V} \setminus \overline{U}_2(\epsilon)$ having the only pole at $z^{(1)}$ and no zeros, and satisfying on $\tilde{V} \setminus (\overline{U}_2(\epsilon) \cup z^{(1)})$ the condition

$$\bar{\partial} f = -e^{-g} f_0 \bar{\partial} g + e^{-g} \bar{\partial} f_0 = e^{-g} \cdot f_0 \left(-\bar{\partial} g + \frac{\bar{\partial} f_0}{f_0} \right) = 0.$$

From (8) we obtain that for $\zeta \in U$, $|\zeta| < r_1$ we have

$$\log |f(\zeta)|^2 = \log |e^{-g(\zeta)}|^2 + \log \left| \zeta - \frac{1}{2} \right|^2 - \log |\zeta|^2,$$

and therefore

$$\begin{aligned} & \frac{1}{2\pi i} \int_{bU_1} \partial \log |f(\zeta)|^2 \\ &= \frac{1}{2\pi i} \left(\int_{bU_1} \partial \log |e^{-g(\zeta)}|^2 + \int_{bU_1} \partial \log \left| \zeta - \frac{1}{2} \right|^2 - \int_{bU_1} \partial \log |\zeta|^2 \right) \\ &= \frac{1}{2\pi i} \left(\int_{bU_1} \frac{-e^{-\bar{g}(\zeta)} e^{-g(\zeta)}}{|e^{-g(\zeta)}|^2} \partial g(\zeta) + \int_{bU_1} \frac{\bar{\zeta} - \frac{1}{2}}{|\zeta - \frac{1}{2}|^2} d\zeta - \int_{bU_1} \frac{\bar{\zeta}}{|\zeta|^2} d\zeta \right) \\ &= \frac{1}{2\pi i} \left(- \int_{bU_1} \partial g(\zeta) + \int_{bU_1} \frac{d\zeta}{\zeta - \frac{1}{2}} - \int_{bU_1} \frac{d\zeta}{\zeta} \right). \end{aligned} \quad (11)$$

Since g is holomorphic in $\{\zeta \in U : |\zeta| < r_1\} \setminus \overline{U}_2(\epsilon)$, we have $\partial g = dg$, and therefore

$$\int_{bU_1} \partial g(\zeta) = \int_{bU_1} dg(\zeta) = 0.$$

Similarly, we have

$$\int_{bU_1} \frac{d\zeta}{(\zeta - 1/2)} = 0 \quad \text{and} \quad \int_{bU_1} \frac{d\zeta}{\zeta} = 2\pi i.$$

Then using the equalities above and similar equalities for $\int_{bU_2}$ we obtain equalities (6). $\square$

Lemma 2 *Let $U_0 \Subset \tilde{V}$ be an open neighborhood in $\tilde{V}$, and let $z^{(1)}, z^{(2)} \in U \Subset \tilde{V} \setminus \overline{U}_0$, where U is a coordinate neighborhood in $\tilde{V}$ with coordinate function ζ such that $\zeta(z^{(1)}) = 0$, $\zeta(z^{(2)}) = 1/2$. Then there exist neighborhoods $U_j \ni z^{(j)}$, $j = 1, 2$, and a meromorphic function f on $\tilde{V} \setminus (\overline{U}_0 \cup z^{(1)} \cup z^{(2)})$ such that*

$$\begin{cases} \frac{1}{2\pi i} \int_{bU_1} \partial \log |f(\zeta)|^2 = -1, \\ \frac{1}{2\pi i} \int_{bU_2} \partial \log |f(\zeta)|^2 = 1. \end{cases} \quad (12)$$

Proof As in the proof of Lemma 1 we consider a smooth function ψ satisfying conditions (7) and define the function f_0 on $U \setminus (z^{(1)} \cup z^{(2)})$ by (8). Again from the definition of the function f_0 it follows that it can be extended to $\tilde{V} \setminus U$ as $f_0(\zeta) = 1$.

We consider the neighborhoods

$$\begin{aligned} U_1 &= \left\{ \zeta \in U : |\zeta| < \frac{1}{4} \right\}, \\ U_2 &= \left\{ \zeta \in U : \left| \zeta - \frac{1}{2} \right| < \frac{1}{4} \right\}, \\ W &= \tilde{V} \setminus \left(\overline{U}_0 \cup \left\{ |\zeta| \leq \frac{1}{8} \right\} \cup \left\{ \left| \zeta - \frac{1}{2} \right| \leq \frac{1}{8} \right\} \right), \end{aligned} \quad (13)$$

satisfying the following equalities

$$\begin{aligned} (U_1 \cup U_2) \cup W &= \tilde{V} \setminus \overline{U}_0, \\ (U_1 \cup U_2) \cap W &= \left\{ \zeta \in U : \frac{1}{8} < |\zeta| < \frac{1}{4}, \frac{1}{8} < \left| \zeta - \frac{1}{2} \right| < \frac{1}{4} \right\}. \end{aligned} \quad (14)$$

To construct a meromorphic function on $\tilde{V} \setminus \overline{U}_0$ satisfying (12) we consider the smooth differential form $\alpha^{(0,1)}$

$$\begin{cases} \alpha^{(0,1)} = \frac{\bar{\partial} f_0}{f_0} & \text{on } W, \\ \alpha^{(0,1)} = 0 & \text{on } (U_1 \cup U_2). \end{cases}$$

Then on the open Riemann surface $\tilde{V} \setminus \overline{U}_0$ we consider a smooth function g such that $\bar{\partial} g = \alpha^{(0,1)}$ and the meromorphic function $f(z) = e^{-g} \cdot f_0$ on $\tilde{V} \setminus (\overline{U}_0 \cup z^{(1)} \cup z^{(2)})$ satisfying

$$\bar{\partial} f = -e^{-g} f_0 \bar{\partial} g + e^{-g} \bar{\partial} f_0 = e^{-g} \cdot f_0 \left(-\bar{\partial} g + \frac{\bar{\partial} f_0}{f_0} \right) = 0.$$

From (8) we obtain that for $\zeta \in U$, $|\zeta| < r_1$ we have

$$\log |f(\zeta)|^2 = \log |e^{-g(\zeta)}|^2 + \log \left| \zeta - \frac{1}{2} \right|^2 - \log |\zeta|^2,$$

and therefore, as in (11)

$$\frac{1}{2\pi i} \int_{bU_1} \partial \log |f(\zeta)|^2 = \frac{1}{2\pi i} \left(- \int_{bU_1} \partial g(\zeta) + \int_{bU_1} \frac{d\zeta}{\zeta - \frac{1}{2}} - \int_{bU_1} \frac{d\zeta}{\zeta} \right).$$

Since g is holomorphic in $\{\zeta \in U : |\zeta| < r_1\} \setminus (U_1 \cup U_2)$, we have $\partial g = dg$, and therefore

$$\begin{cases} \int_{bU_1} \partial g(\zeta) = \int_{bU_1} dg(\zeta) = 0, \\ \int_{bU_2} \partial g(\zeta) = \int_{bU_2} dg(\zeta) = 0. \end{cases}$$

Using the equalities above together with equalities

$$\int_{bU_1} \frac{d\zeta}{\zeta - \frac{1}{2}} = \int_{bU_2} \frac{d\zeta}{\zeta} = 0$$

we obtain equalities (12). $\square$

The following statement formulated in the terminology of the book [1] is a corollary of Lemma 2:

Corollary 1 *Any divisor D with $\deg D = 0$ on an open Riemann surface is solvable.*

Lemma 3 *Let $z^{(1)}, z^{(2)}$ be two points in $\tilde{V}$, let $U_j \ni z^{(j)}$ be two neighborhoods of those points in $\tilde{V}$, and let $U_j(\epsilon) \subset U_j$ be slightly smaller neighborhoods. Then there exists a holomorphic function f on $\tilde{V} \setminus (\overline{U}_1(\epsilon) \cup \overline{U}_2(\epsilon))$ such that*

$$\begin{cases} \frac{1}{2\pi i} \int_{bU_1} \partial \log |f(\zeta)|^2 = -1, \\ \frac{1}{2\pi i} \int_{bU_2} \partial \log |f(\zeta)|^2 = 1. \end{cases} \quad (15)$$

Proof For the points $z^{(1)}, z^{(2)}$ we consider a sequence of points $z^{(1)} = w^{(1)}, \dots, w^{(r)} = z^{(2)}$ such that every two consecutive points in the sequence belong to the same coordinate neighborhood. Applying Lemma 1 we construct two meromorphic functions: g_1 on $\tilde{V} \setminus \overline{U}_1(\epsilon)$ with zero at $w^{(2)}$ and g_{r-1} on $\tilde{V} \setminus \overline{U}_2(\epsilon)$ with pole at $w^{(r-1)}$. Then using Lemma 2 we construct a sequence of meromorphic functions g_j , $j = 2, \dots, r-2$, on $\tilde{V} \setminus (\overline{U}_1(\epsilon) \cup \overline{U}_2(\epsilon))$ such that g_j has a pole at $w^{(j)}$ and a zero at $w^{(j+1)}$. Defining then

$$f = \prod_{j=1}^{r-1} g_j$$

we obtain the desired function. $\square$

Proof of Proposition 1

Proof Using Lemma 3 we construct a set of holomorphic functions $\{f_1, \dots, f_{m-1}\}$ satisfying (15) on $\tilde{V} \setminus (V_r \cup V_{r+1})$, and define functions $h_r = \log |f_r|^2$. Then we prove

the existence for an arbitrary harmonic $u \in C^1(\overline{V})$ of the coefficients satisfying (3). Let $l \in \mathbb{Z}$ be the number such that there exist coefficients a_j for $j > l$ satisfying

$$\int_{bV_j} \partial \left(u - \sum_{l+1}^{m-1} a_j h_j \right) = 0 \quad \text{for } j = l+2, \dots, m. \quad (16)$$

We prove the proposition by induction with respect to l . Namely, assuming that for some l equality (16) is satisfied for $j > l+1$, we define

$$a_l = \int_{bV_{l+1}} \partial \left(u - \sum_{l+1}^{m-1} a_j h_j \right)$$

and obtain that equality

$$\int_{bV_j} \partial \left(u - \sum_l^{m-1} a_j h_j \right) = 0$$

is satisfied for $j \geq l+1$. □

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2. Polyakov, P.: Boundary integral formula for harmonic functions on Riemann surfaces. Comput. Methods Funct. Theory **20**(2), 235–253 (2020)

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