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Power-series solutions of fractional-order compartmental models

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Abstract
Compartmental models based on coupled differential equations of fractional order have been widely employed in the literature for modeling. An abstraction of these models is given by a system with polynomial vector field. We investigate the use of power series for solving generic polynomial differential equations in any dimension, with Caputo fractional derivative. As is well known, power series convert a continuous formulation into a discrete system of difference equations, which are easily solved by recursion. The novelty of this paper is that we rigorously prove that the series converge on a neighborhood of the initial instant, which is an analogue of the Cauchy–Kovalevskaya theorem. Besides, these series are proved to be continuous with respect to the fractional index. For applications, a general-purpose symbolic implementation of truncated power series is developed, and its execution is illustrated for the fractional SIR epidemiological model.

Keywords Compartmental differential equation system · Caputo fractional derivative · Convergent power-series solution · Discrete equation · Symbolic computation · Fractional SIR model

Mathematics Subject Classification 34A08 · 40A05 · 39A60 · 68W30

1 Introduction

When modeling the dynamics of a phenomenon in a population, the interaction between the elements must be considered. For example, when an epidemic is happening in a region (flu, COVID-19, etc.), contacts between susceptible and infected citizens are crucial, because they cause the transmission of the virus (Lotfi et al. 2020). For social behaviors, such as market choices, ideology, criminality, or health habits, contacts with peers are also important, due to attitudes of imitation, stimulation, pressure, persuasion, etc. Harkins et al. (2017), Esiri

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(2016), and Blanchower et al. (2009). Elements in the population could be other than persons; for example, for an ill patient undergoing a clinical follow-up, the dynamics of the disease in the body and the interaction between the virus and uninfected cells are of relevance (Masenga et al. 2023).

Compartmental models consider that a population is divided into different subgroups, according to certain conditions. Elements interact and move between the compartments. Since rates of flow need to be described, differential equations play a key role for compartmental modeling. Interactions are formulated by nonlinear terms, specifically products (multiplications). Simulation of the differential equations, for certain input values, permits understanding the dynamics of the phenomenon under investigation. Some examples of ordinary differential equation models with division into compartments are Brauer et al. (2008), Cooper et al. (2020), Santonja et al. (2010), Acedo et al. (2010a), Cervello et al. (2014), Sánchez et al. (2011), and Calatayud and Jornet (2020). These works deal with Epidemiology, COVID-19, alcohol consumption, respiratory syncytial virus, mobile telecommunications, cocaine use, and excess weight, respectively.

The word "ordinary" refers to the standard derivative. There are operators of fractional order, that extend the integer-order derivative. The literature is vast and many definitions of fractional derivatives have been given (De Oliveira and Tenreiro Machado 2014; Ortigueira and Machado 2015). In general, the operators are based on a convolution with respect to a kernel, so that history effects appear in the model. Some sort of continuous past delay is incorporated. There are a lot of contributions in the literature on compartmental models based on fractional differential equations; some examples are Carvalho et al. (2020), Carvalho et al. (2018), Martinez et al. (2021), Pinto and Carvalho (2017), Biala and Khaliq (2021), Alshomrani et al. (2021), Ndaïrou et al. (2021), and Area et al. (2015), for HIV, COVID-19 and Ebola dynamics, respectively.

Thus, this exposition motivates the study of fractional-order compartmental models, to extend the integer-order counterpart. Specifically, in this paper, we aim at investigating systems of fractional differential equations with polynomial vector field, in the Caputo-derivative sense. The polynomial is multivariate and depends on the state variable (autonomous equation). Polynomial expressions arise in many compartmental models, because nonlinearities are products (i.e., monomials) to account for contacts that occur homogeneously in the region. Thus, the problem proposed in this article is an abstraction of usual compartmental systems.

A great deal of research work in applied mathematics consists in the investigation of explicit, semi-explicit, and numerical solutions to physical models. Our research in this paper is focused on power-series solutions (i.e., analytic solutions). Power series possess appealing properties in terms of algebraic manipulations. When a power series is put into a fractional or ordinary differential equation, one derives an associated discrete model for the coefficients of the expansion. The difference equations can easily be solved by recursion, explicitly or in the computer, so that a simple functional representation of the model's response is obtained. Although in practice the method works, a difficult issue is to establish convergence theoretically, at least around the initial condition.

Of course, the use of power series for differential equations is not new (Teschl 2012, Chapter 4). Many linear ordinary differential equations, such as Airy, Hermite, Legendre, etc. in mathematical physics, have solutions expressed in terms of power series (Fröbenius method) (Koekoek et al. 2010). Even when stochastic effects are incorporated (Calatayud Gregori et al. 2020; Jornet 2021). Nonlinear ordinary differential equations may be solved with power series too Srivastava et al. (2021). The old Cauchy–Kovalevskaya theorem (Himonas and Petronilho 2020) ensures, essentially, that analytic inputs imply analytic output. This theorem applies for a certain class of partial differential equations as well. For fractional differential equations, specific applications of power series are available in the literature; for example, Area and Nieto (2021, 2023), Jornet (2023) for fractional logistic equations, and Villafuerte (2023) for random linear equations.

To the best of my knowledge, the generic fractional model and the mathematical methodology of this paper have not previously been considered in the literature. Only particular cases have been addressed before (Srivastava et al. 2021; Area and Nieto 2021, 2023; Jornet 2023), which are the bases for this submitted contribution. For systems of fractional differential equations with polynomial vector field, we rigorously investigate the convergence of the power series. This provides a foundation for applications. Besides, the continuity of the model with respect to the fractional order is studied. Finally, a general-purpose symbolic implementation of the method is tackled.

The novelties of this work are the following:

- The mathematical proof of the convergence of the power series for generic Caputo fractional differential equations with polynomial input function. This is an analogue of the Cauchy–Kovalevskaya theorem. In the literature, only applications without a rigorous analysis of convergence (Area and Nieto 2023), or specific models (such as logistic) with an analysis of convergence (Area and Nieto 2021), have been addressed. Hence, the present paper is an advance in the mathematical analysis of fractional systems.
- The generic implementation of the power-series solutions to compartmental models. Hence, the present paper is an advance in the computational analysis of fractional systems.

The paper is organized as follows. In Sect. 2, the fractional system of study is introduced. The notation is given and some examples provide the link with compartmental models. In Sect. 3, the formal power-series solution of the system is derived, by obtaining the discrete model for the expansion coefficients. These parts are a preparation (concepts, notation, etc.) for Sect. 4, where convergence of the power series is proved. This is the main novelty of the paper. Later, in Sect. 5, continuity with respect to the fractional index is proved. In Sect. 6, a computer code for the methodology is implemented, with some examples of execution. Finally, in Sect. 7, a summary with the main aspects of the article and possible extensions are presented.

2 Context

We work with a fractional-order differential equation model of the form

$$\mathcal{D}^\alpha x = p(x). \tag{1}$$

The state variable of the system (1) is x, which is a function of time $t \geq 0$ over the temporal range $\mathcal{J} = [0, T]$, where $T > 0$ is a time horizon. Its image takes vector values, on $\mathbb{R}^d$, for dimension $d \geq 1$. By components, it is expressed as $x = (x_1, \ldots, x_d)$. The scalar case corresponds to $d = 1$.

The differential present in (1) is of fractional order, through the Caputo operator $\mathcal{D}^\alpha$ (Gerasimov 1948; De Oliveira and Tenreiro Machado 2014; Ortigueira and Machado 2015; Caputo 1967). For order $0 < \alpha < 1$, it is defined as

$$\mathcal{D}^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau, \tag{2}$$

where f is an absolutely continuous function with the first-order derivative f' and Γ is the gamma function. It extends classical differentiation, which is retrieved for $\alpha \to 1^-$. The convolution in (2) includes memory effects somehow, because f' depends on $\tau < t$.

The autonomous vector field in (1), p, is of polynomial type, with multivariate evaluation, from $\mathbb{R}^d$ to $\mathbb{R}^d$. Its coordinates are $p = (p_1, \ldots, p_d)$. Componentwise, model (1) reads as

$$\mathcal{D}^\alpha x_j = p_j(x), \quad \forall\, j = 1, \ldots, d. \tag{3}$$

The data in (1) are the initial state, $x(0)$, and the real coefficients of p. For Caputo fractional differential equations, initial conditions can be handled analogously to the classical integer-order counterpart.

To express p, we use multi-index notation (Dunkl and Xu 2014). A multi-index is a vector of the form $I = (i_1, \ldots, i_d)$, formed by integer elements $i_1, \ldots, i_d \geq 0$. Associated notation is

$$v^I = \prod_{j=1}^{d} v_j^{i_j}, \quad v \in \mathbb{R}^d,$$

to compactly denote monomials, and

$$|I| = \sum_{j=1}^{d} i_j,$$

to compute its degree. In consequence, each coordinate of p may be expressed as

$$p_j(x) = \sum_{|I|=0}^{\delta_j} a_{j,I} x^I, \tag{4}$$

where $a_{j,I} \in \mathbb{R}$ are the coefficients, x^I are the monomials, δ_j is the degree, and the sum runs over all multi-indices I with associated degree from 0 to δ_j. The number of multi-indices I is $\binom{d+\delta_j}{d} = (d+\delta_j)!/(d!\delta_j!)$. In vector notation, if $a_I = (a_{1,I}, \ldots, a_{d,I})$, then

$$p(x) = \sum_{|I|=0}^{\max_j \delta_j} a_I x^I.$$

Although clear in the formulation, the multi-index notation is cumbersome in practice. For clarity, one may use the graded lexicographic order: given two multi-indices I and L of length d, $I > L$ if and only if $|I| \geq |L|$ and the first nonzero entry in the difference, $I - L$, is positive. For example, when $d = 4$, the multi-indices in ascending order are (0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0), (0, 0, 0, 2), (0, 0, 1, 1), (0, 0, 2, 0), (0, 1, 0, 1), (0, 1, 1, 0), (0, 2, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0), (2, 0, 0, 0), etc. In terms of monomials, these multi-indices translate into 1, x_4, x_3, x_2, x_1, x_4^2, x_3x_4, x_3^2, x_2x_4, x_2x_3, x_2^2, x_1x_4, x_1x_3, x_1x_2, x_1^2, etc.

The investigation of (1) relies on its widespread use in the literature for modeling. Many models have polynomial vector fields, to account for interactions between a finite number of dynamic subgroups.

Example 1 The fractional logistic equation (Area and Nieto 2021)

$$\mathcal{D}^\alpha x = x(1-x) = x - x^2$$

is a particular case of (1), for dimension $d = 1$ and $p(x) = x - x^2$. The degree is $\delta = 2$, and the coefficients are $a_0 = 0$, $a_1 = 1$ and $a_2 = -1$. Logistic growth can be generalized by incorporating the Allee effect (Area and Nieto 2023)

$$\mathcal{D}^\alpha x = x(1-x)(\theta - x) = -x^3 + (\theta+1)x^2 - \theta x,$$

where $\theta \in (0, 1)$. The parameters are $d = 1$, $\delta = 3$, $a_0 = 0$, $a_1 = -\theta$, $a_2 = \theta + 1$, and $a_3 = -1$.

Example 2 The fractional SIR (susceptible-infected-recovered) model (Alshomrani et al. 2021; Area et al. 2015)

$$\begin{aligned}
\mathcal{D}^\alpha S(t) &= -\beta^\alpha S(t)I(t),\\
\mathcal{D}^\alpha I(t) &= \beta^\alpha S(t)I(t) - \gamma^\alpha I(t),\\
\mathcal{D}^\alpha R(t) &= \gamma^\alpha I(t).
\end{aligned}$$

The parameters are positive and are raised to the power of α for dimensional consistency (the units are time$^{-\alpha}$). Another approach to be consistent with dimensionality would consist in introducing an artificial constant, $\mathrm{d}/\mathrm{d}t \to (1/\sigma^{1-\alpha})\mathcal{D}^\alpha$ (Gómez-Aguilar et al. 2014; Popović et al. 2010). The written SIR model is a particular case of (1) for dimension $d = 3$

$$x = (x_1, x_2, x_3) = (S, I, R)$$

and

$$p(x_1, x_2, x_3) = (-\beta^\alpha x_1 x_2, \beta^\alpha x_1 x_2 - \gamma^\alpha x_3, \gamma^\alpha x_3).$$

The degrees are $\delta_1 = 2$, $\delta_2 = 2$ and $\delta_3 = 1$, and the nonzero coefficients are $a_{1,(1,1,0)} = -\beta^\alpha$, $a_{2,(1,1,0)} = \beta^\alpha$, $a_{2,(0,0,1)} = -\gamma^\alpha$ and $a_{3,(0,0,1)} = \gamma^\alpha$. The first subscript denotes $j \in \{1, 2, 3\}$ and the second subscript is the multi-index $I = (i_1, i_2, i_3)$.

Example 3 A fractional model for HIV (human immunodeficiency virus) is Pinto and Carvalho (2017)

$$\begin{aligned}
\mathcal{D}^\alpha U(t) &= s^\alpha - \mu_U^\alpha U - k_1^\alpha VU,\\
\mathcal{D}^\alpha L(t) &= k_1^\alpha \eta VU - a_L^\alpha L - \mu_L^\alpha L,\\
\mathcal{D}^\alpha F(t) &= k_1^\alpha (1-\eta) VU + a_L^\alpha L - \delta_F^\alpha F - k_2^\alpha FZ,\\
\mathcal{D}^\alpha M(t) &= s_M^\alpha - k_M^\alpha MV - \delta_M^\alpha M,\\
\mathcal{D}^\alpha M_F(t) &= k_M^\alpha MV - \delta_M^\alpha M_F - k_3^\alpha M_F Z,\\
\mathcal{D}^\alpha V(t) &= q^\alpha F + q_M^\alpha M_F - c^\alpha V,\\
\mathcal{D}^\alpha Z(t) &= s_C^\alpha + k_4^\alpha FZ + k_5^\alpha M_F Z - \delta_C^\alpha Z.
\end{aligned}$$

Except $\eta \in (0, 1)$, the parameters are positive and are raised to the power of α for dimensional consistency (the units are time$^{-\alpha}$). For the details on the formulation, there reader is referred to Pinto and Carvalho (2017). Essentially, there are seven compartments for a patient under clinical follow-up: the uninfected CD 4^+ T cells, the latently infected CD 4^+ T cells, the productively infected CD 4^+ T cells, the uninfected macrophages, the infected macrophages, the virus, and the CTLs. In our notation

$$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (U, L, F, M, M_F, V, Z)$$

and

$$\begin{aligned}p(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (&s^\alpha - \mu_U^\alpha x_1 - k_1^\alpha x_1 x_6,\\ &k_1^\alpha \eta x_1 x_6 - a_L^\alpha x_2 - \mu_L^\alpha x_2,\\ &k_1^\alpha (1-\eta) x_1 x_6 + a_L^\alpha x_2 - \delta_F^\alpha x_3 - k_2^\alpha x_3 x_7,\\ &s_M^\alpha - k_M^\alpha x_4 x_6 - \delta_M^\alpha x_4,\\ &k_M^\alpha x_4 x_6 - \delta_M^\alpha x_5 - k_3^\alpha x_5 x_7,\\ &q^\alpha x_3 + q_M^\alpha x_5 - c^\alpha x_6,\\ &s_C^\alpha + k_4^\alpha x_3 x_7 + k_5^\alpha x_5 x_7 - \delta_C^\alpha x_7).\end{aligned}$$

The degrees are $\delta_1 = 2$, $\delta_2 = 2$, $\delta_3 = 2$, $\delta_4 = 2$, $\delta_5 = 2$, $\delta_6 = 1$, and $\delta_7 = 2$. The nonzero coefficients are $a_{1,(0,0,0,0,0,0,0)} = s^\alpha$, $a_{1,(1,0,0,0,0,0,0)} = -\mu_U^\alpha$, $a_{1,(1,0,0,0,0,1,0)} = -k_1^\alpha$, $a_{2,(1,0,0,0,0,1,0)} = k_1^\alpha \eta$, $a_{2,(0,1,0,0,0,0,0)} = -a_L^\alpha - \mu_L^\alpha$, $a_{3,(1,0,0,0,0,1,0)} = k_1^\alpha(1-\eta)$, $a_{3,(0,1,0,0,0,0,0)} = a_L^\alpha$, $a_{3,(0,0,1,0,0,0,0)} = -\delta_F^\alpha$, etc. The first subscript denotes $j \in \{1, 2, 3, 4, 5, 6, 7\}$ and the second subscript is the multi-index $I = (i_1, i_2, i_3, i_4, i_5, i_6, i_7)$.

3 Formal power series: from continuous to discrete

As widely studied in the literature, to solve (1), a power series at t^α is proposed

$$x(t) = \sum_{n=0}^{\infty} b_n (t^\alpha)^n = \sum_{n=0}^{\infty} b_n t^{\alpha n}, \tag{5}$$

where $t \in \mathcal{J}$ and $b_n \in \mathbb{R}^d$. If we denote $b_n = (b_{1,n}, \dots, b_{d,n})$, then (5) is given by components as

$$x_j(t) = \sum_{n=0}^{\infty} b_{j,n} t^{\alpha n}. \tag{6}$$

For now, the series (5) is formal, meaning that one uses the operations associated with power series, but ignoring whether convergence holds on an interval $[0, \epsilon) \subseteq \mathcal{J}$, for some $\epsilon > 0$. See Area and Nieto (2021, 2023) for the algebra associated with fractional power series. The aim of this paper is to show that (5), indeed, converges on a relative neighborhood of 0 in $\mathcal{J}$. To solve (1) and analyze the convergence of (5), we need to see how the coefficients b_n behave.

With the fractional differentiation of powers

$$\mathcal{D}^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \quad \gamma > -1,$$

the power series is differentiated, term by term, as

$$\mathcal{D}^\alpha x(t) = \sum_{n=0}^{\infty} b_n \mathcal{D}^\alpha (t^{\alpha n}) = \sum_{n=0}^{\infty} b_{n+1} \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} t^{\alpha n}. \tag{7}$$

When (5) converges on $[0, \epsilon)$, the identity (7) is not merely formal, but holds with convergence on $[0, \epsilon)$ too.

On the other hand, the power series of $p(x)$ is required. Taking the form (4) into account, we calculate $x_j^{i_j}(t)$ and $x^I(t)$. Recall the expression of the Cauchy product, defined by a discrete convolution. Some examples are the following:

$$x_j^2(t) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} b_{j,m}b_{j,n-m}\right)t^{\alpha n},$$
$$x_j^3(t) = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\left(\sum_{k=0}^{m} b_{j,k}b_{j,m-k}\right)b_{j,n-m}\right]t^{\alpha n},$$

etc. For an arbitrary power i_j, we have

$$x_j^{i_j}(t) = \sum_{n=0}^{\infty}\left(\sum_{n_2=0}^{n}\sum_{n_3=0}^{n_2}\cdots\sum_{n_{i_j}=0}^{n_{i_j-1}} b_{j,n_{i_j}}b_{j,n_{i_j-1}-n_{i_j}}\cdots b_{j,n_2-n_3}b_{j,n-n_2}\right)t^{\alpha n}. \tag{8}$$

We employ the shorthand notation

$$C_{j,i_j}(b_{j,0},\ldots,b_{j,n}) = \sum_{n_2=0}^{n}\sum_{n_3=0}^{n_2}\cdots\sum_{n_{i_j}=0}^{n_{i_j-1}} b_{j,n_{i_j}}b_{j,n_{i_j-1}-n_{i_j}}\cdots b_{j,n_2-n_3}b_{j,n-n_2}, \tag{9}$$

so that (8) becomes

$$x_j^{i_j}(t) = \sum_{n=0}^{\infty} C_{j,i_j}(b_{j,0},\ldots,b_{j,n})t^{\alpha n}. \tag{10}$$

For a multi-index power, by applying (10)

$$\begin{aligned}
x^I(t) &= x_1^{i_1}(t)\cdots x_d^{i_d}(t)\\
&= \sum_{n=0}^{\infty}\Bigg[\sum_{n_2=0}^{n}\sum_{n_3=0}^{n_2}\cdots\sum_{n_d=0}^{n_{d-1}} C_{d,i_d}(b_{d,0},\ldots,b_{d,n_d})\\
&\quad\cdot C_{d-1,i_{d-1}}(b_{d-1,0},\ldots,b_{d-1,n_{d-1}-n_d})\\
&\quad\cdots C_{2,i_2}(b_{2,0},\ldots,b_{2,n_2-n_3})C_{1,i_1}(b_{1,0},\ldots,b_{1,n-n_2})\Bigg]t^{\alpha n}.
\end{aligned} \tag{11}$$

If

$$\begin{aligned}
C_I(b_{1,0},\ldots,b_{d,n}) &= \sum_{n_2=0}^{n}\sum_{n_3=0}^{n_2}\cdots\sum_{n_d=0}^{n_{d-1}} C_{d,i_d}(b_{d,0},\ldots,b_{d,n_d})\\
&\quad\cdot C_{d-1,i_{d-1}}(b_{d-1,0},\ldots,b_{d-1,n_{d-1}-n_d})\\
&\quad\cdots C_{2,i_2}(b_{2,0},\ldots,b_{2,n_2-n_3})C_{1,i_1}(b_{1,0},\ldots,b_{1,n-n_2}),
\end{aligned} \tag{12}$$

we can express (11) as

$$x^I(t) = \sum_{n=0}^{\infty} C_I(b_{1,0},\ldots,b_{d,n})t^{\alpha n}. \tag{13}$$

If (5) converges on $[0,\epsilon)$, it is well known that the Cauchy product (13) converges on $[0,\epsilon)$, as well.

With (7) and (13), we equate

$$\sum_{n=0}^{\infty} b_{j,n+1} \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} t^{\alpha n} = \sum_{|I|=0}^{\delta_j} a_{j,I} \sum_{n=0}^{\infty} C_I(b_{1,0},\ldots,b_{d,n}) t^{\alpha n}$$
$$= \sum_{n=0}^{\infty} \left[\sum_{|I|=0}^{\delta_j} a_{j,I} C_I(b_{1,0},\ldots,b_{d,n}) \right] t^{\alpha n}.$$

Then, the coefficients of the power series must be equal. After isolating $b_{j,n+1}$, we arrive at the following scalar recursive equations:

$$b_{j,n+1} = \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} \sum_{|I|=0}^{\delta_j} a_{j,I} C_I(b_{1,0},\ldots,b_{d,n}), \tag{14}$$

for $j=1,\ldots,d$ and $n=0,1,2,\ldots$, with initial data $b_{j,0}=x_j(0)$. For each n, one runs (14) for all $j=1,\ldots,d$, before moving to $n+1$. That is, the loop over j is nested in the loop over n. Recall that C_I is defined by (12) and (9). Given (14), the main question is whether (5) converges around 0. Under convergence, the previous series manipulations are rigorous and we obtain the proper solution to (1).

4 Convergence of the power series

The aim is to prove that (5) converges around 0. We first need to upper bound the absolute value of the coefficients, $|b_{j,n+1}|$, given by (14). These depend on (12) and (9).

By the triangular inequality on (9)

$$|C_{j,i_j}(b_{j,0},\ldots,b_{j,n})| = \left| \sum_{n_2=0}^{n} \sum_{n_3=0}^{n_2} \cdots \sum_{n_{i_j}=0}^{n_{i_j-1}} b_{j,n_{i_j}} b_{j,n_{i_j-1}-n_{i_j}} \cdots b_{j,n_2-n_3} b_{j,n-n_2} \right|$$
$$\le \sum_{n_2=0}^{n} \sum_{n_3=0}^{n_2} \cdots \sum_{n_{i_j}=0}^{n_{i_j-1}} |b_{j,n_{i_j}}||b_{j,n_{i_j-1}-n_{i_j}}| \cdots |b_{j,n_2-n_3}||b_{j,n-n_2}|$$
$$= C_{j,i_j}(|b_{j,0}|,\ldots,|b_{j,n}|).$$

As a consequence, by the triangular inequality on (12)

$$|C_I(b_{1,0},\ldots,b_{d,n})|$$
$$= \Bigg| \sum_{n_2=0}^{n} \sum_{n_3=0}^{n_2} \cdots \sum_{n_d=0}^{n_{d-1}} C_{d,i_d}(b_{d,0},\ldots,b_{d,n_d}) C_{d-1,i_{d-1}}(b_{d-1,0},\ldots,b_{d-1,n_{d-1}-n_d})$$
$$\cdots C_{2,i_2}(b_{2,0},\ldots,b_{2,n_2-n_3}) C_{1,i_1}(b_{1,0},\ldots,b_{1,n-n_2}) \Bigg|$$
$$\le \sum_{n_2=0}^{n} \sum_{n_3=0}^{n_2} \cdots \sum_{n_d=0}^{n_{d-1}} |C_{d,i_d}(b_{d,0},\ldots,b_{d,n_d})||C_{d-1,i_{d-1}}(b_{d-1,0},\ldots,b_{d-1,n_{d-1}-n_d})|$$
$$\cdots |C_{2,i_2}(b_{2,0},\ldots,b_{2,n_2-n_3})||C_{1,i_1}(b_{1,0},\ldots,b_{1,n-n_2})|$$

$$\leq \sum_{n_2=0}^{n} \sum_{n_3=0}^{n_2} \cdots \sum_{n_d=0}^{n_{d-1}} C_{d,i_d}(|b_{d,0}|, \ldots, |b_{d,n_d}|)$$
$$\cdot C_{d-1,i_{d-1}}(|b_{d-1,0}|, \ldots, |b_{d-1,n_{d-1}-n_d}|)$$
$$\cdots C_{2,i_2}(|b_{2,0}|, \ldots, |b_{2,n_2-n_3}|) C_{1,i_1}(|b_{1,0}|, \ldots, |b_{1,n-n_2}|)\Big|$$
$$= C_I(|b_{1,0}|, \ldots, |b_{d,n}|). \tag{15}$$

On the other hand, from a certain $n \geq 0$ (no need to specify it for easiness of notation), the monotone condition

$$\Gamma(n\alpha + 1) \leq \Gamma((n+1)\alpha + 1) \tag{16}$$

holds (Kershaw 1983). In fact

$$\frac{\Gamma(n\alpha + 1)}{\Gamma((n+1)\alpha + 1)} \sim \frac{1}{(n\alpha + 1)^{\alpha}} \xrightarrow{n\to\infty} 0,$$

because it is well known that $\Gamma(y + \alpha) \sim \Gamma(y)y^{\alpha}$ when $y \to \infty$.

By (15) and (16), the coefficients in (14) are upper bounded as follows:

$$|b_{j,n+1}| \leq \sum_{|I|=0}^{\delta_j} |a_{j,I}| C_I(|b_{1,0}|, \ldots, |b_{d,n}|). \tag{17}$$

From (17), we build a "majorizing" series for $\sum_{n=0}^{\infty} |b_{j,n}| t^{\alpha n}$. Let

$$h_{j,0} = |b_{j,0}|$$

and

$$h_{j,n+1} = \sum_{|I|=0}^{\delta_j} |a_{j,I}| C_I(h_{1,0}, \ldots, h_{d,n}) \tag{18}$$

be new coefficients, for $j = 1, \ldots, d$ and $n \geq 0$. Again, for each n, one runs (18) for all $j = 1, \ldots, d$, before advancing to $n + 1$. By induction on n, it is trivially justified that

$$|b_{j,n}| \leq h_{j,n} \tag{19}$$

for all subscripts. Let

$$\psi_j(z) = \sum_{n=0}^{\infty} h_{j,n} z^n, \tag{20}$$

and with vector notation

$$\psi(z) = \sum_{n=0}^{\infty} h_n z^n, \tag{21}$$

where $\psi = (\psi_1, \ldots, \psi_d)$, $z \in \mathbb{R}$ and $h_n = (h_{1,n}, \ldots, h_{d,n}) \in \mathbb{R}^d$. For now, the power series (20) is formal. We need to check its convergence on a neighborhood of 0. In such a case, the power series (5), with coefficients (14), will be well defined around 0, by (19), as wanted.

Notice that, by (20), (18) and the Cauchy product for multi-indices [as in (13)], it holds

$$\begin{aligned}
\psi_j(z) &= h_{j,0} + z\sum_{n=0}^{\infty} h_{j,n+1} z^n \\
&= h_{j,0} + z\sum_{n=0}^{\infty}\left[\sum_{|I|=0}^{\delta_j} |a_{j,I}| C_I(h_{1,0},\ldots,h_{d,n})\right] z^n \\
&= h_{j,0} + z\sum_{|I|=0}^{\delta_j} |a_{j,I}| \left[\sum_{n=0}^{\infty} C_I(h_{1,0},\ldots,h_{d,n}) z^n\right] \\
&= h_{j,0} + z\sum_{|I|=0}^{\delta_j} |a_{j,I}| \psi(z)^I. \qquad (22)
\end{aligned}$$

Viewing this identity (22) for ψ as a functional root, we consider the maps

$$\phi_j(z,w) = w_j - h_{j,0} - z\sum_{|I|=0}^{\delta_j} |a_{j,I}| w^I,$$

for $j = 1, \ldots, d$, which are the coordinates of a map

$$\phi = (\phi_1, \ldots, \phi_d) \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d.$$

By (22), ψ in (21) and its coefficients (18) are characterized by $\psi(0) = h_0$ and $\phi(z, \psi(z)) = 0$. We have

$$\phi(0, h_0) = 0$$

and the Jacobian

$$\begin{aligned}
J_w\phi(z,w) &= \det\begin{pmatrix} \frac{\partial\phi_1}{\partial w_1}(z,w) & \frac{\partial\phi_2}{\partial w_1}(z,w) & \ldots & \frac{\partial\phi_d}{\partial w_1}(z,w) \\ \frac{\partial\phi_1}{\partial w_2}(z,w) & \frac{\partial\phi_2}{\partial w_2}(z,w) & \ldots & \frac{\partial\phi_d}{\partial w_2}(z,w) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi_1}{\partial w_d}(z,w) & \frac{\partial\phi_2}{\partial w_d}(z,w) & \ldots & \frac{\partial\phi_d}{\partial w_d}(z,w) \end{pmatrix} \\
&= \det\begin{pmatrix} 1 - z\cdot * & -z\cdot * & \ldots & -z\cdot * \\ -z\cdot * & 1 - z\cdot * & \ldots & -z\cdot * \\ \vdots & \vdots & \ddots & \vdots \\ -z\cdot * & -z\cdot * & \ldots & 1 - z\cdot * \end{pmatrix},
\end{aligned}$$

with

$$J_w\psi(0, h_0) = \det(\mathrm{Id}_d) = 1 \neq 0.$$

Here, $*$ denotes any expression and Id_d is the identity matrix of size $d \times d$. By the analytic version of the implicit-function theorem (Kaup and Kaup 2011, Section 8, Chapter 0), there exists a unique analytic function $w \equiv w(z)$ on a neighborhood $(-\mu, \mu)$ of zero, $\mu > 0$, $w : (-\mu, \mu) \to \mathbb{R}^d$, such that $w(0) = d_0$ and $\phi(z, w(z)) = 0$. Then, $\psi = w$ is convergent on $(-\mu, \mu)$. By (19)

$$\sum_{n=0}^{\infty} |b_{j,n}| t^{\alpha n} \leq \sum_{n=0}^{\infty} h_{j,n} t^{\alpha n} = \psi(t^\alpha) < \infty$$

for $t \in [0, \mu^{1/\alpha})$. In conclusion, the power series (5), with coefficients (14), is well defined around 0 and defines a suitable solution to (1).

5 Consistency with the integer-order model

When $\alpha = 1$, problem (1) is no more fractional, since $\mathcal{D}^1 x(t) = x'(t)$. Let us denote the coefficients in (5) and (14) as $b_{j,n}|_\alpha$, that is, we make the fractional-order explicit. The aim is to prove consistency, namely

$$\begin{aligned}\lim_{\alpha\to 1^-} \sum_{n=0}^{\infty} b_{j,n}|_\alpha t^{\alpha n} &= \sum_{n=0}^{\infty} \left[\lim_{\alpha\to 1^-} b_{j,n}|_\alpha\right] t^n \\ &= \sum_{n=0}^{\infty} b_{j,n}|_{\alpha=1} t^n. \end{aligned} \tag{23}$$

This last series is the classical integer-order power-series solution.

We distinguish two situations, taking into account the role of α:

Case 1 The coefficients $a_{j,I}$ of p_j, see (4) for $j = 1, \ldots, d$, do not depend on α; that is, they do not change with variations of α.

The coefficients $b_{j,n}|_\alpha$ satisfy (19), i.e., $|b_{j,n}|_\alpha| \leq h_{j,n}$, where the terms $h_{j,n}$ are defined by (18) and are independent of α. Independence holds due to (16).

On the other hand, $t^{\alpha n} \leq t^n$ if $t \geq 1$, and $t^{\alpha n} \leq (\sqrt{t})^n$ if $t < 1$ and $\alpha > 1/2$.

In consequence, if (21) converges on $(-\mu, \mu)$ and $t, \sqrt{t} \in (-\mu, \mu)$, then $|b_n||t|^{\alpha n}$ is upper bounded by summable α-independent terms. By the dominated convergence theorem (Rudin 1976, result 11.32, page 321), which permits interchanging $\lim_{\alpha\to 1^-}$ and $\sum_{n=0}^{\infty}$, (23) holds.

Case 2 The coefficients $a_{j,I}$ of p_j, see (4) for $j = 1, \ldots, d$, depend on α, as $a_{j,I} = \overline{a}_{j,I}^{\alpha}$; this is often the case when dimensional consistency is sought, where $\overline{a}_{j,I} > 0$ is independent of α.

The proof is very similar to Case 1. Actually, we proceed to reduce Case 2 to Case 1.

We start by $|b_{j,n}|_\alpha| \leq h_{j,n}$, see (19), but now the terms $h_{j,n}$ defined by (18) are functions of α. We bound $|a_{j,I}| \leq \tilde{a}_{j,I}$ for all subindices, where the coefficients $\tilde{a}_{j,I}$ are independent of α. This can be done by setting $\tilde{a}_{j,I} = \max\{\overline{a}_{j,I}, \sqrt{\overline{a}_{j,I}}\}$, considering $\alpha > 1/2$ and whether $\overline{a}_{j,I} \leq 1$ or $\overline{a}_{j,I} > 1$. Define new $\tilde{h}_{j,n}$ (independent of α), as (18) but with $\tilde{a}_{j,I}$ instead

$$\begin{aligned}\tilde{h}_{j,0} &= |b_{j,0}|, \\ \tilde{h}_{j,n+1} &= \sum_{|I|=0}^{\delta_j} \tilde{a}_{j,I} C_I(\tilde{h}_{1,0}, \ldots, \tilde{h}_{d,n}).\end{aligned}$$

The inequalities $|b_{j,n}|_\alpha| \leq h_{j,n} \leq \tilde{h}_{j,n}$ hold, by induction on n. The previous case and calculations then apply, with the "majorizing" series $\tilde{\psi}_j(z) = \sum_{n=0}^{\infty} \tilde{h}_{j,n} z^n$ (i.e., with tildes).

6 Symbolic implementation

We implement the computation of a truncated sum of (5)

$$x^{[N]}(t)=\sum_{n=0}^{N} b_n t^{\alpha n}. \tag{24}$$

We use computer algebra to obtain the function $t\mapsto x^{[N]}(t)$, where the variable t is symbolic. If a numerical value is required, then t is substituted accordingly.

Symbolic or algebraic computation is concerned with exact operations and expressions, with variables that have no given value (symbols) or are fractions (no decimal numbers).

A general-purpose algorithm is developed in Mathematica® (Wolfram Research, Inc. 2020) and illustrated here. The function is `expansionFractionalCompartmentODE`, with inputs:

- `d`: The dimension $d\geq 1$ of the system. It has to be a number.
- `NN`: The truncation index N, which is repeated in writing to distinguish from a built-in command `N` of the software. It has to be a number.
- `p`: The polynomial p, as a function of vectors in $\mathbb{R}^d$, defined through `p[v_] := ..., ..., ...`. In the dots, one writes the equations of the compartments, in terms of the components `v[[1]]`, `v[[2]]`, etc. The coefficients in `p` can be symbolic or numeric.
- `alpha`: The fractional order $\alpha\in(0,1]$. It can be symbolic or numeric.
- `x0`: The initial condition $x(0)\in\mathbb{R}^d$. It can be symbolic or numeric.

The output is $x^{[N]}(t)$, in terms of the symbolic variable t.

The code is the following:

```
expansionFractionalCompartmentODE[d_, NN_, p_, alpha_, x0_] :=
  Module[{coeffList, b, x, list, field, j, n, value, expansion},
   Clear@t;
   coeffList[y_] := If[SameQ[y, 0], {0}, CoefficientList[y, t]];
   b = ConstantArray[0, {d, NN + 1}];
   Do[b[[j, 1]] = x0[[j]], {j, d}];
   Do[
    x[t_] = Table[Total[b[[j, 1 ;; n]]*t^Range[0, n - 1]], {j, d}];
    field[t_] = p[x[t]];
    Do[
     list = coeffList[field[t][[j]] // Expand];
     If[Length@list < n, value = 0, value = list[[n]]];
     b[[j, n + 1]] = Gamma[(n - 1)*alpha + 1]/Gamma[n*alpha + 1]*value,
     {j, 1, d}
     ],
    {n, 1, NN}
    ];
   expansion =
    Table[Total[b[[j, 1 ;; NN + 1]]*t^(alpha*Range[0, NN])], {j, d}];
   Return[expansion];
   ];
```

We show some executions for the SIR model of Example 2. This is a very important compartmental system, originated in the work (Kermack and McKendrick 1927), and closed-form solutions are being investigated for it and its extensions (Srivastava et al. 2021; Harko et al. 2014; Heng and Althaus 2020; Acedo et al. 2010b).

Example 4 Set the following variables:

```
d = 3; NN = 5; beta = 1/2; gamma = 1/3; s0 = 99/100; i0 =
 1/100; r0 = 0; alpha = 1;
p[v_] := {-beta^alpha*v[[1]]*v[[2]],
 beta^alpha*v[[1]]*v[[2]] - gamma^alpha*v[[2]],
  gamma^alpha*v[[2]]}
x0 = {s0, i0, r0};
```

We have

$$
\begin{aligned}
x_1^{[5]}(t) = S^{[5]}(t) &= \frac{8257447t^5}{864000000000000} - \frac{266717t^4}{720000000000} \\
&\quad - \frac{42097t^3}{2400000000} - \frac{1551t^2}{4000000} - \frac{99t}{20000} + \frac{99}{100}, \\
x_2^{[5]}(t) = I^{[5]}(t) &= -\frac{229421789t^5}{23328000000000000} + \frac{20221t^4}{4860000000000} + \frac{284819t^3}{64800000000} \\
&\quad + \frac{4259t^2}{36000000} + \frac{97t}{60000} + \frac{1}{100}, \\
x_3^{[5]}(t) = R^{[5]}(t) &= \frac{20221t^5}{72900000000000} + \frac{284819t^4}{777600000000} + \frac{4259t^3}{324000000} \\
&\quad + \frac{97t^2}{360000} + \frac{t}{300}.
\end{aligned}
$$

This example for integer order was conducted in Srivastava et al. (2021) too.

Example 5 Set the following variables:

```
d = 3; NN = 5; beta = 1/2; gamma = 1/9; s0 = 99/100; i0 =
 1/100; r0 = 0; alpha = 1/2;
p[v_] := {-beta^alpha*v[[1]]*v[[2]],
  beta^alpha*v[[1]]*v[[2]] - gamma^alpha*v[[2]],
  gamma^alpha*v[[2]]}
x0 = {s0, i0, r0};
```

We have

$$
\begin{aligned}
x_1^{[5]}(t) = S^{[5]}(t) &= \frac{16}{15\sqrt{\pi}}\left(\frac{150170999}{80000000000} - \frac{48963666719}{18000000000000\sqrt{2}}\right. \\
&\quad + \frac{3267}{1250000000\pi^2} - \frac{1080299}{187500000000\sqrt{2}\pi^2} - \frac{30614837}{180000000000\pi} \\
&\quad \left. + \frac{405649211}{1500000000000\sqrt{2}\pi}\right)t^{5/2} \\
&\quad + \frac{3}{8}\sqrt{\pi}\left(\frac{160083}{1250000000\pi^{3/2}} - \frac{539}{6250000\sqrt{2}\pi^{3/2}} - \frac{77378257}{15000000000\sqrt{\pi}}\right. \\
&\quad \left. + \frac{6066269}{900000000\sqrt{2\pi}}\right)t^2 \\
&\quad + \frac{4\left(\frac{6501}{2000000} - \frac{292699}{50000000\sqrt{2}} - \frac{33}{500000\pi} + \frac{9801}{50000000\sqrt{2}\pi}\right)t^{3/2}}{3\sqrt{\pi}} \\
&\quad + \frac{1}{2}\sqrt{\pi}\left(\frac{33}{5000\sqrt{2\pi}} - \frac{4851}{500000\sqrt{\pi}}\right)t - \frac{99\sqrt{t}}{5000\sqrt{2\pi}} + \frac{99}{100}, \\
x_2^{[5]}(t) = I^{[5]}(t) &= \frac{16}{15\sqrt{\pi}}\left(-\frac{54683730391}{19440000000000} + \frac{72183319219}{18000000000000\sqrt{2}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3267}{1250000000\pi^2}+\frac{1080299}{187500000000\sqrt{2}\pi^2}+\frac{34156331}{180000000000\pi}\\
&-\frac{438154211}{1500000000000\sqrt{2}\pi}\Bigg)t^{5/2}\\
&+\frac{3}{8}\sqrt{\pi}\left(-\frac{590249}{3750000000\pi^{3/2}}+\frac{2167}{12500000\sqrt{2}\pi^{3/2}}\right.\\
&\left.+\frac{9096088817}{1215000000000\sqrt{\pi}}-\frac{9287861}{900000000\sqrt{2}\pi}\right)t^2\\
&+\frac{4\left(-\frac{56569}{10800000}+\frac{402699}{50000000\sqrt{2}}+\frac{33}{500000\pi}-\frac{9801}{50000000\sqrt{2}\pi}\right)t^{3/2}}{3\sqrt{\pi}}\\
&+\frac{1}{2}\sqrt{\pi}\left(\frac{53659}{4500000\sqrt{\pi}}-\frac{33}{2500\sqrt{2}\pi}\right)t+\frac{2\left(\frac{99}{10000\sqrt{2}}-\frac{1}{300}\right)\sqrt{t}}{\sqrt{\pi}}+\frac{1}{100},
\end{aligned}
$$

$$
\begin{aligned}
x_3^{[5]}(t)=R^{[5]}(t)=&\frac{16}{15\sqrt{\pi}}\left(\frac{9096088817}{9720000000000}-\frac{9287861}{7200000000\sqrt{2}}\right.\\
&\left.-\frac{590249}{30000000000\pi}+\frac{2167}{100000000\sqrt{2}\pi}\right)t^{5/2}+\frac{3}{8}\sqrt{\pi}\left(\frac{11}{375000\pi^{3/2}}\right.\\
&\left.-\frac{1089}{12500000\sqrt{2}\pi^{3/2}}-\frac{56569}{24300000\sqrt{\pi}}+\frac{134233}{37500000\sqrt{2}\pi}\right)t^2\\
&+\frac{4\left(\frac{53659}{27000000}-\frac{11}{5000\sqrt{2}}\right)t^{3/2}}{3\sqrt{\pi}}+\frac{1}{2}\sqrt{\pi}\left(\frac{33}{5000\sqrt{2}\pi}-\frac{1}{450\sqrt{\pi}}\right)t\\
&+\frac{\sqrt{t}}{150\sqrt{\pi}}.
\end{aligned}
$$

We report some timings. The computer is an Intel® Core™ i7 CPU 2.9 GHz, in Windows 10. For order $N = 20$, if `beta = 1/2`, `gamma = 1/9`, `s0 = 99/100`, `i0 = 1/100`, `r0 = 0` and `alpha = 1/2` are specified with symbolic fractions, then 4.5 s are required to calculate the three-coordinates function $x^{[20]}(t)$. However, if those parameters are specified with decimals, numerically, then the time is 0.02 s. With decimal numbers, other timings are 0.08 s for $N = 50$, 0.35 s for $N = 100$, 2.2 s for $N = 200$, and 4.5 s for $N = 250$. Once the symbolic expression $x^{[250]}(t)$ is computed, the evaluation at a specific numerical value of t lasts around 0.0005 s.

Example 6 Set the following variables:

```
d = 3; NN = 2;
p[v_] := {-beta^alpha*v[[1]]*v[[2]],
  beta^alpha*v[[1]]*v[[2]] - gamma^alpha*v[[2]],
  gamma^alpha*v[[2]]}
x0 = {s0, i0, r0};
```

We have

$$x_1^{[2]}(t) = S^{[2]}(t) = \frac{t^{2\alpha}\Gamma(\alpha+1)\left(\frac{\gamma^\alpha I_0 S_0 \beta^\alpha}{\Gamma(\alpha+1)} + \frac{I_0^2 S_0 \beta^{2\alpha}}{\Gamma(\alpha+1)} - \frac{I_0 S_0^2 \beta^{2\alpha}}{\Gamma(\alpha+1)}\right)}{\Gamma(2\alpha+1)} - \frac{I_0 S_0 \beta^\alpha t^\alpha}{\Gamma(\alpha+1)} + S_0,$$

$$x_2^{[2]}(t) = I^{[2]}(t) = \frac{t^{2\alpha}\Gamma(\alpha+1)\left(-\frac{2\gamma^\alpha I_0 S_0 \beta^\alpha}{\Gamma(\alpha+1)} - \frac{I_0^2 S_0 \beta^{2\alpha}}{\Gamma(\alpha+1)} + \frac{I_0 S_0^2 \beta^{2\alpha}}{\Gamma(\alpha+1)} + \frac{\gamma^{2\alpha} I_0}{\Gamma(\alpha+1)}\right)}{\Gamma(2\alpha+1)} + \frac{t^\alpha\left(I_0 S_0 \beta^\alpha - \gamma^\alpha I_0\right)}{\Gamma(\alpha+1)} + I_0,$$

$$x_3^{[2]}(t) = R^{[2]}(t) = \frac{t^{2\alpha}\Gamma(\alpha+1)\left(\frac{\gamma^\alpha I_0 S_0 \beta^\alpha}{\Gamma(\alpha+1)} - \frac{\gamma^{2\alpha} I_0}{\Gamma(\alpha+1)}\right)}{\Gamma(2\alpha+1)} + \frac{\gamma^\alpha I_0 t^\alpha}{\Gamma(\alpha+1)} + R_0.$$

We report some timings, with the previous computer's specifications. Compared to the earlier example, the calculations are more costly, because all of the variables are symbolic. For $N = 8$, the code runs in 25 s; for $N = 7$, in 2.6 s; and for $N = 6$, in 0.5 s.

7 Conclusions

Compartmental models, based on coupled differential equations, have been used in a lot of contributions for dynamical modeling. The fields of application include Epidemiology, Ecology and Sociology, for example, because individuals or species interact (nonlinear terms) and transmit diseases, behaviors, etc. This paper is a contribution to the study of this type of models, with fractional derivatives incorporated. We used the Caputo definition.

We investigated the use of power-series expansions to find closed-form expressions for the solution. Of course, the applications of power series for differential equations, even nonlinear, are well known. However, to my knowledge, there is no study in the literature that conducts the presented mathematical treatment for a generic fractional model of the form (1), in any state dimensionality.

The coefficients of the formal power series, for the candidate solution, satisfy difference equations. This process may be viewed as a conversion from a continuous into a discrete model, as is well known in the literature. After introducing the necessary concepts and notation, we rigorously proved that the power series converge on a neighborhood of the initial instant zero. We based on "majorizing" series and the analytic implicit-function theorem, dealing with the cumbersome notation from the Cauchy products. Key facts were the monotone condition of the gamma function and the absolute-value inequality met by the Cauchy products. On the other hand, continuity with respect to the fractional order was established, by applying the dominated convergence theorem. Two scenarios were distinguished, depending on whether the coefficients of the polynomial were related to the fractional parameter or not. This result showed consistency between the fractional- and the integer-order formulations.

The series, which must be truncated sums for applications, were implemented in a computer-algebra software. The code was explicitly given in the article. The output of our developed function depends on the symbolic time variable. Some executions were exhibited and commented for the SIR epidemiological model of fractional order, with different symbolic and numeric parameters (force of infection, recovery rate, initial conditions, and fractional index). Explicit sums and CPU times were reported. Other examples, with other inputs, would be analogous.

Some extensions of this paper could be the following:

- The estimation of the radius of convergence for the power-series solution (5). The analytic version of the implicit-function theorem only provides the existence of a neighborhood of analyticity. The estimation would likely require obtaining more information about the "majorizing" series ψ, through the identity (22). For example, the scalar case $d = 1$ of polynomial degree $\delta = 2$, which corresponds to a logistic equation, gives an explicit root $\psi(z)$ in (22), and the largest interval of convergence for ψ could be calculated or estimated.
- The estimation of the rate of convergence for the power-series solution (5). Also, the rate of convergence as the fractional order tends to integer-order one. Both the implicit-function theorem and the dominated convergence theorem do not give that information. Probably, a proof with specific inequalities would be required.
- A more efficient implementation of the symbolic code presented for the truncated power series (24).
- The analysis of other problems (1) with a general fractional order $\alpha \in (0, \infty)$ or with alternative fractional derivatives.
- The analysis of problems $\mathcal{D}^\alpha x = f(x)$, where f is any analytic function. If f is not a polynomial, then one could use a polynomial approximation $f \approx p$ and solve $\mathcal{D}^\alpha \tilde{x} = p(\tilde{x})$, by truncating the Taylor series, but the new analytic solution $\tilde{x}$ would be biased. The methodology of the paper does not directly apply to $\mathcal{D}^\alpha x = f(x)$.
- The investigation of the corresponding stochastic problem, where $x(0)$ and/or coefficients of the polynomial p are random variables. The power series may be of use for uncertainty-propagation computations, but mean-square convergence should be established to ensure convergence for the second-order statistics.

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Declarations

Conflict of interest The author declares that there is no conflict of interest regarding the publication of this article.

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