Asymmetric Truncated Toeplitz Operators of Rank One

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Abstract A truncated Toeplitz operator is a compression of the multiplication operator to a backward shift invariant subspace of the Hardy space H^2 . An asymmetric truncated Toeplitz operator is a compression of the multiplication operator that acts between two different backward shift invariant subspaces of H^2 . All rank-one truncated Toeplitz operators have been described by Sarason. Here, we characterize all rank-one asymmetric truncated Toeplitz operators. This completes the description given by Łanucha for asymmetric truncated Toeplitz operators on finite-dimensional backward shift invariant subspaces.

Keywords Model space · Truncated Toeplitz operator · Asymmetric truncated Toeplitz operator · Rank-one operator

Mathematics Subject Classification 47B32 · 47B35 · 30H10

1 Introduction

Denote by H^2 the Hardy space of the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ and let P be the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2 .

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A classical Toeplitz operator T_φ with symbol $\varphi \in L^\infty(\partial\mathbb{D})$ is defined on H^2 by

$$T_\varphi f = P(\varphi f).$$

It is clear that a Toeplitz operator with symbol from $L^\infty(\partial\mathbb{D})$ is bounded. Note that if $\varphi \in L^2(\partial\mathbb{D})$, then the above definition produces an operator T_φ , densely defined on $H^\infty = H^2 \cap L^\infty(\partial\mathbb{D})$. It is known that such T_φ extends boundedly to H^2 if and only if $\varphi \in L^\infty(\partial\mathbb{D})$.

The operators $S = T_z$ and $S^* = T_{\bar{z}}$ are called the unilateral shift and the backward shift, respectively. Obviously, $Sf(z) = zf(z)$ and a simple computation reveals that

$$S^* f(z) = \frac{f(z) - f(0)}{z}.$$

A truncated Toeplitz operator is a compression of a classical Toeplitz operator to a model space. Recall, that model spaces are the closed subspaces of H^2 of the form

$$K_\alpha = H^2 \ominus \alpha H^2,$$

where α is an inner function, i.e., $\alpha \in H^\infty$ and $|\alpha| = 1$ a.e. on $\partial\mathbb{D}$. As a consequence of the theorem of Beurling, model spaces are the typical non-trivial S^* -invariant subspaces of H^2 .

Extensive study of the class of truncated Toeplitz operators began in 2007 with Sarason's paper [10]. Generalizations of these operators, the so-called asymmetric truncated Toeplitz operators, were recently introduced in [2, 4, 5].

Let α, β be two inner functions. An asymmetric truncated Toeplitz operator $A_\varphi^{\alpha, \beta}$ with symbol $\varphi \in L^2(\partial\mathbb{D})$ is given by

$$A_\varphi^{\alpha, \beta} f = P_\beta(\varphi f), \quad f \in K_\alpha,$$

where P_β is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto K_β . In particular, a truncated Toeplitz operator is the operator $A_\varphi^\alpha = A_\varphi^{\alpha, \alpha}$.

Since $P_\beta(\varphi f) \in K_\beta$ for $f \in K_\alpha^\infty = K_\alpha \cap H^\infty$ and K_α^∞ is dense in K_α , the operator $A_\varphi^{\alpha, \beta}$ is densely defined. It is known that the boundedness of φ is not necessary for $A_\varphi^{\alpha, \beta}$ to extend boundedly to K_α .

Put

$$\mathcal{T}(\alpha, \beta) = \{A_\varphi^{\alpha, \beta} : \varphi \in L^2(\partial\mathbb{D}) \text{ and } A_\varphi^{\alpha, \beta} \text{ is bounded}\}$$

and $\mathcal{T}(\alpha) = \mathcal{T}(\alpha, \alpha)$.

Sarason showed that the truncated Toeplitz operators of rank one are generated (in a sense to be explained below) by functions of the form

$$k_w^\alpha(z) = \frac{1 - \overline{\alpha(w)}\alpha(z)}{1 - \overline{w}z}$$

and

$$\tilde{k}_w^\alpha(z) = \frac{\alpha(z) - \alpha(w)}{z - w}.$$

The function k_w^α , called the reproducing kernel, is the kernel function for the point evaluation functional $f \mapsto f(w)$, which for $w \in \mathbb{D}$ is bounded on K_α . In other words, $f(w) = \langle f, k_w^\alpha \rangle$ for every $f \in K_\alpha$ and each $w \in \mathbb{D}$. It is easy to check that $k_w^\alpha = P_\alpha k_w$, where $k_w(z) = (1 - \bar{w}z)^{-1}$.

The function $\tilde{k}_w^\alpha$ is the image of the reproducing kernel k_w^α under the map

$$C_\alpha f(z) = \alpha(z) \overline{zf(z)}, \quad |z| = 1.$$

One can verify that C_α is an antilinear and isometric involution on $L^2(\partial\mathbb{D})$ which preserves K_α . The map C_α is called a conjugation and, since $\tilde{k}_w^\alpha = C_\alpha k_w^\alpha$, the function k_w^α is often called the conjugate kernel.

It should be noted that the functional $f \mapsto f(w)$ is bounded on K_α also for $w \in \partial\mathbb{D}$ such that the function α has an angular derivative in the sense of Carathéodory (an ADC) at w . Recall that α is said to have an ADC at $w \in \partial\mathbb{D}$ if there exist complex numbers $\alpha(w)$ and $\alpha'(w)$ such that $\alpha(z) \rightarrow \alpha(w) \in \partial\mathbb{D}$ and $\alpha'(z) \rightarrow \alpha'(w)$ whenever z from $\mathbb{D}$ tends to w non-tangentially (that is, with $|z - w|/(1 - |z|)$ bounded). In that case, every $f \in K_\alpha$ has a non-tangential limit $f(w)$ at w , the functions k_w^α and $\tilde{k}_w^\alpha$ belong to K_α , and $f(w) = \langle f, k_w^\alpha \rangle$ (for more details, see for example [6, Thm. 7.4.1.]).

It was proved in [10] that the only rank-one operators in $\mathcal{T}(\alpha)$ are the scalar multiples of $\tilde{k}_w^\alpha \otimes k_w^\alpha$ and $k_w^\alpha \otimes \tilde{k}_w^\alpha$ for all $w \in \mathbb{D}$ and all $w \in \partial\mathbb{D}$ such that α has an ADC at w .

In the asymmetric case, the authors in [8] showed that the operators $\tilde{k}_w^\beta \otimes k_w^\alpha$ and $k_w^\beta \otimes \tilde{k}_w^\alpha$ belong to $\mathcal{T}(\alpha, \beta)$ for every $w \in \mathbb{D}$ and every $w \in \partial\mathbb{D}$ such that both α and β have an ADC at w . It is natural to ask: *are the scalar multiples of these the only rank-one operators in $\mathcal{T}(\alpha, \beta)$?* This issue was addressed in [9]. The author in [9] proved that if K_α is one dimensional and K_β has a finite dimension larger than two (or K_β is one dimensional and $\dim K_\alpha > 2$), then the answer is negative: there exists a rank-one operator in $\mathcal{T}(\alpha, \beta)$ that is neither a scalar multiple of $\tilde{k}_w^\beta \otimes k_w^\alpha$ nor a scalar multiple of $k_w^\beta \otimes \tilde{k}_w^\alpha$. Nevertheless, it was also showed in [9] that the answer is positive for all other finite-dimensional cases (that is, whenever α and β are two finite Blaschke products):

Theorem 1.1 ([9, Thm. 4.5]) *Let α and β be two finite Blaschke products of degree $m > 0$ and $n > 0$, respectively. The only rank-one operators in $\mathcal{T}(\alpha, \beta)$ are the non-zero scalar multiples of the operators $\tilde{k}_w^\beta \otimes k_w^\alpha$ and $k_w^\beta \otimes \tilde{k}_w^\alpha$, $w \in \mathbb{D}$, if and only if either $mn \leq 2$, or $m > 1$ and $n > 1$.*

In this paper, we resolve the issue in the general case. In particular, we prove that if both K_α and K_β have dimension larger than one (and not necessarily finite), then the only rank-one operators in $\mathcal{T}(\alpha, \beta)$ are the non-zero scalar multiples of $\tilde{k}_w^\beta \otimes k_w^\alpha$ and $k_w^\beta \otimes \tilde{k}_w^\alpha$.

2 Rank-One Asymmetric Truncated Toeplitz Operators

In [10], Sarason gave several characterizations of the operators from $\mathcal{T}(\alpha)$. For example, he proved that a bounded linear operator A on K_α is a truncated Toeplitz operator if and only if $A - S_\alpha A S_\alpha^*$ is a rank-two operator of a special kind, where $S_\alpha = A_z^\alpha$ is the so-called compressed shift. Similar characterizations of asymmetric truncated Toeplitz operators were given in [2, 3] and [9] for some particular cases. Here, we use the following two characterizations of asymmetric truncated Toeplitz operators.

Theorem 2.1 ([7, Thm. 2.1]) *Let A be a bounded linear operator from K_α into K_β . Then, $A \in \mathcal{T}(\alpha, \beta)$ if and only if there exist $\psi \in K_\beta$ and $\chi \in K_\alpha$ such that*

$$A - S_\beta A S_\alpha^* = \psi \otimes k_0^\alpha + k_0^\beta \otimes \chi.$$

Corollary 2.2 ([7, Cor. 2.7(c)]) *Let A be a bounded linear operator from K_α into K_β . Then, $A \in \mathcal{T}(\alpha, \beta)$ if and only if A is shift invariant, that is,*

$$\langle ASf, Sg \rangle = \langle Af, g \rangle$$

for all $f \in K_\alpha$, $g \in K_\beta$ such that $Sf \in K_\alpha$, $Sg \in K_\beta$.

Note here that if $f \in K_\alpha$, then $Sf \in K_\alpha$ if and only if f is orthogonal to $\tilde{k}_0^\alpha$ (see [10, p. 512]).

In his proof of [10, Thm. 5.1], Sarason uses the fact that every operator from $\mathcal{T}(\alpha)$ is C_α -symmetric. Recall that an operator A on K_α is said to be C_α -symmetric if $C_\alpha A C_\alpha = A^*$. In particular, $C_\alpha S_\alpha C_\alpha = S_\alpha^*$. In the case of asymmetric truncated Toeplitz operators it is known that $A \in \mathcal{T}(\alpha, \beta)$ if and only if $C_\beta A C_\alpha \in \mathcal{T}(\alpha, \beta)$ (see [9, p. 9]). So, here we take the approach from [9], that is, we use the following lemma (compare with [9, Lem. 4.4]; see also [10, pp. 503–504]).

Lemma 2.3 *Let α and β be two inner functions such that K_α and K_β have dimension $m > 1$ and $n > 1$, respectively. Let $f \in K_\alpha$, $g \in K_\beta$ be two non-zero functions such that $g \otimes f$ belongs to $\mathcal{T}(\alpha, \beta)$ and let $w \in \overline{\mathbb{D}}$. Then,*

- (a) *g is a scalar multiple of k_w^β if and only if f is a scalar multiple of $\tilde{k}_w^\alpha$,*
- (b) *g is a scalar multiple of $\tilde{k}_w^\beta$ if and only if f is a scalar multiple of k_w^α .*

Proof Let $f \in K_\alpha$, $g \in K_\beta$ be two non-zero functions such that $g \otimes f$ belongs to $\mathcal{T}(\alpha, \beta)$.

We first show that if g is a scalar multiple of k_w^β for some $w \in \overline{\mathbb{D}}$, then f is a scalar multiple of $\tilde{k}_w^\alpha$. Actually, without loss of generality we can assume that $g = k_w^\beta$ for some $w \in \overline{\mathbb{D}}$.

Note here, that for $w \in \partial\mathbb{D}$, we automatically assume that β has an ADC at w (this is implied by the fact that $k_w^\beta \in K_\beta$) and we must prove that α also has an ADC at w .

We consider two cases.

Case 1. $\beta(0) = 0$.

Since the operator $k_w^\beta \otimes f$ belongs to $\mathcal{T}(\alpha, \beta)$, it must be shift invariant in the sense of Corollary 2.2. That is,

$$\langle (k_w^\beta \otimes f)Sh_\alpha, Sh_\beta \rangle = \langle (k_w^\beta \otimes f)h_\alpha, h_\beta \rangle$$

for all $h_\alpha \in K_\alpha$ and $h_\beta \in K_\beta$ such that $Sh_\alpha \in K_\alpha$ and $Sh_\beta \in K_\beta$. Equivalently,

$$\begin{aligned} \langle (k_w^\beta \otimes f)Sh_\alpha, Sh_\beta \rangle - \langle (k_w^\beta \otimes f)h_\alpha, h_\beta \rangle &= \langle Sh_\alpha, f \rangle \langle k_w^\beta, Sh_\beta \rangle - \langle h_\alpha, f \rangle \langle k_w^\beta, h_\beta \rangle \\ &= \langle h_\alpha, \langle h_\beta, S_\beta^* k_w^\beta \rangle S_\alpha^* f - \langle h_\beta, k_w^\beta \rangle f \rangle = 0 \end{aligned}$$

for all $h_\alpha \in K_\alpha$ and $h_\beta \in K_\beta$, such that $h_\alpha \perp \tilde{k}_0^\alpha$ and $h_\beta \perp \tilde{k}_0^\beta$. But it follows from

$$S_\beta^* k_w^\beta = \overline{w} k_w^\beta - \overline{\beta(w)} \tilde{k}_0^\beta$$

(see [10, Lem. 2.2(a)] and its Corollary) that for every $h_\beta \perp \tilde{k}_0^\beta$, we have

$$\langle h_\beta, S_\beta^* k_w^\beta \rangle = w \langle h_\beta, k_w^\beta \rangle.$$

Thus, the shift invariance of $k_w^\beta \otimes f$ means that

$$\langle k_w^\beta, h_\beta \rangle \langle h_\alpha, w S_\alpha^* f - f \rangle = 0 \quad (2.1)$$

for all $h_\alpha \perp \tilde{k}_0^\alpha$ and $h_\beta \perp \tilde{k}_0^\beta$.

Since the dimension of K_β is greater than one, the functions k_w^β and $\tilde{k}_0^\beta$ are linearly independent. Indeed, if k_w^β and $\tilde{k}_0^\beta$ were linearly dependent, then so would be $\tilde{k}_w^\beta$ and $k_0^\beta \equiv 1$. But, here it is easy to verify that if $\tilde{k}_w^\beta = c k_0^\beta = c$, then $\beta(z) = cz$ and K_β is one dimensional—a contradiction.

Therefore, there exists a function $h_\beta \in K_\beta$ orthogonal to $\tilde{k}_0^\beta$ and such that $\langle k_w^\beta, h_\beta \rangle \neq 0$. Putting such h_β into (2.1), we see that

$$\langle h_\alpha, w S_\alpha^* f - f \rangle = 0$$

for all $h_\alpha \in K_\alpha$, $h_\alpha \perp \tilde{k}_0^\alpha$. Thus,

$$(I_{K_\alpha} - w S_\alpha^*) f = c \tilde{k}_0^\alpha$$

or, using C_α -symmetry of S_α ,

$$(I_{K_\alpha} - \overline{w} S_\alpha) C_\alpha f = \overline{c} k_0^\alpha \quad (2.2)$$

for some complex number c (here I_{K_α} denotes the identity operator on K_α).

If $c = 0$ and $|w| < 1$, then, by (2.2) and invertibility of $I_{K_\alpha} - \bar{w}S_\alpha$, we have $C_\alpha f = 0$. This implies that $f = 0$ which contradicts the assumption that f is non-zero.

If $c = 0$ and $|w| = 1$, then by (2.2),

$$S_\alpha C_\alpha f = w C_\alpha f,$$

which again is not possible since S_α has no eigenvalues on $\partial\mathbb{D}$ ([10, Lem. 2.5]).

Therefore, $c \neq 0$ and without loss of generality we can assume that $c = 1$. Hence, instead of (2.2) we consider

$$(I_{K_\alpha} - \bar{w}S_\alpha)C_\alpha f = k_0^\alpha. \quad (2.3)$$

As above, if $|w| < 1$, then $I_{K_\alpha} - \bar{w}S_\alpha$ is invertible. Since $(I_{K_\alpha} - \bar{w}S_\alpha)k_w^\alpha = k_0^\alpha$ (for $w = 0$ this is obvious, for $w \neq 0$ see [10, Lem. 2.2(b)]), we have

$$f = C_\alpha(I_{K_\alpha} - \bar{w}S_\alpha)^{-1}k_0^\alpha = C_\alpha k_w^\alpha = \tilde{k}_w^\alpha.$$

If $|w| = 1$, then applying C_α to both sides of (2.3) and using C_α -symmetry of S_α we get

$$f(z) - w \cdot \frac{f(z) - f(0)}{z} = \frac{\alpha(z) - \alpha(0)}{z}, \quad z \in \mathbb{D}.$$

Hence,

$$\frac{\alpha(z) - (\alpha(0) + wf(0))}{z - w} = f(z) \in H^2$$

and by [6, Thm. 7.4.1.] α has an ADC at w with the non-tangential limit $\alpha(w)$ equal to $\alpha(0) + wf(0)$, $\tilde{k}_w^\alpha$ belongs to K_α and

$$f(z) = \frac{\alpha(z) - \alpha(w)}{z - w} = \tilde{k}_w^\alpha.$$

This completes the proof in the first case.

Case 2. $\beta(0) \neq 0$.

For any two complex numbers a, b and functions $f \in K_\alpha, g \in K_\beta$ define

$$J_a^\alpha f(z) = \frac{\sqrt{1 - |a|^2}}{1 - \bar{a}\alpha(z)} \cdot f(z) \quad \text{and} \quad J_b^\beta g(z) = \frac{\sqrt{1 - |b|^2}}{1 - \bar{b}\beta(z)} \cdot g(z).$$

It is known that J_a^α is a unitary map from K_α onto K_{α_a} , where

$$\alpha_a(z) = \frac{a - \alpha(z)}{1 - \bar{a}\alpha(z)},$$

and that J_b^β is a unitary map from K_β onto K_{β_b} , where

$$\beta_b(z) = \frac{b - \beta(z)}{1 - \bar{b}\beta(z)}$$

(see [10, pp. 521–523] for more details). It was also proved in [8] that $A \in \mathcal{T}(\alpha, \beta)$ if and only if $J_b^\beta A(J_a^\alpha)^{-1} \in \mathcal{T}(\alpha_a, \beta_b)$ ([8, Prop. 2.5]).

Take $a = 0$ and $b = \beta(0)$. Clearly, $\alpha_0 = \alpha$ and $J_0^\alpha = I_{K_\alpha}$. Therefore, since $k_w^\beta \otimes f$ belongs to $\mathcal{T}(\alpha, \beta)$, the rank-one operator

$$J_b^\beta(k_w^\beta \otimes f)(J_a^\alpha)^{-1} = (J_b^\beta k_w^\beta) \otimes f$$

belongs to $\mathcal{T}(\alpha, \beta_b)$. A simple computation reveals that

$$J_b^\beta k_w^\beta = \frac{1 - b\overline{\beta(w)}}{\sqrt{1 - |b|^2}} \cdot k_w^{\beta_b}$$

(this can also be seen using Lemma 2.4 from [8]). Hence, the rank-one operator $k_w^{\beta_b} \otimes f$ belongs to $\mathcal{T}(\alpha, \beta_b)$, and, since $\beta_b(0) = 0$, f is a scalar multiple of $\tilde{k}_w^\alpha$ by Case 1.

This completes the proof in the second case and the proof of the first implication in (a).

To prove the first implication in (b) assume that $g = \tilde{k}_w^\beta$ for some $w \in \overline{\mathbb{D}}$. Then the rank-one operator

$$k_w^\beta \otimes C_\alpha f = C_\beta(g \otimes f)C_\alpha$$

belongs to $\mathcal{T}(\alpha, \beta)$ [9, p. 9], and it follows from the part of (a) proved above that $C_\alpha f$ is a scalar multiple of $\tilde{k}_w^\alpha$ and f is a scalar multiple of k_w^α .

To complete the proof of (a) assume that $f = \tilde{k}_w^\alpha$. Then,

$$\tilde{k}_w^\alpha \otimes g = (g \otimes \tilde{k}_w^\alpha)^*$$

belongs to $\mathcal{T}(\beta, \alpha)$ [2, Lem. 3.2], and it follows from the part of (b) proved above that g is a scalar multiple of k_w^β .

Similarly, if $f = k_w^\alpha$, then

$$k_w^\alpha \otimes g = (g \otimes k_w^\alpha)^*$$

belongs to $\mathcal{T}(\beta, \alpha)$ and it follows from (a) that g is a scalar multiple of $\tilde{k}_w^\beta$. This completes the proof of the lemma. $\square$

Using Lemma 2.3, one can modify the proof of [10, Thm. 5.1(c)] to show the following.

Theorem 2.4 *Let α and β be two inner functions such that K_α and K_β have dimension $m > 1$ and $n > 1$, respectively. Then the only rank-one operators in $\mathcal{T}(\alpha, \beta)$ are the non-zero scalar multiples of the operators $\tilde{k}_w^\beta \otimes k_w^\alpha$ and $k_w^\beta \otimes \tilde{k}_w^\alpha$ where $w \in \mathbb{D}$ or $w \in \partial\mathbb{D}$ and α and β have an ADC at w .*

Note that neither the above nor [9] includes the case when one of the model spaces is one dimensional and the other is infinite dimensional.

Let us now assume that α, β are two inner functions such that $\dim K_\alpha = 1$ and $\dim K_\beta = +\infty$.

Clearly, every bounded linear operator A from K_α into K_β is of rank one. Moreover, since K_α is spanned by k_0^α , $A - S_\beta A S_\alpha^*$ is a rank-one operator of the form $\psi \otimes k_0^\alpha$ for some $\psi \in K_\beta$. So here, by Theorem 2.1, all bounded linear operators from K_α into K_β are asymmetric truncated Toeplitz operators of rank one.

As the finite-dimensional case suggests, one can expect that here there are more rank-one operators in $\mathcal{T}(\alpha, \beta)$ than just scalar multiples of $\tilde{k}_w^\beta \otimes k_w^\alpha$ and $k_w^\beta \otimes \tilde{k}_w^\alpha$. We now prove that this is indeed the case. We do this in two steps: (1) we show that if every rank-one operator from $\mathcal{T}(\alpha, \beta)$ is a scalar multiple of $\tilde{k}_w^\beta \otimes k_w^\alpha$ or $k_w^\beta \otimes \tilde{k}_w^\alpha$ for some $w \in \overline{\mathbb{D}}$, then every non-zero function from K_β must be a scalar multiple of a reproducing kernel or a conjugate kernel; (2) we prove that since $\dim K_\beta = +\infty$, there must exist $f \in K_\beta$ that is neither a scalar multiple of a reproducing kernel nor a scalar multiple of a conjugate kernel.

Step 1. Let $f \in K_\beta \setminus \{0\}$. Then, $f \otimes k_0^\alpha \in \mathcal{T}(\alpha, \beta)$ and there exist $w \in \overline{\mathbb{D}}$ and $c \neq 0$ such that

$$f \otimes k_0^\alpha = c(\tilde{k}_w^\beta \otimes k_w^\alpha) \quad \text{or} \quad f \otimes k_0^\alpha = c(k_w^\beta \otimes \tilde{k}_w^\alpha).$$

Since $\dim K_\alpha = 1$, we can replace k_0^α in the above with k_w^α or $\tilde{k}_w^\alpha$ (multiplying both sides of the assumed equality by a constant if necessary). Then it follows easily that f is a scalar multiple of either $\tilde{k}_w^\beta$ or k_w^β .

Step 2. Here, we use [1, Prop. 2.8]. Fix $w_1, w_2 \in \mathbb{D}$, $w_1 \neq w_2$ and let $f = k_{w_1}^\beta + k_{w_2}^\beta \in K_\beta \setminus \{0\}$. Since $\dim K_\beta = +\infty$, by [1, Prop. 2.8], the functions $k_{w_1}^\beta$, $k_{w_2}^\beta$ and $\tilde{k}_w^\beta$ are linearly independent for any fixed $w \in \mathbb{D}$ and for any fixed $w \in \partial\mathbb{D}$ such that β has an ADC at w . Thus, f cannot be a scalar multiple of $\tilde{k}_w^\beta$. A similar reasoning shows that f cannot be a scalar multiple of k_w^β for any $w \in \overline{\mathbb{D}}$.

We proved that if $\dim K_\alpha = 1$ and $\dim K_\beta = +\infty$, then there is a rank-one operator from $\mathcal{T}(\alpha, \beta)$ that is neither a scalar multiple of $\tilde{k}_w^\beta \otimes k_w^\alpha$ nor a scalar multiple of $k_w^\beta \otimes \tilde{k}_w^\alpha$. The case $\dim K_\alpha = +\infty$ and $\dim K_\beta = 1$ can be treated analogously.

Theorem 2.5 *Let α and β be two inner functions such that $\dim K_\alpha = m$ and $\dim K_\beta = n$ (with m or n possibly infinite). The only rank-one operators in $\mathcal{T}(\alpha, \beta)$ are the non-zero scalar multiples of the operators $\tilde{k}_w^\beta \otimes k_w^\alpha$ and $k_w^\beta \otimes \tilde{k}_w^\alpha$, $w \in \overline{\mathbb{D}}$, if and only if either $mn \leq 2$, or $m > 1$ and $n > 1$.*

Proof Assume that the only rank-one operators in $\mathcal{T}(\alpha, \beta)$ are the non-zero scalar multiples of the operators $\tilde{k}_w^\beta \otimes k_w^\alpha$ and $k_w^\beta \otimes \tilde{k}_w^\alpha$. If both m and n are finite, then α

and β are two finite Blaschke products of degree m and n , respectively, and the claim follows from Theorem 1.1. If m or n is infinite, then $m > 1$ and $n > 1$ as already observed before.

To complete the proof, assume that either $mn \leq 2$, or $m > 1$ and $n > 1$. If both m and n are finite, then the only rank-one operators in $\mathcal{T}(\alpha, \beta)$ are the non-zero scalar multiples of the operators $\tilde{k}_w^\beta \otimes k_w^\alpha$ and $k_w^\beta \otimes \tilde{k}_w^\alpha$ by Theorem 1.1. If m or n is infinite, then clearly $m > 1$ and $n > 1$ and the claim follows from Theorem 2.4. $\square$

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