On Balayage and B-Balayage Operators

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Abstract

Here, we consider the balayage operator in the setting of H^p spaces and its Bergman space version (B-balayage) introduced by Wulan et al. (Complex Var Ellipt Equ 59(12):1775–1782, 2014), and extend some known results on these operators.

Keywords Carleson measure · Balayage · BMO · Bergman spaces · Analytic Besov spaces

Mathematics Subject Classification 30H25 · 30H35

1 Introduction

Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T}$ the unit circle. For $0 < p < \infty$, the Hardy space H^p consists of all functions f which are holomorphic on $\mathbb{D}$ and satisfy

$$\|f\|_{H^p} = \sup_{0 < r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right\}^{\frac{1}{p}} < \infty.$$

It is known that each function $f \in H^p$ has the radial limit $f(e^{it}) = \lim_{r \rightarrow 1^-} f(re^{it})$ a.e. on $\mathbb{T}$ and $f(e^{it}) \in L^p(\mathbb{T})$.

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For $\phi \in L^1(\mathbb{T})$, we say that $\phi \in \text{BMO}(\mathbb{T})$ if

$$\|\phi\|_* = \sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |\phi(e^{it}) - \phi_I| dt < \infty,$$

where I denotes any arc of $\mathbb{T}$, $|I|$ is its arc length and

$$\phi_I = \frac{1}{|I|} \int_I \phi(e^{it}) dt.$$

In [7], the authors have recently considered Campanato spaces $\mathcal{L}^{p,\lambda}(\mathbb{T})$ defined as follows. For $\lambda \geq 0$ and $1 \leq p < \infty$, the space $\mathcal{L}^{p,\lambda}(\mathbb{T})$ consists of all functions $\phi \in L^p(\mathbb{T})$ for which

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^\lambda} \int_I |\phi(e^{it}) - \phi_I|^p dt < \infty.$$

We note that $\text{BMO}(\mathbb{T}) = \mathcal{L}^{p,1}$, $1 \leq p < \infty$, (see [3, pp. 222-235]).

For a finite positive Borel measure μ on $\mathbb{D}$, the function

$$S_\mu(e^{it}) = \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - ze^{-it}|^2} d\mu(z), \quad (1)$$

is called the balayage of μ . It follows from Fubini's theorem that $S_\mu(e^{it}) \in L^1(\mathbb{T})$ (see [3, p. 229]).

If I is an arc of $\mathbb{T}$, the Carleson square $S(I)$ is defined as

$$S(I) = \left\{ re^{it} : e^{it} \in I, 1 - \frac{|I|}{2\pi} \leq r < 1 \right\}.$$

A positive Borel measure μ is called an s -Carleson measure, $0 < s < \infty$, if there exists a positive constant $C = C(\mu)$, such that

$$\mu(S(I)) \leq C(\mu)|I|^s, \quad \text{for any arc } I \subset \mathbb{T}.$$

A 1-Carleson measure is simply called a Carleson measure. In [1], Carleson proved that if μ is a positive Borel measure in $\mathbb{D}$, then, for $0 < p < \infty$, $H^p \subset L^p(d\mu)$ if and only if μ is a Carleson measure.

It has been proved in [3, p. 229] that if μ is the Carleson measure, then S_μ belongs to $\text{BMO}(\mathbb{T})$. However, the Carleson property of measure μ is not a necessary condition for S_μ being a $\text{BMO}(\mathbb{T})$ function [5].

In the next section, we obtain an extension of the result mentioned above. More precisely, we prove that if μ is an s -Carleson measure, $0 < s \leq 1$, then S_μ belongs to $\mathcal{L}^{1,s}$.

In [6], H. Wulan, J. Yang, and K. Zhu introduced the Bergman space version of the balayage operator on the unit disk that was called B-balayage. The B-balayage of a finite complex measure μ on $\mathbb{D}$ is given by

$$G_{\mu}(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^2}{|1 - \bar{z}w|^4} d\mu(w), \quad z \in \mathbb{D}.$$

It has been proved in [6] that if μ is a 2-Carleson measure, then there exists a constant $C > 0$, such that

$$|G_{\mu}(z) - G_{\mu}(w)| \leq C\beta(z, w), \quad z, w \in \mathbb{D}, \quad (2)$$

where β is the hyperbolic metric on $\mathbb{D}$. Here, applying a similar idea to that used in the proof of this result, we prove the following theorem.

Theorem 1 Assume that $1 < p < \infty$ and μ is a positive Borel measure on $\mathbb{D}$. If μ is a $2p$ -Carleson measure, then there exists a positive constant $C = C(p)$, such that

$$|G_{\mu}(z) - G_{\mu}(w)| \leq C (\beta(z, w))^{\frac{1}{p}}$$

for all $z, w \in \mathbb{D}$.

Actually, this theorem is a special case of a more general theorem stated in Sect. 3.

Here, C will denote a positive constant which can vary from line to line.

2 Balayage Operators and Campanato Spaces $\mathcal{L}^{1,s}$

We start with the following result.

Theorem 2 If μ is an s -Carleson measure, $0 < s \leq 1$, S_{μ} is given by (1) and $0 \leq \gamma < 1$, then there exists a positive constant C , such that for any $I \subset \mathbb{T}$:

$$\frac{1}{|I|^{1+s-\gamma}} \int_I \int_I \frac{|S_{\mu}(e^{i\theta}) - S_{\mu}(e^{i\varphi})|}{|e^{i\theta} - e^{i\varphi}|^{\gamma}} d\theta d\varphi \leq C.$$

Proof Without loss of generality, we can assume that $|I| < 1$.

Let, for $z \in \mathbb{D}$ and $\theta \in \mathbb{R}$:

$$P_z(\theta) = \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} = \operatorname{Re} \left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \right)$$

be the Poisson kernel for the disk $\mathbb{D}$. By the Fubini theorem:

$$\begin{aligned} \int_I \int_I \frac{|S_{\mu}(e^{i\theta}) - S_{\mu}(e^{i\varphi})|}{|e^{i\theta} - e^{i\varphi}|^{\gamma}} d\theta d\varphi &\leq \int_I \int_I \int_{\mathbb{D}} \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^{\gamma}} d\mu(z) d\theta d\varphi \\ &= \int_{\mathbb{D}} \int_I \int_I \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^{\gamma}} d\theta d\varphi d\mu(z). \end{aligned} \quad (3)$$

For a subarc I of $\mathbb{T}$, let $2^n I$, $n \in \mathbb{N}$ denote the subarc of $\mathbb{T}$ with the same center as I and the length $2^n |I|$.

In view of the equality

$$\int_0^{2\pi} P_z(\theta) d\theta = 2\pi,$$

we have

$$\int_I \int_I \frac{P_z(\theta)}{|e^{i\theta} - e^{i\varphi}|^\gamma} d\theta d\varphi = \int_I P_z(\theta) \int_I \frac{d\varphi}{|e^{i\theta} - e^{i\varphi}|^\gamma} d\theta \leq C|I|^{1-\gamma}.$$

Consequently

$$\int_{S(2I)} \int_I \int_I \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^\gamma} d\theta d\varphi d\mu(z) \leq 2C|I|^{1-\gamma} \int_{S(2I)} d\mu(z) \leq C|I|^{1+s-\gamma}. \quad (4)$$

Since $P_z(\theta) \leq 4$ for $|z| \leq \frac{1}{2}$, we get

$$\begin{aligned} \int_{|z| \leq \frac{1}{2}} \int_I \int_I \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^\gamma} d\theta d\varphi d\mu(z) &\leq 8\mu(\mathbb{D}) \int_I \int_I \frac{d\theta d\varphi}{|e^{i\theta} - e^{i\varphi}|^\gamma} \\ &\leq C|I|^{2-\gamma} \leq C|I|^{1+s-\gamma}. \end{aligned}$$

Now, we assume that $\frac{1}{2} \leq |z| < 1$ and $z = |z|e^{i\omega} \in S(2^{n+1}I) \setminus S(2^n I)$. We consider two cases: (i) $e^{i\omega} \in 2^n I$ and (ii) $e^{i\omega} \in 2^{n+1}I \setminus 2^n I$.

In case (i), we have

$$\frac{2^n |I|}{2\pi} < 1 - |z| \leq \frac{2^{n+1} |I|}{2\pi}.$$

Thus

$$\begin{aligned} |P_z(\theta) - P_z(\varphi)| &= \frac{(1 - |z|^2)2|z| |\cos(\theta - \omega) - \cos(\varphi - \omega)|}{((1 - |z|)^2 + 4|z| \sin^2 \frac{\theta - \omega}{2})((1 - |z|)^2 + 4|z| \sin^2 \frac{\varphi - \omega}{2})} \\ &\leq \frac{8|\sin \frac{(\theta - \omega) + (\varphi - \omega)}{2}| |\sin \frac{(\theta - \varphi)}{2}|}{(1 - |z|)^3} \\ &\leq 2 \frac{(|\theta - \omega| + |\varphi - \omega|) |\theta - \varphi|}{(1 - |z|)^3}. \end{aligned}$$

Therefore, if $e^{i\theta}, e^{i\varphi} \in I$, then

$$\begin{aligned} \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^\gamma} &\leq C \frac{(|\theta - \omega| + |\varphi - \omega|) |\theta - \varphi|^{1-\gamma}}{(1 - |z|)^3} \\ &\leq C \frac{2^n |I| |I|^{1-\gamma}}{(2^n |I|)^3} = C \frac{|I|^{-1-\gamma}}{2^{2n}}. \end{aligned} \quad (5)$$

Now, we turn to case (ii). Then, for $e^{i\psi} \in I$,

$$2^{n-2} |I| \leq |\psi - \omega| \leq 2^n |I|.$$

Consequently, for $e^{i\theta}, e^{i\varphi} \in I$, we get

$$\begin{aligned} \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^\gamma} &\leq 2 \frac{|(1 - ze^{-i\theta})^2 - (1 - ze^{-i\varphi})^2|}{|e^{i\theta} - e^{i\varphi}|^\gamma |1 - ze^{-i\theta}| |1 - ze^{-i\varphi}|^2} \\ &\leq C \frac{(|\theta - \omega| + |\varphi - \omega|) |\theta - \varphi|^{1-\gamma}}{|\theta - \omega| |\varphi - \omega|^2} \\ &\leq C \frac{|I|^{-1-\gamma}}{2^{2n}}. \end{aligned} \quad (6)$$

Now, we put $Q_n = S(2^n I)$, $n = 1, 2, \dots$. Then, by (5) and (6),

$$\int_{Q_{n+1} \setminus Q_n} \int_I \int_I \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^\gamma} d\theta d\varphi d\mu(z) \leq C \frac{|I|^{1-\gamma}}{2^{2n}} \int_{Q_{n+1}} d\mu(z) \leq C \frac{|I|^{1+s-\gamma}}{2^{n(2-s)}}.$$

The above inequality and (4) imply

$$\begin{aligned} \int_{\mathbb{D}} \int_I \int_I \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^\gamma} d\theta d\varphi d\mu(z) &\leq \int_{Q_1} \int_I \int_I \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^\gamma} d\theta d\varphi d\mu(z) \\ &\quad + \sum_{n=1}^{\infty} \int_{Q_{n+1} \setminus Q_n} \int_I \int_I \frac{|P_z(\theta) - P_z(\varphi)|}{|e^{i\theta} - e^{i\varphi}|^\gamma} d\theta d\varphi d\mu(z) \\ &\leq C |I|^{s+1-\gamma} \sum_{n=1}^{\infty} \frac{1}{2^{n(2-s)}} = C |I|^{1+s-\gamma}. \end{aligned}$$

□

The next theorem shows that if μ is an s -Carleson measure, $0 < s \leq 1$, then S_μ is in the Campanato space $\mathcal{L}^{1,s}$.

Theorem 3 *If μ is an s -Carleson measure on $\mathbb{D}$, $0 < s \leq 1$ and $S_\mu(t) = S_\mu(e^{it})$ is the balayage operator of μ given by (1), then there exists a positive constant C , such that for any $I \subset \mathbb{T}$*

$$\frac{1}{|I|^s} \int_I |S_\mu(t) - (S_\mu)_I| dt \leq C.$$

Proof It is enough to observe that

$$\frac{1}{|I|^s} \int_I |S_\mu(t) - (S_\mu)_I| dt \leq \frac{1}{|I|^{s+1}} \int_I \int_I |S_\mu(t) - S_\mu(u)| dt du$$

and the inequality follows from Theorem 2 with $\gamma = 0$. $\square$

3 B-Balayage for Weighted Bergman Spaces A_α^p

Recall that, for $0 < p < \infty$, $-1 < \alpha < \infty$, the weighted Bergman space A_α^p is the space of all holomorphic functions in $L^p(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$$

and dA is the normalized Lebesgue measure on $\mathbb{D}$; that is, $\int_{\mathbb{D}} dA = 1$. If f is in $L^p(\mathbb{D}, dA_\alpha)$, we write

$$\|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z).$$

It is well known that, for $1 < p < \infty$, the Bergman projection P_α given by

$$P_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^{2+\alpha}} dA_\alpha(w)$$

is a bounded operator from $L^p(\mathbb{D}, dA_\alpha)$ onto A_α^p .

Let for $z, w \in \mathbb{D}$, the function

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}$$

denote the automorphism of the unit disk $\mathbb{D}$. The hyperbolic metric on $\mathbb{D}$ is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

For $z \in \mathbb{D}$ and $r > 0$, the hyperbolic disk with center z and radius r is

$$D(z, r) = \{w \in \mathbb{D} : \beta(z, w) < r\}.$$

For $s > 1$, the condition for an s -Carleson measure given in Introduction is equivalent to the condition where Carleson squares are replaced by hyperbolic disks. More exactly, the following result is known.

Proposition [2,10] *Let μ be a positive Borel measure on $\mathbb{D}$ and $1 < s < \infty$. Then, the following statements are equivalent*

- (i) μ is an s -Carleson measure,
- (ii) $\mu(D(z, r)) \leq C(1 - |z|^2)^s$ for some constant C depending only on r for all hyperbolic disk $D(z, r)$, $z \in \mathbb{D}$.

A positive Borel measure μ on $\mathbb{D}$ is called an A_α^p -Carleson measure if there exists a positive constant C , such that

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)$$

for all $f \in A_\alpha^p$.

It is well known that μ is an A_α^p -Carleson measure if and only if μ is $(2+\alpha)$ -Carleson measure (see [10, p. 133]). This means that A_α^p -Carleson measures are independent of p .

The next corollary is an immediate consequence of the last proposition.

Corollary [6] *For $\alpha > -1$, $\sigma > 0$, let μ, ν be positive Borel measures on $\mathbb{D}$, such that*

$$d\nu(z) = (1 - |z|)^\sigma d\mu(z).$$

Then, μ is an A_α^p -Carleson measure if and only if ν is an $A_{\alpha+\sigma}^p$ -Carleson measure.

Recall that, for $1 < p < \infty$, the Besov space B_p is the space of all functions f analytic on $\mathbb{D}$, such that

$$\|f\|_{B_p}^p = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p d\tau(z) < \infty,$$

where

$$d\tau(z) = \frac{dA(z)}{(1 - |z|^2)^2}$$

is the Möbius invariant measure on $\mathbb{D}$.

We will use the fact that the Besov space $B_p = P_\alpha(L^p, d\tau)$. The proof of this equality is given in [9, p. 119]. Moreover, if $f = P_\alpha g$, where $g \in L^p(d\tau)$, then

$$(1 - |z|^2)f'(z) = (\alpha + 2)(1 - |z|^2) \int_{\mathbb{D}} \frac{g(w)\bar{w}}{(1 - z\bar{w})^{3+\alpha}} dA_\alpha(w).$$

It then follows from [4, Thm. 1.9] that

$$\|f\|_{B_p} \leq C_{p,\alpha} \|g\|_{L^p(d\tau)}. \quad (7)$$

The next theorem gives a Lipschitz type estimate for functions in the analytic Besov space.

Theorem 4 [8] *For any $1 < p < \infty$, there exists a constant $C_p > 0$, such that*

$$|f(z) - f(w)| \leq C_p \|f\|_{B_p} (\beta(z, w))^{\frac{1}{q}}$$

for all $f \in B_p$ and $z, w \in \mathbb{D}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

In [6], the authors also consider a version of the balayage of a measure μ on $\mathbb{D}$ defined by

$$G_{\mu, \alpha}(z) = \int_{\mathbb{D}} \frac{(1 - |z|^2)^{2+\alpha}}{|1 - \bar{z}w|^{4+2\alpha}} d\mu(w).$$

They have proved the following generalization of inequality (2).

If μ is an A_{α}^p -Carleson measure, then the generalized balayage $G_{\mu, \alpha}$ satisfies the Lipschitz condition:

$$|G_{\mu, \alpha}(z) - G_{\mu, \alpha}(w)| \leq C \beta(z, w), \quad z, w \in \mathbb{D},$$

where C is independent of z and w .

It is worth noting here that the balayage given by (1) is in a certain sense a limit case of $G_{\mu, \alpha}$ as $\alpha \rightarrow -1$. Since an A_{α}^p -Carleson measure is actually a $(2 + \alpha)$ -Carleson measure, the last inequality gives a necessary condition for a measure μ to be an s -Carleson measure, as $1 < s < \infty$.

Theorem 1 is a special case of the following more general theorem.

Theorem 5 *Assume that $1 < p < \infty$, $-1 < \alpha < \infty$, and μ is a positive Borel measure on $\mathbb{D}$. If μ is a $p(2 + \alpha)$ -Carleson measure, then there exists a positive constant $C = C(p, \alpha)$, such that*

$$|G_{\mu, \alpha}(z) - G_{\mu, \alpha}(w)| \leq C (\beta(z, w))^{\frac{1}{p}}$$

for all $z, w \in \mathbb{D}$.

Proof For z, w , we have

$$\begin{aligned} |G_{\mu, \alpha}(z) - G_{\mu, \alpha}(w)| &\leq \int_{\mathbb{D}} \left| \frac{(1 - |a|^2)^{2+\alpha}}{|1 - a\bar{z}|^{4+2\alpha}} - \frac{(1 - |a|^2)^{2+\alpha}}{|1 - a\bar{w}|^{4+2\alpha}} \right| d\mu(a) \\ &\leq \int_{\mathbb{D}} \left| \frac{(1 - |a|^2)^{2+\alpha}}{(1 - a\bar{z})^{4+2\alpha}} - \frac{(1 - |a|^2)^{2+\alpha}}{(1 - a\bar{w})^{4+2\alpha}} \right| d\mu(a). \end{aligned}$$

Since μ is a finite measure on $\mathbb{D}$, the Jensen's inequality yields

$$\begin{aligned} |G_{\mu}(z) - G_{\mu}(w)|^p &\leq C \int_{\mathbb{D}} \left| \frac{(1 - |a|^2)^{2+\alpha}}{(1 - a\bar{z})^{4+2\alpha}} - \frac{(1 - |a|^2)^{2+\alpha}}{(1 - a\bar{w})^{4+2\alpha}} \right|^p d\mu(a) \\ &= C \int_{\mathbb{D}} \left| \frac{1}{(1 - a\bar{z})^{4+2\alpha}} - \frac{1}{(1 - a\bar{w})^{4+2\alpha}} \right|^p (1 - |a|^2)^{(2+\alpha)p} d\mu(a). \end{aligned}$$

By the Corollary, $(1 - |a|^2)^{p(2+\alpha)} d\mu(a)$ is an $A_{2p(2+\alpha)-2}^p$ -Carleson measure, because μ is an $A_{p(2+\alpha)-2}^p$ -Carleson measure. Consequently,

$$\begin{aligned} & \int_{\mathbb{D}} \left| \frac{1}{(1 - a\bar{z})^{4+2\alpha}} - \frac{1}{(1 - a\bar{w})^{4+2\alpha}} \right|^p (1 - |a|^2)^{p(2+\alpha)} d\mu(a) \\ & \leq C \int_{\mathbb{D}} \left| \frac{1}{(1 - a\bar{z})^4} - \frac{1}{(1 - a\bar{w})^4} \right|^p (1 - |a|^2)^{2p(2+\alpha)-2} dA(a) \\ & = C \int_{\mathbb{D}} \left| \frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}z)^{4+2\alpha}} - \frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}w)^{4+2\alpha}} \right|^p dA_{\frac{2}{q-1}}(a), \end{aligned}$$

where q is the conjugate index for p , that is, $\frac{1}{p} + \frac{1}{q} = 1$.

Now, set $\beta = \frac{2}{q-1}$ and note that

$$\begin{aligned} & \int_{\mathbb{D}} \left| \frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}z)^{4+2\alpha}} - \frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}w)^{4+2\alpha}} \right|^p dA_{\beta}(a) \\ & = \left(\sup_{\|f\|_{q,\beta} \leq 1} \left| \int_{\mathbb{D}} \left(\frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}z)^{4+2\alpha}} - \frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}w)^{4+2\alpha}} \right) f(a) dA_{\beta}(a) \right| \right)^p. \end{aligned}$$

Put $g = (\alpha+1)^{\frac{1}{q}} (1 - |a|^2)^{\frac{\beta+2}{q}} f$ and observe that $\|f\|_{q,\beta} \leq 1$ if and only if $\|g\|_{L^q(d\tau)} \leq 1$. Moreover, since $\beta = \frac{2}{q-1}$ satisfies $\frac{\beta+2}{q} = \beta$, we get

$$\begin{aligned} & \sup_{\|f\|_{q,\beta} \leq 1} \left| \int_{\mathbb{D}} \left(\frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}z)^{4+2\alpha}} - \frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}w)^{4+2\alpha}} \right) f(a) dA_{\beta}(a) \right| \\ & = C \sup_{\|g\|_{L^q(d\tau)} \leq 1} \left| \int_{\mathbb{D}} \left(\frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}z)^{4+2\alpha}} - \frac{(1 - |a|^2)^{2+2\alpha}}{(1 - \bar{a}w)^{4+2\alpha}} \right) g(a) dA(a) \right| \\ & = C \sup_{\|g\|_{L^q(d\tau)} \leq 1} \left| \int_{\mathbb{D}} \left(\frac{g(a)}{(1 - \bar{a}z)^{4+2\alpha}} - \frac{g(a)}{(1 - \bar{a}w)^{4+2\alpha}} \right) dA_{2+\alpha}(a) \right| \\ & = C \sup_{\|g\|_{L^q(d\tau)} \leq 1} |P_{2+\alpha}g(z) - P_{2+\alpha}g(w)| \leq C(\beta(z, w))^{\frac{1}{p}}, \end{aligned}$$

where the last inequality follows from Theorem 4 and inequality (7). $\square$

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