Generalization of Pólya's Zero Distribution Theory for Exponential Polynomials, and Sharp Results for Asymptotic Growth

Janne M. Heittokangas^{1,3} · Zhi-Tao Wen^{2,3}

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Abstract

An exponential polynomial of order q is an entire function of the form

$$f(z) = P_1(z)e^{Q_1(z)} + \dots + P_k(z)e^{Q_k(z)},$$

where the coefficients $P_j(z)$, $Q_j(z)$ are polynomials in z such that

$$\max\{\deg(Q_j)\} = q.$$

In 1977 Steinmetz proved that the zeros of f lying outside of finitely many logarithmic strips around so called critical rays have exponent of convergence $\leq q - 1$. This result does not say anything about the zero distribution of f in each individual logarithmic strip. Here, it is shown that the asymptotic growth of the non-integrated counting function of zeros of f is asymptotically comparable to r^q in each logarithmic strip. The result generalizes the first order results by Pólya and Schwengeler from the 1920's, and it shows, among other things, that the critical rays of f are precisely the Borel directions of order q of f . The error terms in the asymptotic equations for $T(r, f)$ and $N(r, 1/f)$ originally due to Steinmetz are also improved.

Keywords Asymptotic growth · critical ray · Error term · Exponential polynomial · Exponential sum · Logarithmic strip · Number of zeros · Value distribution

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✉ Janne M. Heittokangas
janne.heittokangas@uef.fi

Extended author information available on the last page of the article

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1 Background on Exponential Sums

An exponential sum is an entire function of the form

$$f(z) = P_0(z)e^{w_0z} + \cdots + P_n(z)e^{w_mz}, \quad (1.1)$$

where the coefficients $P_j(z)$ are polynomials, $P_j(z) \not\equiv 0$ for $j = 1, \dots, m$, and if $P_0(z) \not\equiv 0$, we set $w_0 = 0$. The constants $w_j \in \mathbb{C}$ are pairwise distinct, and they are called the leading coefficients of f . Thus $w_j \neq 0$ for $j = 1, \dots, m$.

Every exponential sum (or more generally every exponential polynomial, see (1.7) below) satisfies a linear differential equation with polynomial coefficients [24]. In the literature, exponential sums can also be found as solutions to differential-difference equations [2, 21], and even to linear differential equations of infinite order [2].

The functions $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ and $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ are typical examples of exponential sums, their quotient being $\tan z$. In 1929 Ritt [19] showed that if the ratio g of two exponential sums with constant coefficients is entire, then g is an exponential sum also. The proof of this result was simplified by Lax in 1949 [14]. In 1958 Shapiro [21] conjectured that if two exponential sums with polynomial coefficients have infinitely many zeros in common, then they are both multiples of some third exponential sum. This claim is known as Shapiro's conjecture, and it remains unsolved.

We note that distribution of a -points and zero distribution of exponential sums are essentially the same thing because f is an exponential sum if and only if $f - a$ is. Zero distribution of functions f of the form (1.1) has been investigated for roughly a century by several authors, starting with the case where the coefficients P_j are constants and the leading coefficients w_j are real and commensurable. The latter means that $w_j = \alpha p_j$, where $\alpha \in \mathbb{R}$ and $p_j \in \mathbb{Z}$. The early developments, with references, are surveyed in [13].

Zero distribution results in the general case are traced back to the work of Pólya [17, 18] and Schwengeler [20]. We summarize these findings in Theorem A below, for which Dickson [3] has given a generalization in the sense that the coefficients are asymptotically polynomials. Before stating the actual result, we need to agree on the notation.

If $P_0(z) \not\equiv 0$, set $W_f = \{\bar{w}_0 = 0, \bar{w}_1, \dots, \bar{w}_m\}$, while if $P_0(z) \equiv 0$, set $W_f = \{\bar{w}_1, \dots, \bar{w}_m\}$. Moreover, set $W_f^0 = W_f \cup \{0\}$. Thus, if $P_0(z) \not\equiv 0$, then $W_f = W_f^0$. For any finite set $G \subset \mathbb{C}$, let $C(\text{co}(G))$ denote the circumference of the convex hull $\text{co}(G)$. Around a given ray $\arg(z) = \varphi$, we define a logarithmic strip

$$\Lambda(\varphi, c) = \left\{ re^{i\theta} : r > 1, |\theta - \varphi| < c \frac{\log r}{r} \right\}, \quad (1.2)$$

where $c > 0$. Finally, we let $\theta^\perp$ denote the argument of a ray parallel to one of the outer normals of $\text{co}(W_f)$. Thus the ray $\arg(z) = \theta^\perp$ is orthogonal to a line passing through one of the sides of the polygon $\text{co}(W_f)$.

Theorem A [3,17,18,20] *Let f be an exponential sum of the form (1.1). Then f has at most finitely many zeros outside of the finitely many domains $\Lambda(\theta^\perp, c)$. Moreover, let $\bar{w}_j, \bar{w}_k$ be two consecutive vertex points of $\text{co}(W_f)$, and let $\theta_0^\perp$ denote the argument of the ray parallel to the outer normal of $\text{co}(W_f)$ corresponding to its side $[\bar{w}_j, \bar{w}_k]$. Then the number $n(r, \Lambda)$ of zeros of f in $\Lambda(\theta_0^\perp, c) \cap \{|z| < r\}$ satisfies*

$$n(r, \Lambda) = |w_j - w_k| \frac{r}{2\pi} + O(1). \quad (1.3)$$

Let $n(r, 1/f)$ denote the number of zeros of f in $|z| < r$. We recall that the integrated counting function for the zeros of f and the proximity function of f are defined by

$$N\left(r, \frac{1}{f}\right) = \int_0^r \frac{n\left(t, \frac{1}{f}\right) - n\left(0, \frac{1}{f}\right)}{t} dt + n\left(0, \frac{1}{f}\right) \log r,$$

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

respectively. The Nevanlinna characteristic function of f is $T(r, f) = m(r, f) + N(r, f)$, where $N(r, f)$ is the integrated counting function for the poles of f .

By summing over all sides of $\text{co}(W_f)$ in Theorem A, we get $\sum |w_j - w_k| = C(\text{co}(W_f))$. Hence Theorem A yields the asymptotic equations

$$n\left(r, \frac{1}{f}\right) = C(\text{co}(W_f)) \frac{r}{2\pi} + O(1), \quad (1.4)$$

$$N\left(r, \frac{1}{f}\right) = C(\text{co}(W_f)) \frac{r}{2\pi} + O(\log r), \quad (1.5)$$

where $r \rightarrow \infty$ without any exceptional set. In addition, Wittich [25] has proved the asymptotic equation

$$T(r, f) = C(\text{co}(W_f^0)) \frac{r}{2\pi} + o(r), \quad (1.6)$$

where $r \rightarrow \infty$ without any exceptional set.

Exponential sums can be generalized to exponential polynomials, which are higher order entire functions of the form

$$f(z) = P_0(z)e^{Q_0(z)} + \dots + P_n(z)e^{Q_n(z)}, \quad (1.7)$$

where $P_j(z), Q_j(z)$ are polynomials for $0 \leq j \leq n$. A polynomial can be considered to be a special case of an exponential polynomial. We assume that the polynomials $Q_j(z)$ are pairwise distinct and normalized such that $Q_j(0) = 0$. This convention forces the

functions $g_j(z) = P_j(z)e^{Q_j(z)}$ to be linearly independent. Let $q = \max\{\deg(Q_j)\}$ denote the order of f . As observed in [9,22], we may then write f in the normalized form

$$f(z) = H_0(z)e^{w_0 z^q} + H_1(z)e^{w_1 z^q} + \cdots + H_m(z)e^{w_m z^q}, \quad (1.8)$$

where the functions $H_0(z), H_1(z), \dots, H_m(z)$ are non-vanishing exponential polynomials of order $\leq q - p$ for some $1 \leq p \leq q$, and $m \leq n$. The leading coefficients w_j , $0 \leq j \leq m$, are pairwise distinct and non-zero, except possibly w_0 . Thus $H_0(z) \equiv 0$ precisely when $q = \deg(Q_j)$ holds for every j .

Steinmetz [22] has proved analogues of (1.5) and (1.6) for exponential polynomials. We will improve the error terms in these asymptotic equations. So far there has been no attempt to find a higher order analogue of (1.3). The main focus of this paper is to find one. Then a higher order analogue of (1.4) will be found in two different ways as simple consequences.

The main results are stated and further motivated in Sects. 2–3. The proofs of the main results are given in Sects. 4–8.

2 Zero Distribution

So far there have been no attempts to prove a higher order analogue of Theorem A. The major part of this paper focuses on proving such a result. Before stating the actual result, we need to recall a few concepts.

If f is an exponential polynomial of the form (1.8), then its Phragmén–Lindelöf indicator function

$$h_f(\theta) = \limsup_{r \rightarrow \infty} r^{-q} \log |f(re^{i\theta})|$$

takes the form

$$h_f(\theta) = \max_j \left\{ \operatorname{Re} \left(w_j e^{iq\theta} \right) \right\},$$

see [10, p. 993]. The indicator $h_f(\theta)$ is 2π -periodic and differentiable everywhere except for cusps θ^* at which

$$h_f(\theta^*) = \operatorname{Re} \left(w_j e^{iq\theta^*} \right) = \operatorname{Re} \left(w_k e^{iq\theta^*} \right) \quad (2.1)$$

for a pair of indices j, k such that $j \neq k$. These cusps are called the critical angles for f , and the corresponding rays are called critical rays for f [9,22].

An alternative way to find the critical angles is by means of the convex hull $\operatorname{co}(W_f)$ of the conjugated leading coefficients of f . Namely, let $\arg(z) = \theta_{j,k}^\perp$ be the ray that is parallel to the outer normal of $\operatorname{co}(W_f)$ determined by two successive vertex points $\bar{w}_j, \bar{w}_k$ of $\operatorname{co}(W_f)$. Then the rays

$$\arg(z) = \frac{1}{q} \left(\theta_{j,k}^\perp + 2\pi n \right), \quad n \in \mathbb{Z}, \quad (2.2)$$

Fig. 1 Assuming that the positive real axis is a critical ray for f , the zeros of f have been considered in different regions as follows: Pólya in sectors $|\arg(z)| < \varepsilon$, Schwengeler in logarithmic strips $|\arg(z)| \leq c \log |z|/|z|$, and Heittokangas–Ishizaki–Tohge–Wen in modified logarithmic strips $|\arg(z)| \leq c \log |z|/|z|^p$. Here $\varepsilon > 0$ and $c > 0$ are real, and $p \geq 1$ is an integer

are critical rays for f , see [9, Lem. 3.2]. Thus, if $\text{co}(W_f)$ has s sides/vertices, then f has sq critical rays all together. In the particular case when $q = 1$ the critical rays and the orthogonal rays are one and the same.

Let f be given in the normalized form (1.8), where we suppose that $\rho(H_j) \leq q - p$ for $1 \leq p \leq q$. The case $p = 1$ corresponds to the standard case, while the identity $p = q$ reduces the coefficients $H_j(z)$ to be polynomials. In [9] it is proved that most of the zeros of f are inside of modified logarithmic strips

$$\Lambda_p(\theta^*, c) = \left\{ z = re^{i\theta} : r > 1, |\arg(z) - \theta^*| < c \frac{\log r}{r^p} \right\},$$

where $c > 0$ and $\arg(z) = \theta^*$ represents one of the critical rays for f . For $p \geq 2$ these domains approach the corresponding critical rays asymptotically, which does not happen in the standard case $p = 1$, see [9, Sect. 2].

Let $N_{\Lambda_p}(r)$ denote the integrated counting function of the zeros of f in $|z| < r$ lying outside of the union of the finitely many domains $\Lambda(\theta^*, c)$. Then

$$N_{\Lambda_p}(r) = O(r^{q-p} + \log r). \quad (2.3)$$

The case $p = 1$ was proved earlier in [22], while partial results for the general case were proved in [5, 16]. For (2.3) to hold, the constant $c > 0$ needs to be large enough (depending on f). In what follows, we will assume that this is always the case without further notice whenever discussing the domains $\Lambda_p(\theta^*, c)$.

The discussion above says nothing about the zero distribution in each individual modified logarithmic strip Λ_p . This motivates us to prove the following generalization of Theorem A.

Theorem 2.1 *Let f be an exponential polynomial of the form (1.8), where the coefficients $H_j(z)$ are exponential polynomials of growth $\rho(H_j) \leq q - p$ for some integer*

$1 \leq p \leq q$. Then the number $n_{\lambda_p}(r)$ of zeros of f outside of the finitely many domains $\Lambda_p(\theta^*, c)$ satisfies

$$n_{\lambda_p}(r) = O(r^{q-p} + \log r).$$

Moreover, let $\bar{w}_j, \bar{w}_k$ be two consecutive vertex points of $\text{co}(W_f)$, and let $\theta_0^\perp$ denote the argument of the ray parallel to the outer normal of $\text{co}(W_f)$ corresponding to its side $[\bar{w}_j, \bar{w}_k]$. Let $\varepsilon > 0$, and let θ^* be any critical angle for f associated with the angle $\theta_0^\perp$ by means of (2.2). Then the number of zeros $n(r, \Lambda_p)$ of f in $\Lambda_p(\theta^*, c) \cap \{|z| < r\}$ satisfies

$$n(r, \Lambda_p) = |w_j - w_k| \frac{r^q}{2\pi} + O(r^{q-p} \log^{3+\varepsilon} r). \quad (2.4)$$

We make some comments regarding Theorem 2.1, and discuss some immediate consequences of it. First, according to (2.1), at least two leading coefficients are needed to induce a critical ray. The example

$$f(z) = \sum_{j=0}^n e^{jz}, \quad n \geq 2, \quad (2.5)$$

having critical rays at $\arg(z) = \pm\pi/2$ shows that more than two leading coefficients can induce the same critical ray. It follows that the conjugate of any leading coefficient inducing a critical ray must be a boundary point of $\text{co}(W_f)$. Indeed, if (2.1) holds, then

$$\text{Re}(w_j e^{iq\theta^*}) = \text{Re}(w_k e^{iq\theta^*}) \geq \max_{l \neq k, j} \text{Re}(w_l e^{iq\theta^*}).$$

From (2.2) we get that $\arg(z) = q\theta^*$ is an orthogonal ray for a line through the points $\bar{w}_j$ and $\bar{w}_k$. Hence this line must be a support line for $\text{co}(W_f)$, and so $\bar{w}_j$ and $\bar{w}_k$ are boundary points of $\text{co}(W_f)$. However, being a boundary point of $\text{co}(W_f)$ is not enough. In proving Theorem 2.1 it is necessary that in between two consecutive Λ_p -domains precisely one exponential term is dominant. This property is explained in more detail in Lemma 7.2 below. Only those exponential terms that are associated with vertex points of $\text{co}(W_f)$ have this property. For the function f in (2.5), the dominant term in the right half-plane is e^{nz} , while the dominant term in the left half-plane is 1.

Second, keeping in mind that $\bar{w}_j, \bar{w}_k$ are two consecutive vertex points of $\text{co}(W_f)$, we get $\sum |w_j - w_k| = C(\text{co}(W_f))$, where the summation is taken over all sides of $\text{co}(W_f)$. From (2.2) we see that the orthogonal ray associated with the pair $\bar{w}_j, \bar{w}_k$ induces q critical rays. This gives rise to the asymptotic equation

$$n(r, 1/f) = qC(\text{co}(W_f)) \frac{r^q}{2\pi} + O(r^{q-p} \log^{3+\varepsilon} r). \quad (2.6)$$

Third, since a domain $\Lambda_p(\theta^*, c)$ for any p is essentially contained in an arbitrary ε -angle $\{z : |\arg(z) - \theta^*| < \varepsilon\}$, it follows that the critical rays of f are precisely

the Borel directions of order q of f . Note that f has no Borel directions of order $\rho \in (q - p, q)$. In the case $q = 1$ the critical rays of f (which are the same as the orthogonal rays of f) are precisely the Julia directions for f in a strong sense. Indeed, for any $a \in \mathbb{C}$, f has at most finitely many a -points outside of the domains $\Lambda_1(\theta^*, c)$ by Theorem A.

Finally, it seems that the best possible error term in (2.4) might be $O(r^{q-p})$, but our method does not give this.

3 Asymptotic Growth of $T(r, f)$ and $N(r, 1/f)$

For an exponential polynomial f of the normalized form (1.8), Steinmetz [22] has proved the asymptotic equations

$$N\left(r, \frac{1}{f}\right) = C(\text{co}(W_f)) \frac{r^q}{2\pi} + o(r^q), \quad (3.1)$$

$$T(r, f) = C\left(\text{co}\left(W_f^0\right)\right) \frac{r^q}{2\pi} + o(r^q). \quad (3.2)$$

These are higher order analogues of (1.5) and (1.6). The next result improves the error term in (3.2) even in the standard case $p = 1$. It is obvious that the logarithmic term in (3.3) below is only needed when $q = p$, that is, when the coefficients $H_j(z)$ are ordinary polynomials.

Theorem 3.1 *Let f be an exponential polynomial in the normalized form (1.8), where we suppose that $\rho(H_j) \leq q - p$ for $1 \leq p \leq q$. Then*

$$T(r, f) = C\left(\text{co}\left(W_f^0\right)\right) \frac{r^q}{2\pi} + O\left(r^{q-p} + \log r\right). \quad (3.3)$$

Let f be the exponential polynomial in (1.7), and suppose that f can be written in the normalized form (1.8), where $\rho(H_j) \leq q - p$ for $1 \leq p \leq q$. If one of the polynomials $Q_j(z)$ vanishes identically, we may suppose that it is $Q_0(z)$, and that no other polynomial $Q_j(z)$ vanishes identically. In this case $H_0(z)$ reduces to an ordinary polynomial in z . Under these conditions, we have the following result that improves the error term in (3.1).

Theorem 3.2 *If $H_0(z) \not\equiv 0$, we have*

$$m\left(r, \frac{1}{f}\right) = O\left(r^{q-p} + \log r\right). \quad (3.4)$$

If $Q_0(z) \equiv 0$, then we have the stronger estimate

$$m\left(r, \frac{1}{f}\right) = O(\log r). \quad (3.5)$$

Moreover, we have

$$N\left(r, \frac{1}{f}\right) = C(\operatorname{co}(W_f)) \frac{r^q}{2\pi} + O\left(r^{q-p} + \log r\right). \quad (3.6)$$

We conclude this section with the following quick consequence of (3.1). This is a weaker form than (2.6), but it is needed in proving Theorem 2.1, whereas (2.6) is a consequence of Theorem 2.1.

Corollary 3.3 *Let f be an exponential polynomial of order q such that $C = C(\operatorname{co}(W_f)) > 0$. Then*

$$n\left(r, \frac{1}{f}\right) = qC(\operatorname{co}(W_f)) \frac{r^q}{2\pi} + o(r^q).$$

Proof Write (3.1) in the form

$$C(\operatorname{co}(W_f)) \frac{r^q}{2\pi} - \varepsilon(r)r^q \leq N\left(r, \frac{1}{f}\right) \leq C(\operatorname{co}(W_f)) \frac{r^q}{2\pi} + \varepsilon(r)r^q,$$

where $0 < \varepsilon(r) < 1$ is a decreasing function such that $\varepsilon(r) \rightarrow 0$ as $r \rightarrow \infty$. Let $0 < \delta(r) < 1$ be any decreasing function such that $\delta(r) \rightarrow 0$ as $r \rightarrow \infty$. Using $\log x \geq \frac{x-1}{x}$ for $x \geq 1$, we get

$$\begin{aligned} n\left(r, \frac{1}{f}\right) &= \frac{n\left(r, \frac{1}{f}\right)}{\log(1+\delta(r))} \int_r^{r(1+\delta(r))} \frac{dt}{t} \leq \frac{1+\delta(r)}{\delta(r)} \int_r^{r(1+\delta(r))} \frac{n\left(t, \frac{1}{f}\right)}{t} dt \\ &= \frac{1+\delta(r)}{\delta(r)} \left(N\left(r(1+\delta(r)), \frac{1}{f}\right) - N\left(r, \frac{1}{f}\right) \right) \\ &= \frac{1+\delta(r)}{\delta(r)} \left(\frac{Cr^q}{2\pi} \sum_{j=1}^q \binom{q}{j} \delta(r)^j + O(\varepsilon(r)r^q) \right) \\ &= qC \frac{r^q}{2\pi} + O(\delta(r)r^q) + O(\varepsilon(r)r^q) + O\left(\frac{\varepsilon(r)r^q}{\delta(r)}\right). \end{aligned}$$

Similarly, using $\log x \leq x - 1$ for $x \geq 1$, we get

$$\begin{aligned} n\left(r, \frac{1}{f}\right) &= \frac{n\left(r, \frac{1}{f}\right)}{-\log(1-\delta(r))} \int_{r(1-\delta(r))}^r \frac{dt}{t} \geq \frac{1-\delta(r)}{\delta(r)} \int_{r(1-\delta(r))}^r \frac{n\left(t, \frac{1}{f}\right)}{t} dt \\ &= qC \frac{r^q}{2\pi} + O(\delta(r)r^q) + O(\varepsilon(r)r^q) + O\left(\frac{\varepsilon(r)r^q}{\delta(r)}\right). \end{aligned}$$

Choose $\delta(r) = \sqrt{\varepsilon(r)}$, and all error terms above are of the form $o(r^q)$. $\square$

4 Lemmas for Theorem 3.1

Let f be an exponential polynomial with no zeros. Then $f = e^P$, where P is a polynomial of degree p , and there exist constants $C > 0$ and $R > 0$ such that

$$\log |f(z)| \geq \log e^{-|P(z)|} \geq -Cr^p \quad (4.1)$$

for $|z| > R$. If f has finitely many zeros, then $f = Qe^P$, where Q is also a polynomial. The conclusion in (4.1) holds in this case also, but possibly for different constants C and R .

For an exponential polynomial f with infinitely many zeros, a lower bound for $\log |f(z)|$ is given in [9, Lem. 5.1], which originates from [23, Thm V. 19]. This estimate is valid for all z outside of certain discs centered at the zeros of f . In our applications, however, we need to be more flexible with the size of these discs, as is described next.

Lemma 4.1 *Let f be an entire function of finite order $\rho = \rho(f)$, and let $\{z_n\}$ denote the zeros of f . Suppose that the exponent of convergence of $\{z_n\}$ is $\lambda = \lambda(f) \geq 1$, and that the number of zeros of f in $|z| < r$ satisfies*

$$n\left(r, \frac{1}{f}\right) = C(r)r^\lambda,$$

where $C(r)$ is bounded from above as $r \rightarrow \infty$. Moreover, let $k : (0, \infty) \rightarrow [1, \infty)$ be any non-decreasing function. Then there exists a constant $A > 0$ such that

$$\log |f(z)| \geq -A \max \left\{ r^\rho, r^\lambda \log(rk(2r)) \right\} \quad (4.2)$$

whenever $z = re^{i\theta}$ lies outside of the discs $|z - z_n| \leq 1/k(|z_n|)$ and $|z| \leq e$.

Proof We may factorize the entire f as $f = Qe^P$, where P is a polynomial and Q is a canonical product formed with the zeros z_n of f . Set $p = \deg(P)$, in which case $\rho = \max\{p, \lambda\}$, and $q = \lfloor \lambda \rfloor$. Since the integral $\int_1^\infty N(r, 1/f)/r^{q+1} dr$ diverges, the sum $\sum_{n=1}^\infty |z_n|^{-q}$ also diverges, see [8, Lem. 1.4]. Hence q is the genus of the sequence $\{z_n\}$, see [23, p. 216] for the definition. This allows us to write Q in the form

$$Q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \cdots + \frac{1}{q} \left(\frac{z}{z_n}\right)^q\right).$$

We also write

$$\Psi(z, z_n) = \log \frac{z_n}{z_n - z} - \left(\frac{z}{z_n} + \cdots + \frac{1}{q} \left(\frac{z}{z_n}\right)^q\right),$$

so that

$$\log \frac{1}{Q(z)} = \sum_{n=1}^{\infty} \Psi(z, z_n), \quad z \neq z_n.$$

Taking positive real parts on both sides of the above equation results in

$$\log^+ \frac{1}{|Q(z)|} \leq \sum_{r_n \leq 2r} \operatorname{Re}^+(\Psi(z, z_n)) + \sum_{r_n > 2r} |\Psi(z, z_n)| =: \Sigma_1 + \Sigma_2,$$

where $r_n = |z_n|$. Similarly as in the proof of [9, Lem. 5.1], we obtain $\Sigma_2 = O(r^q)$. This estimate has nothing to do with the size of the discs around the points z_n . In Σ_1 , we have

$$\begin{aligned} \operatorname{Re}^+(\Psi(z, z_n)) &\leq \log^+ \left| \frac{z_n}{z - z_n} \right| + \sum_{j=1}^q \left(\frac{r}{r_n} \right)^j \\ &\leq \log^+ \left| \frac{z_n}{z - z_n} \right| + \left(\frac{r}{r_n} \right)^q \sum_{j=0}^{q-1} \left(\frac{r_n}{r} \right)^j \\ &\leq \log^+ \left| \frac{z_n}{z - z_n} \right| + 2^q \left(\frac{r}{r_n} \right)^q. \end{aligned}$$

If z lies outside of the discs $|z - z_n| \leq 1/k(|z_n|)$ and $|z| \leq e$, it follows that

$$\log^+ \left| \frac{z_n}{z - z_n} \right| \leq \log(r_n k(r_n)) \leq \log(2rk(2r)),$$

because $r_n \leq 2r$ and k is non-decreasing. Thus

$$\begin{aligned} \Sigma_1 &\leq n(2r) \log(2rk(2r)) + 2^q r^q \int_e^{2r} \frac{dn(t)}{t^q} + O(1) \\ &\leq O(r^\lambda \log(rk(2r))) + 2^q q r^q \int_e^{2r} \frac{n(t)}{t^{q+1}} dt = O(r^\lambda \log(rk(2r))). \end{aligned}$$

The reasoning above shows that there exists a constant $B > 0$ such that $\log |Q(z)| \geq -Br^\lambda \log(rk(2r))$ whenever $z = re^{i\theta}$ lies outside of the aforementioned closed discs. Since P is a polynomial of degree p , there is a constant $C > 0$ such that $\log |e^{P(z)}| \geq -Cr^p$ for $|z| > e$. The assertion now follows by choosing $A = B + C$. $\square$

Corollary 4.2 *Let f be an exponential polynomial of order $\rho = \rho(f)$, let $\{z_n\}$ denote the zeros of f , and let $\lambda = \lambda(f) \geq 1$ be the exponent of convergence of $\{z_n\}$. Suppose that $k : (0, \infty) \rightarrow [1, \infty)$ is any non-decreasing function. Then there exists a constant $A > 0$ such that*

$$\log |f(z)| \geq -A \max \{r^\rho, r^\lambda \log(rk(2r))\}$$

whenever $z = re^{i\theta}$ lies outside of the discs $|z - z_n| \leq 1/k(|z_n|)$ and $|z| \leq e$.

Remark 1 We may choose $k(x) = x^{\lambda+\varepsilon}$, $\varepsilon > 0$, in Lemma 4.1 and $k(x) = x^q \log^2 x$ in Corollary 4.2. For these choices of k , it follows by Riemann–Stieltjes integration that

$$\sum_{|z_n| \geq e} \frac{1}{k(|z_n|)} = \int_e^\infty \frac{dn(t)}{k(t)} < \infty.$$

Thus the projection I of the discs $|z - z_n| \leq 1/k(|z_n|)$ on the positive real axis has finite linear measure.

The following lemma follows implicitly from the discussions in [11, Sect. 3]. For the convenience of the reader, we give a direct proof.

Lemma 4.3 Suppose that $T : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is eventually a non-decreasing function. Suppose further that there exist constants $C > 0$, $q > 0$, $0 \leq p < q$ and $s \geq 0$ such that, as $r \rightarrow \infty$, T can be written in the form

$$T(r) = Cr^q + O(r^p \log^s r), \quad r \notin E \cup [0, 1], \quad (4.3)$$

where $E \subset [1, \infty)$ has finite logarithmic measure (or finite linear measure). Then the asymptotic equality in (4.3) holds for all r large enough.

Proof Let $\varepsilon \in (0, \min\{1, C\})$. Then there exist constants $R_1 = R_1(\varepsilon) > 1$ and $C_1 > 0$ such that $T(r)$ is increasing for $r > R_1$ and

$$(C - \varepsilon)r^q - C_1 r^p \log^s r \leq T(r) \leq (C + \varepsilon)r^q + C_1 r^p \log^s r,$$

where $r \in (R_1, \infty) \setminus E$. Moreover, due to $p < q$ we may suppose that R_1 is chosen large enough so that the lower bound and the upper bound for $T(r)$ are increasing functions of r . Then, using [7, Lem. 5], we can find an $R_2 = R_2(\varepsilon) > \max\{2, R_1\}$ such that, for $r > R_2$,

$$\begin{aligned} T(r) &\leq (C + \varepsilon)(1 + \varepsilon)^q r^q + C_1 (2r)^p \log^s (2r) \\ &\leq (C + \varepsilon)(1 + \varepsilon)^q r^q + 2^{p+s} C_1 r^p \log^s r, \end{aligned}$$

and similarly

$$T(r) \geq \frac{C - \varepsilon}{(1 + \varepsilon)^q} r^q - C_1 r^p \log^s r.$$

In other words,

$$(C - \varepsilon')r^q - C_2 r^p \log^s r \leq T(r) \leq (C + \varepsilon')r^q + C_2 r^p \log^s r, \quad r > R_2,$$

where $R_2 = R_2(\varepsilon') > 2$ and $C_2 = 2^{p+s} C_1$. This yields the assertion. $\square$

Lemma 4.4 [9] *Let f be an exponential polynomial of the form (1.8), $\psi_k(\theta) = \operatorname{Re}(w_k e^{iq\theta})$, $p \leq q$ and*

$$2a = \min \left\{ |\psi'_k(\theta) - \psi'_j(\theta)| : \psi_k(\theta) = \psi_j(\theta), k \neq j \right\}. \quad (4.4)$$

For a given $c > 0$ there exists an $R_0 > 0$, with the following property: If $z = re^{i\theta} \notin \Lambda_p(\theta^, c)$ for any θ^* and any $r \geq R_0$, and if $h_f(\theta) = \psi_j(\theta)$, then*

$$h_f(\theta) > \max_{k \neq j} \psi_k(\theta) + ac \frac{\log r}{r^p}.$$

5 Proof of Theorem 3.1

We divide the proof into four steps for clarity.

5.1 Preliminaries

Let $\arg(z) = \theta_j$, $j = 1, \dots, k$, be the critical rays of f arranged such that $0 \leq \theta_1 < \theta_2 < \dots < \theta_k < 2\pi$. Set $\theta_{k+1} = \theta_1 + 2\pi$. For $j = 1, \dots, k$, define $F_j = (\theta_j, \theta_{j+1})$ and

$$\begin{aligned} \Lambda_p(\theta_j, c) &= \left\{ z = re^{i\theta} : r > 1, |\arg(z) - \theta_j| < c \frac{\log r}{r^p} \right\}, \\ F_j(r) &= \left(\theta_j + c \frac{\log r}{r^p}, \theta_{j+1} - c \frac{\log r}{r^p} \right), \end{aligned}$$

where $c > 0$ is sufficiently large. The domain $\Lambda_p(\theta_j, c)$ curves asymptotically towards the critical ray $\arg(z) = \theta_j$, provided that $p \geq 2$, see [9, Sect. 2]. The interval $F_j(r)$ corresponds to the arguments in between two consecutive domains $\Lambda_p(\theta_j, c)$ and $\Lambda_p(\theta_{j+1}, c)$. Clearly $F_j(r) \rightarrow (\theta_j, \theta_{j+1}) = F_j$ as $r \rightarrow \infty$ for any $p \geq 1$.

Since f has no poles, we have

$$T(r, f) = m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

For $j = 0, \dots, m$, we define

$$\begin{aligned} E_j &= \left\{ \theta \in [0, 2\pi) : \operatorname{Re}(w_j e^{iq\theta}) > \max_{k \neq j} \operatorname{Re}(w_k e^{iq\theta}) \right\}, \\ E_j(r) &= \left\{ \theta \in [0, 2\pi) : \operatorname{Re}(w_j e^{iq\theta}) > \max_{k \neq j} \operatorname{Re}(w_k e^{iq\theta}) + c \frac{\log r}{r^p} \right\}. \end{aligned}$$

By Lemma 4.4, each set $E_j(r)$ is a finite union of intervals $F_i(r)$. Thus the set E_j is a finite union of intervals F_i . If $E_0 \neq \emptyset$, then $P_0(z) \not\equiv 0$ and $w_0 = 0$, so the definition

of the set E_j implies that the terms $|H_j(z)e^{w_j z^q}|$, $j = 1, \dots, m$, are bounded for $\arg(z) = \theta \in E_0$. On the other hand, if $\theta \notin E_j$ for some j , then $|H_j(z)e^{w_j z^q}| \leq M(r, H_j)$. Keeping in mind that $\rho(H_j) \leq q - p$, we conclude in both cases $E_0 = \emptyset$ and $E_0 \neq \emptyset$ that

$$T(r, f) = \sum_{j=1}^m \frac{1}{2\pi} \int_{E_j} \log^+ |f(re^{i\theta})| d\theta + O(r^{q-p} + \log r). \quad (5.1)$$

5.2 Upper Bound for $T(r, f)$

To estimate the integrals in (5.1) over the sets E_j , we write

$$f(z) = e^{w_j z^q} \left(H_j(z) + \sum_{k \neq j} H_k(z) e^{(w_k - w_j) z^q} \right). \quad (5.2)$$

If $z = re^{i\theta}$ and $\theta \in E_j$, it follows that

$$\begin{aligned} \log^+ |f(z)| &\leq \operatorname{Re} \left(w_j e^{iq\theta} \right) + \sum_{j=0}^m \log^+ M(r, H_j) + \log(m+1) \\ &= k_f(q\theta) r^q + O(r^{q-p} + \log r), \end{aligned} \quad (5.3)$$

where $k_f(\theta)$ is the support function for the convex set $\operatorname{co}(W_f^0)$ [15, p. 74]. Therefore

$$\begin{aligned} T(r, f) &\leq \sum_{j=1}^m \frac{r^q}{2\pi} \int_{E_j} k_f(q\theta) d\theta + O(r^{q-p} + \log r) \\ &\leq \frac{r^q}{2\pi} \int_0^{2\pi} k_f(q\theta) d\theta + O(r^{q-p} + \log r) \\ &= \frac{r^q}{2\pi} \int_0^{2\pi} k_f(\varphi) d\varphi + O(r^{q-p} + \log r) \\ &= C(\operatorname{co}(W_f^0)) \frac{r^q}{2\pi} + O(r^{q-p} + \log r), \end{aligned}$$

where the last identity follows from Cauchy's formula for convex curves. This formula can be found in many books on integral geometry.

5.3 Lower Bound for $T(r, f)$

If $q = 1$ or if $p = q$, then the coefficients $H_j(z)$ are polynomials, and no exceptional set occurs when estimating $T(r, f)$ from below. Hence we suppose that $1 \leq p \leq q - 1$. Observe that

$$\log^+(xy) \geq \max \left\{ \log^+ x - \log^+ \frac{1}{y}, 0 \right\}$$

for all $x \geq 0$ and $y > 0$. So, if $z = re^{i\theta}$ and $\theta \in E_j(r)$, by applying this observation to (5.2), we have

$$\begin{aligned} \log^+ |f(z)| &\geq \max \left\{ \operatorname{Re} \left(w_j e^{iq\theta} \right) - \log^+ \left| |H_j(z)| - \sum_{k \neq j} M(r, H_k) \left| e^{(w_k - w_j)z^q} \right| \right|^{-1}, 0 \right\} \\ &\geq \max \left\{ h_f(\theta)r^q - \log^+ \left| |H_j(z)| - \sum_{k \neq j} M(r, H_k) e^{-cr^{q-p} \log r} \right|^{-1}, 0 \right\}. \end{aligned}$$

The functions $H_j(z)$ are of order $\leq q - p \leq q - 1$ and of finite type. Therefore there exists a constant $A > 0$ such that

$$\log M(r, H_j) \leq Ar^{q-p} \log r \quad \text{and} \quad \log |H_j(z)| \geq -Ar^{q-p} \log r.$$

The latter inequality is valid for all $|z| = r$ outside of a set $I \subset (0, \infty)$ of finite linear measure. This follows by applying Lemma 4.1 with $k(x) = x^q$ to each of the coefficients $H_j(z)$, and then taking the union of all the exceptional sets involved. See also the remark following Lemma 4.1. We note that $k(x)$ could be chosen to have less growth at this point, but the particular growth rate fixed here is needed later on.

Choosing $c > 2a$ in Lemma 4.4, we have

$$\log^+ |f(z)| \geq \max\{h_f(\theta)r^q, 0\} - O(r^{q-p} + \log r), \quad (5.4)$$

where $|z| = r \notin I$. Recall that each set E_j is a finite union of intervals F_i . In fact, $E(r) := \bigcup_j E_j(r) = \bigcup_i F_i(r)$, so that

$$\begin{aligned} \int_{[0, 2\pi] \setminus E(r)} \log^+ |f(re^{i\theta})| d\theta &= \sum_{j=1}^n \int_{\theta_j + cr^{-p} \log r}^{\theta_{j+1} - cr^{-p} \log r} \log^+ |f(re^{i\theta})| d\theta \\ &= O(r^{q-p} + \log r). \end{aligned}$$

From (5.1), (5.3) and (5.4), it then follows that

$$\begin{aligned} T(r, f) &\geq \sum_{j=1}^m \frac{r^q}{2\pi} \int_{E_j(r)} \max\{h_f(\theta), 0\} d\theta + O(r^{q-p} + \log r) \\ &= \sum_{j=1}^m \frac{r^q}{2\pi} \int_{E_j} \max\{h_f(\theta), 0\} d\theta + O(r^{q-p} + \log r) \end{aligned}$$

$$\begin{aligned}
&= \frac{r^q}{2\pi} \int_0^{2\pi} \max\{h_f(\theta), 0\} d\theta + O(r^{q-p} + \log r) \\
&= C(\operatorname{co}(W_f^0)) \frac{r^q}{2\pi} + O(r^{q-p} + \log r), \quad r \notin I,
\end{aligned}$$

where the last identity follows again by Cauchy's formula and by the fact that $h_f(\theta) = k_f(q\theta)$.

5.4 Conclusion of the Proof

The upper and lower estimates just obtained for $T(r, f)$ have the same magnitude of growth, but the lower estimate is valid outside of an exceptional set I of finite linear measure. The set I can be avoided by means of Lemma 4.3. This completes the proof of (3.3).

6 Proof of Theorem 3.2

Let us begin with the particular case $Q_0(z) \equiv 0$. As the functions $g_j(z) = P_j(z)e^{Q_j(z)}$, $j = 1, \dots, n$, are linearly independent, they form a fundamental solution base for a linear differential equation

$$L_n(g) := g^{(n)} + a_{n-1}(z)g^{(n-1)} + \dots + a_0(z)g = 0, \quad (6.1)$$

see [12, Prop. 1.4.6]. The coefficient functions $a_j(z)$ can be expressed as quotients of Wronskian and modified Wronskian determinants of the functions $g_j(z)$. Note that $g_j'(z) = R_j(z)e^{Q_j(z)}$, where

$$R_j(z) = P_j'(z) + Q_j'(z)P_j(z) \neq 0.$$

This means that the exponential term $e^{Q_j(z)}$ remains in differentiation, and further differentiations will not change that. Therefore, from each column of each determinant we can pull out the single exponential term as a common factor. Thus $\exp(Q_1(z) + \dots + Q_n(z))$ is a common factor for each determinant, and they cancel out in the quotients. The entries of the remaining determinants are polynomials. Hence the coefficients $a_j(z)$ in (6.1) are rational functions.

If $P_0(z)$ were to be a solution of (6.1), it would have to be linearly dependent with the solutions $g_1, \dots, g_n$. Thus $P_0(z)$ would be transcendental, which is a contradiction. It follows that $a_n(z) := L_n(P_0) \neq 0$ is a rational function. We conclude that f is a solution of the non-homogeneous equation

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \dots + a_0(z)f = a_n(z).$$

Dividing both sides by $f a_n(z)$ and using the lemma on the logarithmic derivative together with the fact that the coefficients are rational gives us (3.5).

Suppose next that $Q_0(z) \not\equiv 0$. Define

$$g(z) = e^{-Q_0(z)} f(z) = P_0(z) + \cdots + P_n(z) e^{Q_n(z) - Q_0(z)}. \quad (6.2)$$

Now the discussion above can be applied to g giving us $m(r, 1/g) = O(\log r)$ and, a fortiori,

$$N\left(r, \frac{1}{f}\right) = N\left(r, \frac{1}{g}\right) = T(r, g) + O(\log r). \quad (6.3)$$

Suppose in addition that $H_0(z) \not\equiv 0$. Due to the assumption $\rho(H_j) \leq q - p$, we may assume, without loss of generality, that $\deg(Q_0) \leq q - p$. Thus, applying Theorem 3.1 to f and using (6.2), we get

$$\begin{aligned} T(r, f) &= C\left(\operatorname{co}\left(W_f^0\right)\right) \frac{r^q}{2\pi} + O\left(r^{q-p} + \log r\right) \\ &= T(r, g) + O\left(r^{q-p} + \log r\right). \end{aligned} \quad (6.4)$$

We deduce the estimate (3.4) by combining (6.3) and (6.4). Finally, we suppose that $H_0(z) \equiv 0$, in which case $\deg(Q_j) = q$ for every j and $m = n$. Let $U = \{\bar{w}_1 - \bar{w}_0, \dots, \bar{w}_m - \bar{w}_0\}$ be the set of conjugate leading coefficients of g , and set $U_0 = U \cup \{0\}$. By a simple vector calculus, we conclude the following: If $\bar{w}_0$ is a boundary point of $\operatorname{co}(W_f)$, then 0 is a boundary point of $\operatorname{co}(U)$, while if $\bar{w}_0$ is an interior point of $\operatorname{co}(W_f)$, then 0 is an interior point of $\operatorname{co}(U)$. Thus, applying Theorem 3.1 to g , we get

$$\begin{aligned} T(r, g) &= C(\operatorname{co}(U_0)) \frac{r^q}{2\pi} + O\left(r^{q-p} + \log r\right) \\ &= C(\operatorname{co}(W_f)) \frac{r^q}{2\pi} + O\left(r^{q-p} + \log r\right). \end{aligned}$$

The final assertion (3.6) follows from this and (6.3).

7 Lemmas for Theorem 2.1

The following lemma is considered for more general curves in [4] but in Cartesian coordinates. However, our situation is very delicate, and we need the representation in polar coordinates.

Lemma 7.1 *Let $p \geq 1$ be any real number, and let U be any collection of Euclidean discs $D_n = D(z_n, r_n)$, where the center points $z_n \in \mathbb{C}$ are ordered according to increasing moduli, $|z_n| \rightarrow \infty$, $r_n > 0$, $r_n \rightarrow 0$ and*

$$\sum_{|z_n| > e} \frac{|z_n|^{p-1} r_n}{\log |z_n|} < \infty. \quad (7.1)$$

Fig. 2 The disc D_n and the asymptotic curves in the case $p = 1$

Then the set $C \subset (0, \infty)$ of values of c for which the curve

$$\partial \Lambda_p(0, c) = \left\{ z = r e^{i\theta} : r \geq 1, |\arg(z)| = c \frac{\log r}{r^p} \right\}$$

meets infinitely many discs D_n has linear measure zero.

Proof If $z \in \partial \Lambda(0, c)$, then

$$z = r \cos \left(c \frac{\log r}{r^p} \right) \pm i r \sin \left(\frac{\log r}{r^p} \right) \sim r \pm i c \frac{\log r}{r^p}.$$

Asymptotically this corresponds to the Cartesian curves $y = \pm c \frac{\log x}{x^{p-1}}$. From this representation we see that the cases $p = 1$ and $p > 1$ have a different geometry. Nevertheless our proof works for both cases simultaneously.

By the Cartesian representation we may suppose that the points z_n lie in the right half-plane in between the lines $y = \pm x$. Since $|z_n| \rightarrow \infty$ and $r_n \rightarrow 0$, there exists a positive integer N such that

$$|z_n| - r_n \geq \frac{|z_n|}{2} \geq 2e, \quad n \geq N.$$

Note that the function $x \mapsto \log x / x^p$ is strictly decreasing for $x \geq e$. Moreover, it suffices to consider the portion of $\partial \Lambda_p(0, c)$ located in the upper half-plane, call it $\Lambda^+(c)$ for short.

Suppose that a disc D_n lies asymptotically in between the curves $\Lambda^+(c_1)$ and $\Lambda^+(c_2)$, where $c_1 < c_2$, and let ζ_1 and ζ_2 denote the respective intersection points. The disc D_n can be seen from the origin at an angle $2\theta_n$, where $\sin \theta_n = r_n / |z_n|$.

We have $\theta_n \leq 2r_n/|z_n| \rightarrow 0$ as $n \rightarrow \infty$, and

$$\begin{aligned} c_2 - c_1 &= \frac{|\zeta_2|^p \arg(\zeta_2)}{\log |\zeta_2|} - \frac{|\zeta_1|^p \arg(\zeta_1)}{\log |\zeta_1|} \\ &\leq \frac{(|z_n| + r_n)^p \arg(\zeta_2)}{\log(|z_n| + r_n)} - \frac{(|z_n| - r_n)^p \arg(\zeta_1)}{\log(|z_n| - r_n)} \\ &\leq \frac{|z_n|^p \left(1 + \frac{r_n}{|z_n|}\right)^p (\arg(z_n) + \theta_n)}{\log(|z_n| + r_n)} - \frac{|z_n|^p \left(1 - \frac{r_n}{|z_n|}\right)^p (\arg(z_n) - \theta_n)}{\log(|z_n| - r_n)} \\ &\leq \frac{|z_n|^p \arg(z_n) + 2|z_n|^p \theta_n}{\log(|z_n| - r_n)} \left(\left(1 + \frac{r_n}{|z_n|}\right)^p - \left(1 - \frac{r_n}{|z_n|}\right)^p \right). \end{aligned}$$

Using the estimate

$$\begin{aligned} \left(1 + \frac{r_n}{|z_n|}\right)^p - \left(1 - \frac{r_n}{|z_n|}\right)^p &\leq 2^p \left(1 - \left(\frac{|z_n| - r_n}{|z_n| + r_n}\right)^p\right) \\ &= 2^p p \int_{\frac{|z_n| - r_n}{|z_n| + r_n}}^1 x^{p-1} dx \leq \frac{2^{p+1} p r_n}{|z_n|}, \end{aligned}$$

we conclude that

$$\begin{aligned} c_2 - c_1 &\leq \frac{2^{p+1} p |z_n|^{p-1} r_n (\arg(z_n) + 2\theta_n)}{\log(|z_n| - r_n)} \\ &\leq \frac{2^p p \pi |z_n|^{p-1} r_n}{\log\left(\frac{|z_n|}{2}\right)} \leq \frac{2^{p+1} p \pi |z_n|^{p-1} r_n}{\log |z_n|}, \quad n \geq N. \end{aligned} \quad (7.2)$$

Let $\varepsilon > 0$. By the assumption (7.1) there exists an integer $N(\varepsilon) \geq N$ such that

$$\sum_{n \geq N(\varepsilon)} \frac{|z_n|^{p-1} r_n}{\log |z_n|} < \frac{\varepsilon}{2^{p+1} p \pi}. \quad (7.3)$$

Let then $C_\varepsilon^+ \subset (0, \infty)$ denote the set of values c for which the curve $\Lambda^+(c)$ meets at least one disc D_n , where $n \geq N(\varepsilon)$. By (7.2) and (7.3),

$$\int_{C_\varepsilon^+} dx \leq \sum_{n \geq N(\varepsilon)} \frac{2^{p+1} p \pi |z_n|^{p-1} r_n}{\log |z_n|} < \varepsilon.$$

Analogously, if $C_\varepsilon^- \subset (0, \infty)$ denotes the set of values c for which the curve $\Lambda^-(c)$ in the lower half-plane meets at least one disc D_n , where $n \geq N(\varepsilon)$, then $\int_{C_\varepsilon^-} dx < \varepsilon$. Since C is a subset of $C_\varepsilon^+ \cup C_\varepsilon^-$ for any $\varepsilon > 0$, we have $\int_C dx = 0$. $\square$

Through the rest of this section, let $k : (0, \infty) \rightarrow [1, \infty)$ denote any non-decreasing function. To simplify the notation, we restrict to the growth rate $k(r) = O(r^\sigma)$ for

some $\sigma > 0$, even though the results that will follow would allow for even faster growth. Let f be an exponential polynomial in the normalized form (1.8), and let $\{z_n\}$ denote the sequence of zeros of f and of all of its transcendental coefficients H_j , listed according to multiplicities and ordered according to increasing moduli. Finally, let $\mathcal{C}(f, k)$ denote the collection of discs $|z - z_n| \leq 1/k(|z_n|)$. For simplicity, we may also assume that a disc $|z| < R$ for a suitably large $R > 0$ is included in $\mathcal{C}(f, k)$.

The next auxiliary result is obtained by modifying the proof of [9, Thm. 2.4], see also the remark following [9, Thm. 2.4].

Lemma 7.2 *Let f be an exponential polynomial in the normalized form (1.8), where we suppose that $\rho(H_j) \leq q - p$ for $1 \leq p \leq q$. Then there exist constants $A > q$ and $c > 0$ with the following properties: If Δ denotes the domain in between any two consecutive zero-domains $\Delta_p(\theta_1^*, c)$ and $\Delta_p(\theta_2^*, c)$ of f , and if $z \in \Delta \setminus \mathcal{C}(f, k)$, then there exists a unique index j_0 such that*

$$f(z) = H_{j_0}(z)e^{w_{j_0}z^q}(1 + \varepsilon_{q-p}(r)), \quad (7.4)$$

where $|\varepsilon_k(r)| \leq B \exp(-Ar^k \log r)$, $k \geq 0$, $|z| = r$, and $B > 0$ is some constant. Denote

$$G_j(z) = H'_j(z) + qw_jz^{q-1}H_j(z).$$

For the same values of z and j_0 as above, we have

$$f'(z) = \begin{cases} G_{j_0}(z)e^{w_{j_0}z^q}(1 + \varepsilon_{q-p}(r)), & \text{if } j_0 \neq 0, \\ H'_0(z) + \varepsilon_{q-p}(r), & \text{if } j_0 = 0. \end{cases} \quad (7.5)$$

Proof (1) In order to prove (7.4), we first suppose that $q \geq 2$ and that $1 \leq p \leq q - 1$. The latter means that all coefficients $H_j(z)$ are assumed to be transcendental. There exist constants $A > q$ and $r_0 > 0$ such that

$$\log M(r, H_j) \leq Ar^{q-p}, \quad j \in \{0, \dots, m\}, \quad (7.6)$$

whenever $r \geq r_0$. We conclude by Corollary 4.2 that there exists a constant $A > q$ such that

$$\log |H_j(z)| \geq -Ar^{q-p} \log r, \quad j \in \{0, \dots, m\}, \quad (7.7)$$

whenever $z \notin \mathcal{C}(f, k)$. We may suppose that the constant A in (7.7) is the same as that in (7.6) by choosing the larger of the two. Noting that $r^q \psi_k(\theta) = \operatorname{Re}(w_k z^q)$, we make use of Lemma 4.4 by choosing $c > 4A/a$: If $z = re^{i\theta} \in \Delta$ and if r is large enough, then there exists a unique index j_0 such that

$$\operatorname{Re}(w_{j_0}z^q) \geq \operatorname{Re}(w_kz^q) + 4Ar^{q-p} \log r \quad (7.8)$$

for all $k \neq j_0$. If in addition $z \notin \mathcal{C}(f, k)$, then the estimates (7.6)–(7.8) applied to (1.8) yield

$$\begin{aligned} \left| f(z)e^{-w_{j_0}z^q} \right| &\geq |H_{j_0}(z)| - \sum_{k \neq j_0} |H_k(z)| e^{(w_k - w_{j_0})z^q} \\ &\geq |H_{j_0}(z)| - \exp(\log m + Ar^{q-p} - 4Ar^{q-p} \log r), \end{aligned}$$

from which

$$\frac{|f(z)|}{|H_{j_0}(z)e^{w_{j_0}z^q}|} \geq 1 - \exp(-Ar^{q-p} \log r).$$

On the other hand,

$$\frac{|f(z)|}{|H_{j_0}(z)e^{w_{j_0}z^q}|} \leq 1 + \sum_{k \neq j_0} \frac{|H_k(z)|}{|H_{j_0}(z)|} \left| e^{(w_k - w_{j_0})z^q} \right| \leq 1 + \exp(-Ar^{q-p} \log r),$$

where $z \in \Lambda \setminus \mathcal{C}(f, k)$. This discussion covers the case $w_{j_0} = w_0 = 0$ also, that is, the case when $f(z) - H_0(z) = \varepsilon_{p-q}(r)$ in $\lambda \setminus \mathcal{C}(f, k)$.

If some (but not all) coefficients are polynomials, the previous reasoning can be simplified. Indeed, if a particular coefficient $H_j(z)$ is a polynomial, then the growth of $|H_j(z)|$ is comparable to $|z|^{\deg(H_j)}$ for r large enough. Consideration of the set $\mathcal{C}(f, k)$ is not needed for this particular coefficient, as we only need to assume that $|z|$ is large enough.

Suppose then that $q \geq 1$ is any integer and $p = q$, that is, all of the coefficients $H_j(z)$ are polynomials. This covers the remaining case. There exist constants $A_1 > 0$ and $A_2 > 0$ such that

$$|H_j(z)| \leq A_1|z|^d \quad \text{and} \quad |H_j(z)| \geq A_2|z|^{d_j},$$

where $d_j = \deg(H_j)$ and $d = \max\{d_j\}$. Suppose that $z = re^{i\theta} \in \Lambda$, and that r is large enough. Let $A = \max\{d, 2\}$. We choose $c > 4A/a$ in Lemma 4.4, and find that there exists a unique index j_0 such that

$$\left| f(z)e^{-w_{j_0}z} \right| \geq |H_{j_0}(z)| - A_1mr^d e^{-4A \log r},$$

from which

$$\frac{|f(z)|}{|H_{j_0}(z)e^{w_{j_0}z^q}|} \geq 1 - \exp(-A \log r), \quad z \in \Lambda \setminus \mathcal{C}.$$

On the other hand,

$$\frac{|f(z)|}{|H_{j_0}(z)e^{w_{j_0}z^q}|} \leq 1 + \sum_{k \neq j_0} \frac{|H_k(z)|}{|H_{j_0}(z)|} \left| e^{(w_k - w_{j_0})z^q} \right| \leq 1 + \exp(-A \log r),$$

where $z \in \Lambda \setminus \mathcal{C}(f, k)$. This completes the proof of (7.4).

(2) In order to prove (7.5), we first note that

$$f' = G_0(z)e^{w_0 z^q} + G_1(z)e^{w_1 z^q} + \cdots + G_m(z)e^{w_m z^q},$$

where $G_j(z) = H'_j(z) + q w_j z^{q-1} H_j(z)$ for $j \geq 0$. If $G_j(z) \equiv 0$, then either $H_j(z)$ is a constant and $w_j = 0$ or $\rho(H_j) = q$. The latter is clearly impossible, while the former is possible only in the case when $j = 0$. Hence f' is an exponential polynomial with coefficients being of order $\leq q - p$, and shares the leading coefficients with f , except possibly w_0 . Thus, by considering separately the cases $j_0 \neq 0$ and $j_0 = 0$, the proof of Part (1) applies to f' , and we obtain (7.5) for the same index j_0 that appears in (7.4). This completes the proof. $\square$

If any of the coefficients H_j in (1.8) is transcendental, it also can be represented in an analogous normalized form as f , where the coefficients are either polynomials or exponential polynomials. Proceeding from one generation of transcendental coefficients to the next, the order of growth decreases at least by one. Obviously there are at most $q - 1$ generations of transcendental descendant coefficients all together.

Let $\{z_n\}$ denote the sequence of zeros of f , of the transcendental coefficients H_j of f , and of all transcendental descendants of these coefficients, listed according to their multiplicities and ordered according to increasing moduli. Let then $\mathcal{C}_0(f, k)$ denote the collection of all discs $|z - z_n| \leq 1/k(|z_n|)$, and suppose that a suitably large disc $|z| < R$ is also included in $\mathcal{C}_0(f, k)$. Clearly, $\mathcal{C}(f, k)$ is a sub-collection of $\mathcal{C}_0(f, k)$.

Let Θ denote the set consisting of the critical angles of f in (1.8), the critical angles of the transcendental coefficients H_j of f , and the critical angles of all transcendental descendants of these coefficients. Some of the critical angles of f may coincide with those of its coefficients or descendant coefficients. Nevertheless, Θ is a finite subset of $[0, 2\pi)$, so the elements θ_j of Θ can be ordered, say $0 \leq \theta_1 < \theta_2 < \cdots < \theta_l < 2\pi$. Set $\theta_{l+1} = \theta_1$.

Lemma 7.3 *Let f be an exponential polynomial in the normalized form (1.8), where we suppose that $\rho(H_j) \leq q - p$ for $1 \leq p \leq q$. Let θ_j, θ_{j+1} be two consecutive elements of Θ , and let Λ be the domain in between the domains $\Lambda_p(\theta_j, c)$ and $\Lambda_p(\theta_{j+1}, c)$, where $c > 0$ is sufficiently large. If $z \in \Lambda \setminus \mathcal{C}_0(f, k)$, then there exists a unique index j_0 and constants $C_0, \dots, C_{q-p-1} \in \mathbb{C}$ such that*

$$\frac{f'(z)}{f(z)} = q w_{j_0} z^{q-1} + C_{q-p-1} z^{q-p-1} + \cdots + C_0 + O(r^{-1}). \quad (7.9)$$

Proof We will make use of the proof of Lemma 7.2 not only for f but also for the coefficients and the descendant coefficients of f . For each function for which the proof is used, new constants $A > q$ and $c > 0$ are found. Since there are at most finitely many functions involved, we may choose A and c to be the maxima of all these corresponding coefficients.

Independently of whether $w_{j_0} = 0$ or $w_{j_0} \neq 0$, it follows directly from (7.4) and (7.5) that

$$\frac{f'(z)}{f(z)} = \left(q w_{j_0} z^{q-1} + \frac{H'_{j_0}(z)}{H_{j_0}(z)} \right) (1 + \varepsilon_{q-p}(r)). \quad (7.10)$$

Suppose that $H_{j_0}(z)$ is a polynomial. Then (7.10) yields

$$\begin{aligned} \frac{f'(z)}{f(z)} &= q w_{j_0} z^{q-1} + O\left(r^{-1}(1 + \varepsilon_{q-p}(r))\right) \\ &\quad + O\left(\exp((q-1)\log r - Ar^{q-p}\log r)\right) \\ &= q w_{j_0} z^{q-1} + O(r^{-1}), \quad z \in \Lambda \setminus \mathcal{C}_0(f, k), \end{aligned}$$

because of $A > q$. This proves (7.9) in the case where $H_{j_0}(z)$ is a polynomial.

Suppose next that $H_{j_0}(z)$ is transcendental. Then $q - p \geq 1$, so we may write $H_{j_0}(z)$ in the normalized form with coefficients being either exponential polynomials of order $\leq q - p - 1$ or ordinary polynomials. Analogously as in (7.10), the proof of Lemma 7.2, applied to $H_{j_0}(z)$ instead of f , yields

$$\frac{H'_{j_0}(z)}{H_{j_0}(z)} = \left(C_{q-p-1} z^{q-p-1} + \frac{K'(z)}{K(z)} \right) (1 + \varepsilon_{q-p-1}(r)), \quad (7.11)$$

where $C_{q-p-1} \in \mathbb{C}$ and where $K(z)$ is either an exponential polynomial of order $\leq q - p - 1$ or an ordinary polynomial in z .

If $K(z)$ in (7.11) is a polynomial, then we combine (7.10) and (7.11) to get

$$\frac{f'(z)}{f(z)} = q w_{j_0} z^{q-1} + C_{q-p-1} z^{q-p-1} + O(r^{-1}), \quad z \in \Lambda \setminus \mathcal{C}_0(f, k).$$

If $K(z)$ is transcendental, then we continue inductively in this fashion by reducing the order on each step by one. Eventually the descendant coefficient must reduce down to a polynomial, and, as such, its logarithmic derivative is of growth $O(r^{-1})$. This gives us the representation (7.9). $\square$

Finally, we remind the reader of the following well-known standard growth estimate for logarithmic derivatives by Gundersen [6].

Lemma 7.4 *Let f be a meromorphic function, let k, j be integers such that $k > j \geq 0$, and let $\alpha > 1$. Then there exists a set $E \subset (1, \infty)$ that has finite logarithmic measure, and there exists a constant $A > 0$ depending only on α, k, j , such that for all z satisfying $|z| \notin E \cup [0, 1]$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq A \left(\frac{T(\alpha r, f)}{r} + \frac{n_j(\alpha r)}{r} \log^\alpha r \log n_j(\alpha r) \right)^{k-j}, \quad (7.12)$$

where $r = |z|$ and $n_j(r)$ denotes the number of zeros and poles of $f^{(j)}$ in $|z| < r$.

8 Proof of Theorem 2.1

The first assertion $n_{\lambda_p}(r) = O(r^{q-p} + \log r)$ is a simple consequence of (2.3) and the standard estimate $n(r) \leq (\log 2)^{-1}(N(2r) - N(r))$ between any counting function $n(r)$ and its integrated counterpart $N(r)$. Thus it suffices to prove (2.4).

Let Γ denote any piecewise smooth positively oriented Jordan curve, and let $n(\Gamma)$ denote the zeros of f in the domain bounded by Γ . By the argument principle, we have

$$n(\Gamma) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz, \quad (8.1)$$

provided that f has no zeros on Γ . In addition, the curve Γ needs to be separated from the zeros of f so that we can use the minimum modulus estimate in Corollary 4.2 as well as its further consequences in Sect. 7. Hence we need to choose Γ carefully.

In order to simplify the situation, we may suppose that the positive real axis is a critical ray for f by appealing to the exponential polynomial $g(z) = f(e^{-i\theta^*} z)$, if necessary. Thus, we assume that $\theta^* = 0$ is a critical angle for f , determined by its leading coefficients w_j, w_k as in (2.1). It follows that w_j and w_k are the unique dominant leading coefficients of f on the sides of the positive real axis. Without loss of generality, we may assume that w_j determines the dominant term of f below the x -axis and that w_k determines the dominant term of f above the x -axis. The uniqueness of the constants w_j, w_k carries over to the application of Lemma 7.3.

Let $k(x) = x^{q+p-1} \log x$ and $r_n = k(|z_n|)$ for $x \geq e$ and $|z_n| \geq e$. For this particular k , we let $\mathcal{C}_0(f, k)$ denote the set discussed in Sect. 7. Then (7.1) clearly holds, and so, by Lemma 7.1, we can find $c > 0$ such that $\partial\Lambda_p(0, c)$ meets at most finitely many discs $|z - z_n| \leq r_n$. This constant c also takes into account the p -generalization (2.3) of the result by Steinmetz. Denote by $\Lambda^+(c)$ and $\Lambda^-(c)$ the portions of $\Lambda_p(0, c)$ in the upper half-plane and the lower half-plane, respectively.

Let $\alpha > 1$, and let $E \subset (1, \infty)$ denote the exceptional set of finite logarithmic measure in Lemma 7.4, where f is our exponential polynomial of order q , and $k = 1, j = 0$. Choose $R \in (e, \infty) \setminus E$ large enough such that the curves $\Lambda^+(c)$ and $\Lambda^-(c)$ do not meet any of the discs $|z - z_n| \leq r_n$ for $r \geq R$. Then choose any $r \in (R, \infty) \setminus E$. Finally, let $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, where

$$\begin{aligned} \Gamma_1 : z &= te^{-ic \frac{\log t}{t^p}}, & \text{where } t \text{ goes from } R \text{ to } r, \\ \Gamma_2 : z &= re^{i\theta}, & \text{where } \theta \text{ goes from } -c \frac{\log r}{r^p} \text{ to } c \frac{\log r}{r^p}, \\ \Gamma_3 : z &= te^{ic \frac{\log t}{t^p}}, & \text{where } t \text{ goes from } r \text{ to } R, \\ \Gamma_4 : z &= Re^{i\theta}, & \text{where } \theta \text{ goes from } c \frac{\log R}{R^p} \text{ to } -c \frac{\log R}{R^p}. \end{aligned}$$

The idea is to keep R fixed and eventually to let r increase. Due to the construction, it is clear that Γ contains no zeros of f , even when $r \in (R, \infty) \setminus E$ is arbitrarily large. We find that

$$I := \int_{\Gamma_1} w_j q z^{q-1} dz = \int_R^r w_j q t^{q-1} e^{-iqc \frac{\log t}{t^p}} \left(1 - \frac{1-p \log t}{t^{p+1}} ci\right) dt.$$

Fig. 3 The domain bounded by the curve $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

If we set $w_j = a_j + ib_j$, then

$$\begin{aligned} \operatorname{Im} I &= \int_R^r b_j q t^{q-1} \left\{ \cos \left(qc \frac{\log t}{t^p} \right) - \sin \left(qc \frac{\log t}{t^p} \right) \frac{1-p \log t}{t^{p+1}} c \right\} dt \\ &\quad - \int_R^r a_j q t^{q-1} \left\{ \sin \left(qc \frac{\log t}{t^p} \right) + \cos \left(qc \frac{\log t}{t^p} \right) \frac{1-p \log t}{t^{p+1}} c \right\} dt. \end{aligned}$$

Using the standard estimates $1 - x^2/2 \leq \cos x \leq 1$ and $x - x^3/6 \leq \sin x \leq x$, we get

$$\operatorname{Im} I = b_j r^q + O(r^{q-p} \log r).$$

Using Lemma 7.3, it follows that

$$\operatorname{Im} \int_{\Gamma_1} \frac{f'(z)}{f(z)} dz = b_j r^q + O(r^{q-p} \log r).$$

Analogously, if $w_k = a_k + ib_k$, then

$$\operatorname{Im} \int_{\Gamma_3} \frac{f'(z)}{f(z)} dz = -b_k r^q + O(r^{q-p} \log r).$$

The minus-sign here is a result of integrating in the opposite direction. Next we apply Lemma 7.4 together with Corollary 3.3, and obtain the estimate

$$\left| \int_{\Gamma_2} \frac{f'(z)}{f(z)} dz + \int_{\Gamma_4} \frac{f'(z)}{f(z)} dz \right| = O(r^{q-p} \log^{2+\alpha} r).$$

By putting everything together, we conclude that

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz = \frac{b_j - b_k}{2\pi} r^q + O\left(r^{q-p} \log^{2+\alpha} r\right). \quad (8.2)$$

Keeping (8.1) in mind, the left-hand side of (8.2) is the number of zeros $n(r, \Gamma)$ of f in a domain bounded by the closed curve Γ that depends on r . Since counting functions are always non-negative, it follows that $b_j - b_k = |b_j - b_k|$. The situation would be symmetric if we would have chosen the leading coefficients w_j and w_k the other way around in the reasoning above. Moreover, from (2.1), we get $\operatorname{Re} w_j e^{iq\theta^*} = \operatorname{Re} w_k e^{iq\theta^*}$, which can be written as

$$a_j \cos q\theta^* - b_j \sin q\theta^* = a_k \cos q\theta^* - b_k \sin q\theta^*.$$

Since we have chosen $\theta^* = 0$, it follows that $a_j = a_k$. Therefore, we have $|w_j - w_k| = |b_j - b_k|$, so that (8.2) now reads as

$$n(r, \Gamma) = \frac{|w_j - w_k|}{2\pi} r^q + O\left(r^{q-p} \log^{2+\alpha} r\right).$$

The assertion follows from this via Lemma 4.3 by writing $\alpha = 1 + \varepsilon$.

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Affiliations

Janne M. Heittokangas^{1,3} · Zhi-Tao Wen^{2,3}

Zhi-Tao Wen
zhtwen@stu.edu.cn

¹ Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, 80101 Joensuu, Finland

² Department of Mathematics, Shantou University, Daxue Road No. 243, Shantou 515063, China

³ Department of Mathematics, Taiyuan University of Technology, Yingze West Street No. 79, Taiyuan 030024, China