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Multidimension: a dimensionality extension of simple games

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Abstract

In voting theory and social choice theory, decision systems can be represented as simple games, i.e., cooperative games defined through their players or voters and their set of winning coalitions. The weighted voting games form a well-known strict subclass of simple games, where each player has a voting weight so that a coalition wins if the sum of weights of their members exceeds a given quota. Since the number of winning coalitions can be exponential in the number of players, simple games can be represented much more compactly as intersections or unions of weighted voting games. A simple game's dimension (codimension) is the minimum number of weighted voting games such that their intersection (union) is the given game. It is known there are voting systems with a high (co)dimension. This work introduces the multidimension as the minimum size of an expression with intersections and unions on weighted voting games necessary to obtain the considered simple game. We generalize this notion to subclasses of weighted voting games and analyze the generative properties of these subclasses. We also characterize the simple games with finite generalized

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multidimension over the set of weighted voting games without dummy players. We provide a comprehensive classification for simple games up to a certain number of players. These results complement similar classification results for generalized (co)dimensions. Our results show how generalized multidimension allows representing more simple games and more compactly, even for a small number of players and for subclasses.

Keywords Game theory · Weighted voting games · Dimensionality · Codimensionality · Canonical minimum representation

Mathematics Subject Classification 91 · 90

1 Introduction

Simple games play a significant role across various disciplines, including mathematics, computer science, and social sciences. They have applications in solving and representing problems related to politics, voting theory, decision theory, social choice theory, threshold logic, circuit complexity, network reliability, linear programming, artificial intelligence, Sperner theory, and order theory, among others (Brams 1975; Taylor and Zwicker 1999; Eiter et al. 2008; Engel 1997; Judson et al. 2005). Additionally, they exhibit close connections with various mathematical and computational structures, such as dual hypergraphs, Sperner families, antichains, monotone Boolean functions, free distributive lattices, monotone collective decision-making systems, and multi-agent systems, to name a few (Taylor and Zwicker 1999; Eiter et al. 2008; Engel 1997). This article is particularly interested in voting system applications.

Voting systems or electoral systems are the set of rules that different governments, social organizations, and other sociopolitical groups adopt for collective decision-making. There are numerous and diverse voting systems worldwide, and new voting systems are continually being created, looking for more fair, representative, or adequate systems for each context.

A usual way to study voting systems is through simple games, i.e., a type of monotonous cooperative games with either 0 or 1 payoffs (von Neumann and Morgenstern 1944). A simple game is defined by a set of players or voters and a set of winning coalitions. A winning coalition is a subset of players that manages to win a motion (election, referendum, etc.), i.e., a coalition with payoff 1. The set of losing coalitions, i.e., those with payoff 0, is formed by all coalitions that are not winners. Simple games are monotonous in the sense that the superset of any winning coalition is also a winner, while the subset of any losing coalition is also a loser (Taylor and Zwicker 1999).

A problem with simple games is that the set of winning (or losing) coalitions of a game can be exponential in the number of players (Molinero et al. 2015). Therefore, explicitly storing all the information needed to describe the game can be costly. Fortunately, it is known that any simple game can be represented as an intersection (Taylor and Zwicker 1993) or union (Freixas and Marciniak 2010) of weighted voting games. Weighted voting games are a subclass of simple games in which each player has a voting weight so that a coalition wins if the sum of its members' weights manages to exceed a given quota for the game. The representation form for weighted voting games is a vector containing the quota and the player's weights, so its size is linear in the number of players. Weighted voting games have garnered attention and investigation in multiple contexts, sometimes referred to by different names, including linearly separated truth functions in contact and rectifier nets (McNaughton

1961), linearly separable switching functions or threshold Boolean functions, for separating circuits in switching circuit theory, and analyzing the threshold synthesis problem (Hu 1965; Freixas and Molinero 2008), trade robustness in voting theory and trade exchanges (Taylor and Zwicker 1992), or threshold hypergraphs, for synchronizing parallel processes (Golumbic 1980; Reiterman et al. 1985). Besides, its more succinct vectorial representation constitutes an advantage for computational treatment.

The votes of the legislative power in countries with proportional representation party-list, mainly from Latin America, Europe, and Africa, can be represented as weighted voting games, where each player represents a political party, and the weights their number of parliamentary seats. The quota depends on the country and type of voting (e.g., common law, organic constitutional law, constitutional reform, etc.) (Riquelme and Gonzalez-Cantergiani 2017; Riquelme et al. 2019). However, it is important to remark that there are voting systems that cannot be represented as weighted voting games. For instance, the European Union Council under the Lisbon rules can be represented as the intersection of between 8 (Kober and Weltge 2021) and 25 weighted voting games (Chen et al. 2019) and as the union of at least 2000 weighted voting games (Kurz and Napel 2015). The above leads us to two key concepts: dimension and codimension. The dimension is the minimum number of weighted voting games whose intersections generate the considered simple game. Simple games formed by the intersection of several weighted voting games are called vector weighted voting games and were initially defined to represent voting systems in multicameralism (Taylor and Zwicker 1993). Similarly, the codimension is the minimum number of weighted voting games whose unions generate the considered simple game. Hence, the EU Council under the Lisbon rules has a dimension between 8 (Kober and Weltge 2021) and 25 (Chen et al. 2019) and a codimension of at least 2000 (Kurz and Napel 2015). Note that simple games with small dimensions can be described with a reasonable amount of bits, and thus analyzing some of their properties might became computationally tractable.

Several studies about dimension and codimension of simple games have appeared during the last two decades (Freixas and Molinero 2010; Taylor and Zwicker 1999; Taylor and Pacelli 2008). Some authors have focused on computing the dimension of simple games theoretically (Olsen et al. 2016; Freixas and Puente 2001, 2008), others on finding high dimensions for voting systems in real life (Kurz and Napel 2015), and others on computational complexity results (Molinero et al. 2016; Freixas et al. 2011). It is known that finding exact values for dimension and codimension are NP-hard computational problems (Deĭneko and Woeginger 2006). Moreover, dimensions in simple games can be exponential in the number of players (Olsen et al. 2016; Taylor and Zwicker 1999). The same occurs for codimensions since dimension and codimension are dual concepts (Kurz et al. 2016).

Recently, in Molinero et al. (2023) it was introduced a generalization of dimension and codimension to subclasses of simple games. The main objective of this generalization is to analyze the ability to express simple games as intersections or unions of games from subclasses of weighted voting games, in particular for subclasses of *pure* games, i.e., games without dummy players. Besides the definitions, Molinero et al. (2023) characterizes the simple games that can be expressed as either intersection or union of pure weighted voting games. The same work also provides a systematic classification of generalized dimension and codimension for all simple games up to six players and all simple games obtained by weighted voting games with linear restrictions until seven players.

In this article, we focus on another generalization of dimension that we call *multidimension*. Our idea is to use a well-formed expression (on unions and intersections) of weighted voting games to describe a simple game. The *multidimension* of a simple game Γ measures the minimum number of operations in an expression on intersections and unions describing Γ

(see definitions in Sect. 3). Besides analyzing other properties, the multidimension of a simple game remains smaller or equal than the minimum of its dimension and codimension. In this way, we expect to provide a form of representation amenable to computational treatment for a bigger number of games. Our definition of multidimension is related to the *minimum size of Boolean weighted voting games* introduced in Faliszewski et al. (2009). In fact, as we will see later, the multidimension of a given simple game Γ is the minimum size of a Boolean weighted voting game equivalent to Γ. On the other hand, there are other similar concepts of Boolean dimension related to our multidimension (O'Dwyer and Slinko 2017; Kurz 2021). We will comment on them in Sect. 3 were we introduce their definitions.

As for dimension and codimension, we also consider generalized notions of multidimension by restricting the subclasses of weighted voting games allowed in an expression. As we will see, some subclasses of weighted voting games are not enough to represent some simple games through expressions over the union and intersection operations. In those cases, when a game cannot be obtained in such a way, following the notation used in Molinero et al. (2023), we say that its generalized multidimension is ∞.

Given a simple game with (generalized) dimension d and (generalized) codimension c, it is clear that its (generalized) multidimension is at most $\min\{c, d\}$. However, for some simple games, the multidimension could be much smaller. For example, the EU Council under the Lisbon rules can be represented as the union of one weighted voting game with the intersection of two weighted voting games (Kurz and Napel 2015), so it has multidimension (or Boolean dimension O'Dwyer and Slinko 2017) 3. As expected, we show that, for subclasses of weighted voting games closed under duality, the generalized multidimension of a game and that of its dual coincide.

Another result is a characterization of the simple games with finite multidimension with respect to the class of pure weighted voting games. We show that all simple games except *singleton games*, i.e., games having only a singleton as a minimal winning coalition, have finite generalized multidimension on pure weighted voting games. In this way, we show that the expressiveness of expressions on intersections and unions is higher than when using only unions or only intersections. Interestingly enough, we show that expressions over a subclass of weighted voting games can generate all simple games if and only if they can generate all singleton games. In particular, this result allows us to show that the Boolean dimension of a game with n players, over the set of all singleton games, according to the definition in O'Dwyer and Slinko (2017), is upper bounded by n.

Finally, we provide a systematic classification of generalized multidimension for several subclasses of weighted voting games with up to six players. Our results are compared to those provided for generalized dimension and codimension in Molinero et al. (2023). Our study shows that the generalized multidimension can be smaller than the generalized dimension and codimension, even for simple games with very few players. Surprisingly, the experiments also show that the generalized multidimension presents a discontinuity for some classes. This latter result differs from the generalized dimension and codimension continuity established in Molinero et al. (2023).

The paper continues as follows. Section 2 presents the theoretical framework of this work. Section 3 introduces the (generalized) multidimension of simple games and gives some properties and examples. The main theoretical results are shown in Sect. 4, and the experimental ones are in Sect. 5. Section 6 is devoted to conclusions and future work.

2 Preliminaries

We start introducing basic notions and terminology for simple games. We also present the concepts of generalized dimension and codimension and recall some basic properties. We follow notation from Taylor and Zwicker (1999) and Molinero et al. (2023).

2.1 Simple games

Let N be a finite set, we denote $\mathcal{P}(N)$ the power set of N. A *simple game* Γ is a pair $(N, \mathcal{W})$, where $N = [n] = \{1, \ldots, n\}$ is a finite set of *players* or *voters* and $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotonic family of subsets of N, so that if $S \subseteq T \subseteq N$ and $S \in \mathcal{W}$, then $T \in \mathcal{W}$. As usual, we assume that $\emptyset \notin \mathcal{W}$ and $N \in \mathcal{W}$. $\mathcal{SG}$ denotes the family of all simple games.

A subset $S \subseteq N$ is called a *coalition*. N is called the *grand coalition*. $\mathcal{W}$ is the set of *winning coalitions*. The set of *losing coalitions*, denoted by $\mathcal{L}$, is formed by those coalitions that are not winning, i.e., $\mathcal{L} = \mathcal{P}(N) \setminus \mathcal{W}$. The set of *minimal winning coalitions*, denoted by $\mathcal{W}^m$, is formed by those winning coalitions whose strict subsets are losing coalitions, i.e., $\mathcal{W}^m = \{S \in \mathcal{W} \mid \forall T \in \mathcal{W}, T \not\subset S\}$. Analogously, the set of *maximal losing coalitions*, denoted by $\mathcal{L}^M$, is formed by those losing coalitions whose strict supersets are winning coalitions, i.e., $\mathcal{L}^M = \{S \in \mathcal{L} \mid \forall T \in \mathcal{L}, S \not\subset T\}$. Each one of these set families, $\mathcal{W}$, $\mathcal{L}$, $\mathcal{W}^m$ and $\mathcal{L}^M$, determine uniquely the game Γ and constitute different forms of representations of simple games (Taylor and Zwicker 1999). Note that the size of these representations may not be polynomial in the number of players (Molinero et al. 2015).

The operations of intersection and union are defined in a natural way over simple games. Let $\Gamma_1 = (N, \mathcal{W}_1)$ and $\Gamma_2 = (N, \mathcal{W}_2)$ be simple games. The *intersection* of Γ_1 and Γ_2 is the game with set of players N and winning coalitions $\mathcal{W}_1 \cap \mathcal{W}_2$, i.e., the game $\Gamma_1 \cap \Gamma_2 = (N, \mathcal{W}_1 \cap \mathcal{W}_2)$. The *union* of Γ_1 and Γ_2 is the game with set of players N and winning coalitions $\mathcal{W}_1 \cup \mathcal{W}_2$, i.e., the game $\Gamma_1 \cup \Gamma_2 = (N, \mathcal{W}_1 \cup \mathcal{W}_2)$. As the set of winning coalitions is monotone, the intersection and the union of simple games are simple games.

We represent a permutation $\sigma : [n] \to [n]$, by a vector with n components indicating the image of the n values. Given a game $\Gamma = (N, \mathcal{W})$ and a permutation σ on N, the game $\sigma(\Gamma)$ is the simple game obtained from Γ by replacing, in each winning coalition, each player i for $\sigma(i)$. We say that two simple games $\Gamma_1 = (N, \mathcal{W}_1)$ and $\Gamma_2 = (N, \mathcal{W}_2)$ are *isomorphic* if there is a permutation σ on N such that $\Gamma_2 = \sigma(\Gamma_1)$.

Example 1 Let be $\Gamma_1 = (N = [4], \mathcal{W}_1^m = \{\{1, 2\}, \{3, 4\}\})$ and $\Gamma_2 = (N = [4], \mathcal{W}_2^m = \{\{1, 3\}, \{2, 4\}\}$. Both games are isomorphic, i.e., $\Gamma_1 \simeq \Gamma_2$, because $\Gamma_1 = \sigma(\Gamma_2)$ where $\sigma = (1\ 3\ 2\ 4)$.

Further, every simple game has an associated dual game. Let $\Gamma = (N, \mathcal{W})$ be a simple game, its *dual* is the game $\Gamma^* = (N, \mathcal{W}^*)$ such that $\mathcal{W}^* = \{S \subseteq N \mid N \setminus S \notin \mathcal{W}\}$. Γ is said to be *self-dual* or *decisive* if $\Gamma = \Gamma^*$. We say that a class of games $\mathcal{G} \subseteq \mathcal{SG}$ is *closed under duality*, if, for each $\Gamma \in \mathcal{SG}$, $\Gamma^* \in \mathcal{G}$.

We recall now some player properties in a simple game, a player i is:

- *Dummy* if $i \in S$ implies $S \notin \mathcal{W}^m$, i.e., $i \notin S$, for all $S \in \mathcal{W}^m$;
- *Passer* if $i \in S$ implies $S \in \mathcal{W}$, i.e., $\{i\} \in \mathcal{W}^m$;
- *Vetoer* if $i \notin S$ implies $S \notin \mathcal{W}$, i.e., $N \setminus \{i\} \in \mathcal{L}$;
- *Dictator* if $i \in S \Leftrightarrow S \in \mathcal{W}$, i.e., if it is passer and vetoer, i.e., if $\mathcal{W}^m = \{\{i\}\}$.

A *pure simple game* is a simple game without dummy players. p-$\mathcal{SG}$ denotes the family of all pure simple games.

As we mentioned in Sect. 1, many voting systems can be represented as weighted voting games, one of the most relevant subclasses of simple games. A simple game $\Gamma = (N, \mathcal{W})$ is a *weighted voting game* if there exists a *weighted function* on the real numbers, $w : N \to \mathbb{R}$, and a real *quota* $q \in \mathbb{R}$, such that for any coalition $S \subseteq N$, $S \in \mathcal{W} \Leftrightarrow w(S) = \sum_{i \in S} w(i) \geq q$. $\mathcal{WVG}$ and p-$\mathcal{WVG}$ denote the families of all weighted voting games and pure weighted voting games, respectively.

In simple game theory, the weights of the players $i \in N$ are usually denoted as w_i instead of $w(i)$. Furthermore, every weighted voting game with a weighted function w, a quota q and a set of players $N = \{1, \ldots, n\}$ can be represented by a vector $[q; w_1, \ldots, w_n]$. Moreover, it is well known that both quota and weights can be restricted to be non-negative integer numbers, without losing expressiveness (Taylor and Zwicker 1999). In the following, we will only consider such integer representations. In particular, it is worth mentioning that $\Gamma = [q; w_1, \ldots, w_n]$ if and only if $\Gamma^* = [w(N) - q + 1; w_1, \ldots, w_n]$. Thus, Γ is self-dual if and only if $2q = w(N) + 1$.

Although there exist simple games that are not weighted simple games, i.e., $\mathcal{WVG} \subset \mathcal{SG}$, it is known that any simple game can be represented as either intersection (Taylor and Zwicker 1993) or union (Freixas and Marciniak 2010) of a finite number of weighted voting games. This property leads to the introduction of the dimension and the codimension concepts. Given a simple game Γ, the *dimension* of Γ ($\mathsf{dim}(\Gamma)$, in short) is the least number of weighted voting games whose intersection is equal to Γ, and the *codimension* of Γ ($\mathsf{codim}(\Gamma)$) is the least number of weighted voting games whose union is equal to Γ.

It is well known that $\mathsf{dim}(\Gamma) \leq |\mathcal{L}^M|$ and $\mathsf{codim}(\Gamma) \leq |\mathcal{W}^m|$. Furthermore, $\mathsf{dim}(\Gamma) = \mathsf{codim}(\Gamma^*)$ because $(\Gamma_1 \cap \Gamma_2)^* = \Gamma_1^* \cup \Gamma_2^*$. See Kurz et al. (2016) for further details.

We finish this section by defining a subclass of simple games containing weighted voting games. It is defined from a desirability relation that orders the players according to their influence (Isbell 1958). Let $\Gamma = (N, \mathcal{W})$ be a simple game, $a, b \in N$, and $S \subseteq N \backslash \{a, b\}$. We say that a player a is *at least as desirable* as another player b in Γ if $S \cup \{a\} \in \mathcal{W}$ implies $S \cup \{b\} \in \mathcal{W}$. A *complete game* is a simple game in which the desirability relation over their players is a complete preorder (Taylor and Pacelli 2008), i.e., a reflexive and transitive binary relation on their players in which any two elements are comparable. $\mathcal{CSG}$ and p-$\mathcal{CSG}$ denote the families of all complete games and all pure complete games, respectively. Note that $\mathcal{WVG} \subset \mathcal{CSG} \subset \mathcal{SG}$ and p-$\mathcal{WVG} \subset p$-$\mathcal{CSG} \subset p$-$\mathcal{SG}$.

2.2 Representations

Now, we recall some known definitions described by Molinero et al. (2023). Those definitions will be applied later to develop our experiments.

Definition 1 Let $\Gamma \in \mathcal{WVG}$. A representation $[q; w_1, w_2, \ldots, w_n]$ of Γ is:

- A *minimum* representation if, for any representation $[q'; w'_1, w'_2, \ldots, w'_n]$ of Γ, we have that $w_i \leq w'_i$, for all $i \in [n]$;
- A *minimum sum* representation (*min-sum*, for short) if, for any representation $[q'; w'_1, w'_2, \ldots, w'_n]$ of Γ, we have $\sum_{i=1}^n w_i \leq \sum_{i=1}^n w'_i$;
- A *canonical* representation if and only if $w_i \geq w_j$ whenever $i < j$;
- An *anti-canonical* representation if and only if $w_i \leq w_j$ whenever $i < j$. Note that it is a canonical representation but with the weights in reversed order;
- The *canonical minimum* representation (*down*, for short) of Γ if it is canonical, min-sum, and the vector $(w_1, w_2, \ldots, w_n)$ is lexicographically minimum among all player's weight vectors of canonical and min-sum representations of Γ. The last condition is

equivalent to, for any other canonical min-sum representation $[q'; w'_1, \ldots, w'_n]$ of Γ, if $i = \min\{j \in [n] \mid w_j \neq w'_j\}$ then $w_i < w'_i$;
- The *anti-canonical minimum representation* (*up*, for short) of Γ if it is the canonical minimum representation but with the weights in reversed order;
- A *down-up* representation if it is the *down* or the *up* representation.

Note that the canonical representation identifies exactly one game for a set of isomorphic games as the definition forces a particular isomorphism. Nevertheless, as a weighted voting game can have more that one min-sum representation, canonical representations might keep more than one game from each class of isomorphic games. According to (Molinero et al. 2023, Prop. 3), a min-sum representation $[q; w_1, w_2, \ldots, w_n]$ verifies that, for $i \in [n]$, $w_i = 0$ if and only if player i is dummy. Note that a simple game $(N, \mathcal{W})$ with a dummy player $i \in N$ could be reduced to a simple game with a smaller grand coalition, $(N \backslash \{i\}, \mathcal{W})$, keeping the same winning (and, therefore, losing) coalitions. Hence, we can generate infinite simple games with dummy players from a simple game without dummies only by increasing the set of players N. We add the suffix -p to a representation name to denote the type of representation with the additional condition that all values (quota and weights) must be positive. Observe that such subclasses hold only representations of pure games.

We also associate with a type of representation $\mathcal{R}$ the corresponding subclass of $\mathcal{WVG}$, denoted as $\mathcal{R}$-$\mathcal{WVG}$. Observe that, in general, a game can have infinite representations and more than one representation of a particular type. However, a representation defines only one game. In this way, min-sum-$\mathcal{WVG} = \mathcal{WVG}$, but down-$\mathcal{WVG}$ (or up-$\mathcal{WVG}$) contains only one game from each class of isomorphic $\mathcal{WVG}$. In the same way, down-p-$\mathcal{WVG}$ (or up-p-$\mathcal{WVG}$) contains only one game from each class of isomorphic p-$\mathcal{WVG}$.

We now recall the definitions of the closure under intersection and union, and the generalization of dimension and codimension to subfamilies of $\mathcal{WVG}$ introduced and studied in Molinero et al. (2023).

Definition 2 Let $\mathcal{G}$ be a subclass of weighted voting games. The *closure under intersection* of $\mathcal{G}$, denoted by $\mathcal{SG}_\cap(\mathcal{G})$, is the set of simple games that can be obtained as the intersection of a finite set of games in $\mathcal{G}$. That is,

$$\mathcal{SG}_\cap(\mathcal{G}) = \{\Gamma \in \mathcal{SG} \mid \exists \Gamma_1, \ldots, \Gamma_k \in \mathcal{G}, k \in \mathbb{N}, \text{ and } \Gamma = \Gamma_1 \cap \ldots \cap \Gamma_k\}.$$

In a similar way, the *closure under union* of $\mathcal{G}$, denoted by $\mathcal{SG}_\cup(\mathcal{G})$, is defined using union instead of intersection, i.e.,

$$\mathcal{SG}_\cup(\mathcal{G}) = \{\Gamma \in \mathcal{SG} \mid \exists \Gamma_1, \ldots, \Gamma_k \in \mathcal{G}, k \in \mathbb{N}, \text{ and } \Gamma = \Gamma_1 \cup \ldots \cup \Gamma_k\}.$$

Definition 3 Let $\Gamma \in \mathcal{SG}$ and $\mathcal{G} \subseteq \mathcal{WVG}$. The *generic dimension of* Γ *over* $\mathcal{G}$ ($\mathsf{g\text{-}dim}\,(\Gamma, \mathcal{G})$, in short) is

$$\mathsf{g\text{-}dim}\,(\Gamma, \mathcal{G}) = \begin{cases} \min\{t \mid \exists \Gamma_1, \ldots, \Gamma_t \in \mathcal{G} \text{ and } \Gamma = \Gamma_1 \cap \ldots \cap \Gamma_t\}, & \text{if } \Gamma \in \mathcal{SG}_\cap(\mathcal{G}) \\ +\infty, & \text{otherwise.} \end{cases}$$

The *generic codimension of* Γ *over* $\mathcal{G}$ (denoted by $\mathsf{g\text{-}codim}\,(\Gamma, \mathcal{G})$) is

$$\mathsf{g\text{-}codim}\,(\Gamma, \mathcal{G}) = \begin{cases} \min\{t \mid \exists \Gamma_1, \ldots, \Gamma_t \in \mathcal{G} \text{ and } \Gamma = \Gamma_1 \cup \ldots \cup \Gamma_t\}, & \text{if } \Gamma \in \mathcal{SG}_\cup(\mathcal{G}) \\ +\infty, & \text{otherwise.} \end{cases}$$

In Molinero et al. (2023), among other results comparing the generalized dimension and codimension over different subclasses of weighted voting games, the authors provide a characterization of the closure under intersection and union of the class p-$\mathcal{WVG}$. Before stating it, we need to introduce some notation. For, $1 \leq i \leq k \leq n$, define $\Gamma_{n,[k]:i}$ to be the simple game with set of players $N = [n]$ such that all minimal winning coalitions are the subsets of $[k]$ with i elements. Let $\mathcal{S}_n$ be the set of all permutations from $[n]$ to $[n]$. We also consider the games obtained from $\Gamma_{n,[k]:i}$ after permuting the players according to $\sigma \in \mathcal{S}_n$, denoted as $\Gamma_{n,\sigma([k]):i}$. Note that $\Gamma_{n,[k]:i}$ and $\Gamma_{n,\sigma([k]):i}$ are isomorphic and, moreover, $\Gamma^*_{n,\sigma([k]):i} = \Gamma_{n,\sigma([k]):k-i+1}$.

Theorem 1 (Molinero et al. 2023) *Let $\Gamma \in \mathcal{SG}$ with $n > 1$ players. Then,*

$$\textit{g-dim}\,(\Gamma,\, p\text{-}\mathcal{WVG}) \begin{cases} = \infty, & \text{if } \Gamma = \Gamma_{n,\sigma([k]):1},\ \text{for} 1 \leq k < n \text{ and } \sigma \in \mathcal{S}_n. \\ < \infty, & \text{otherwise.} \end{cases}$$

Furthermore, by duality,

$$\textit{g-codim}\,(\Gamma,\, p\text{-}\mathcal{WVG}) \begin{cases} = \infty, & \text{if } \Gamma = \Gamma_{n,\sigma([k]):k}, \text{for } 1 \leq k < n \text{and } \sigma \in \mathcal{S}_n. \\ < \infty, & \text{otherwise.} \end{cases}$$

3 Generalized multidimension

Now, we introduce an extension of the concepts of generalized dimension and codimension that we call *generalized multidimension*. To do so, we first generalize the type of expressions that can be used to generate a simple game.

Definition 4 Let $\mathcal{G}$ be a subclass of weighted voting games with a set of players N. An *N-$\mathcal{G}$-expression* is recursively defined by the following rules:

- $\Gamma \in \mathcal{G}$ is an N-$\mathcal{G}$-expression.
- If E' and E'' are N-$\mathcal{G}$-expressions, then so are $(E' \cap E'')$ and $(E' \cup E'')$.
- Nothing else is an N-$\mathcal{G}$-expression.

Note that we formally combine expressions with the intersection or the union operator using parentheses. However, when there is no risk of ambiguity, we can say $E' \cap E''$ instead of $(E' \cap E'')$, and $E' \cup E''$ instead of $(E' \cup E'')$. The *size* of an N-$\mathcal{G}$-expression E, denoted by $\mathsf{size}(E)$, is the number of operators appearing in E plus one.

To each N-$\mathcal{G}$-expression E, we associate a simple game with set of players N, denoted by $\Gamma(N, E)$, recursively as follows:

- If $E = \Gamma \in \mathcal{G}$, then $\Gamma(N, E) = \Gamma$.
- If $E = (E' \cap E'')$, then $\Gamma(N, E) = \Gamma(N, E') \cap \Gamma(N, E'')$.
- If $E = (E' \cup E'')$, then $\Gamma(N, E) = \Gamma(N, E') \cup \Gamma(N, E'')$.

Observe that an N-$\mathcal{G}$-expression can be represented by a binary tree whose internal nodes are labeled by either $\cap$ or $\cup$ and whose leaves are labeled by representations of games in $\mathcal{G}$, all of them defined over the same set of players N. Furthermore, the size of an $\mathcal{G}$-expression E coincides with the number of leaves in the binary tree. Now, we present a particular example where $\mathcal{G}$ is $\mathcal{WVG}$.

Example 2 Let $\Gamma_1 = [3; 1, 1, 2]$, $\Gamma_2 = [2; 1, 1, 2]$, $\Gamma_3 = [1; 0, 1, 0]$ and $\Gamma_4 = [3; 1, 2, 1]$ be four different weighted voting games, and $E_1 = (\Gamma_1 \cup \Gamma_2) \cap \Gamma_3$, $E_2 = (\Gamma_1 \cap \Gamma_3) \cup (\Gamma_2 \cap \Gamma_3)$ and $E_3 = \Gamma_4$ be three N-$\mathcal{WVG}$-expressions. Let $N = \{a, b, c\}$ be the set of players of the games. Note that $\Gamma(N, E_1) = \Gamma(N, E_2) = \Gamma(N, E_3)$ since in the three cases we obtain the simple game $(N, \mathcal{W})$ with $\mathcal{W}^m = \{\{a, b\}, \{b, c\}\}$. Furthermore, the three expressions can be represented by the binary trees illustrated in Fig. 1. Although the three expressions generate Γ_4, note that $\mathsf{size}(E_1) = 3$, $\mathsf{size}(E_2) = 4$ and $\mathsf{size}(E_3) = 1$.

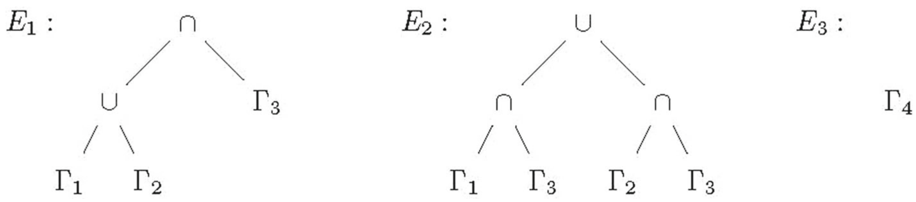

Fig. 1 Binary trees of three N-$\mathcal{WVG}$-expressions generating Γ_4

Note that a simple game with dimension d has an associated N-$\mathcal{WVG}$-expression formed by the intersection of d different weighted voting games, $\Gamma_1 \cap \cdots \cap \Gamma_d$. The same occurs for a simple game with codimension d, replacing $\cap$ by $\cup$ in the N-$\mathcal{WVG}$-expression. The size of both expressions is d, which is the number of operators $\cap$ plus one, and also the number of weighted voting games in the expressions. Thus, our definition of the size of an N-$\mathcal{WVG}$-expression is consistent with the concepts of dimension and codimension.

It is useful to consider the set of all N-$\mathcal{G}$-expressions.

Definition 5 A *$\mathcal{G}$-expression* is a N-$\mathcal{G}$-expression, for some set of players N.

To analyze the expressiveness of $\mathcal{G}$-expressions for subclasses $\mathcal{WVG}$, we introduce the closure concept.

Definition 6 Let $\mathcal{G} \subseteq \mathcal{WVG}$ be a subclass of weighted voting games. The *closure under $\mathcal{G}$-expression* of $\mathcal{G}$, denoted by $\mathcal{SG}_{\mathcal{E}}(\mathcal{G})$, is the set of simple games associated to $\mathcal{G}$-expressions, i.e., $\mathcal{SG}_{\mathcal{E}}(\mathcal{G}) = \cup_{n\in\mathbb{N},|N|=n}\{\Gamma(N, E) \mid E \text{ is a } N\text{-}\mathcal{G}\text{-expression}\}$.

Using this association, we define the generalized multidimension of a simple game over a subclass of weighted voting games $\mathcal{G} \subseteq \mathcal{WVG}$ as follows.

Definition 7 Let $\Gamma = (N, \mathcal{W}) \in \mathcal{SG}$. The *generalized multidimension over a subclass of weighted voting games* $\mathcal{G}$ of $\Gamma \in \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$ is the minimum size of an N-$\mathcal{G}$-expression E with $\Gamma = \Gamma(N, E)$. We denote such generalized multidimension over $\mathcal{G}$ of Γ by $\mathsf{g\text{-}mdim}\,(\Gamma, \mathcal{G})$. When $\Gamma \notin \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$, we say that its multidimension is infinite, i.e., $\mathsf{g\text{-}mdim}\,(\Gamma, \mathcal{G}) = \infty$.

Observe that the game defined on Example 2 has generic dimension, generic codimension and generic multidimension over $\mathcal{WVG}$ equal to 1. It is clear that the generalized multidimension depends on the considered subclass.

Example 3 Let $\Gamma_1 = [1; 1, 1, 0, 0, 0]$ and $\Gamma_2 = [1; 0, 0, 1, 1, 1]$ be two weighted voting games. Note that $[1; 1, 1, 0, 0, 0]$ is the down representation of Γ_1, and $[1; 0, 0, 1, 1, 1]$ is the up representation of Γ_2. Furthermore, Γ_2 does not admit any down representation as player 1 is a dummy player. Therefore, we have the following result: $1 = \mathsf{g\text{-}mdim}\,(\Gamma_1, \text{down-}\mathcal{WVG}) < \mathsf{g\text{-}mdim}\,(\Gamma_2, \text{down-}\mathcal{WVG})$. In a similar way, $\mathsf{g\text{-}mdim}\,(\Gamma_1, \text{up-}\mathcal{WVG}) >$ $\mathsf{g\text{-}mdim}\,(\Gamma_2, \text{up-}\mathcal{WVG}) = 1$. Moreover, it is clear that $\mathsf{g\text{-}mdim}\,(\Gamma_1, \text{down-up-}\mathcal{WVG}) = \mathsf{g\text{-}mdim}\,(\Gamma_2, \text{down-up-}\mathcal{WVG}) = 1$.

As every simple game can be defined as intersection or union of a finite number of weighted voting games, we always have a finite N-$\mathcal{WVG}$-expression describing it. In particular, as every simple game has a finite dimension and codimension (Taylor and Zwicker 1999), every simple game Γ also has a finite $\mathsf{g\text{-}mdim}\,(\Gamma, \mathcal{WVG})$. Moreover, given a simple game Γ, $\mathsf{g\text{-}mdim}\,(\Gamma, \mathcal{WVG}) \leq \min\{\mathsf{dim}(\Gamma), \mathsf{codim}(\Gamma)\}$.

As we have mentioned before, Boolean weighted voting games were introduced by Faliszewski et al. (2009). These games are defined by means of monotone Boolean formulas (those using only conjunction and disjunction). Let us review this definition and relate it with the definition of multidimension.

Definition 8 (Faliszewski et al. 2009, Sec.3) A simple *Boolean weighted voting game* is a tuple $\Gamma = (N, \mathcal{G}, \Phi, \varphi)$, where:

1. $N = \{1, \ldots, n\}$ is a set of players;
2. $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_m\}$ is a set of m weighted voting games over N;
3. $\Phi = \{p_1, \ldots, p_m\}$ is a set of m propositional variables with each variable p_j corresponding to the game Γ_j; and
4. φ is a monotone propositional formula over Φ.

A coalition S wins in Γ if and only if the assignment $\sigma(p_j) = (S \in \mathcal{W}(\Gamma_j))$, for $1 \leq j \leq m$, satisfies φ.

In Faliszewski et al. (2009), the size of a Boolean formula φ is defined as *the number of variable occurrences in* φ. Observe that, there is a one-to-one correspondence among monotone Boolean formulas and expressions over union and intersection. Thus, the multidimension is the minimum size of a Boolean weighted voting game that generates a given simple game. Faliszewski et al. (2009) does not define explicitly the multidimension of a simple game, but it provides results on the number of Boolean weighted voting games that can be described by monotone formulas of a given size. In particular (Faliszewski et al. 2009, Corollary 2) states that, for large enough n, there are simple games that cannot be described by Boolean formulas with size smaller than $2^n/n^5$.

Other related definitions were introduced in O'Dwyer and Slinko (2017) and Kurz (2021) as *Boolean dimension*. On the one hand, O'Dwyer and Slinko (2017, Def.7) defines the Boolean dimension of a simple game Γ as the smallest number d such that a Boolean weighted voting game on d weighted voting games generates Γ. In this way, the Boolean dimensions of Γ and Γ' in Example 9 are at most 3 and 6, respectively, but their multidimensions are 5 and 18, respectively. On the other hand, Kurz (2021, Def.2.12) defines the Boolean dimension of a simple game Γ as the smallest integer d such that Γ is a Boolean combination of d weighted voting games using the operators $\wedge$ and $\vee$, where a Boolean combination of d weighted voting games $v_1, \ldots, v_d$ is given by $v_1 \wedge v'$ or $v_1 \vee v'$, where v' is a Boolean combination of $v_2, \ldots, v_d$.

Several properties can be obtained directly from the definitions. It is clear that $\Gamma \in \mathcal{WVG}$ implies $\mathsf{dim}(\Gamma) = \mathsf{codim}(\Gamma) = \mathsf{g\text{-}mdim}(\Gamma, \mathcal{WVG}) = 1$. Moreover, for any $\Gamma \in \mathcal{SG}$, $\mathsf{dim}(\Gamma) = 2$ or $\mathsf{codim}(\Gamma) = 2$ implies that $\mathsf{g\text{-}mdim}(\Gamma, \mathcal{WVG}) = 2$, and vice versa. When $\Gamma \in \mathcal{SG}$ has $\mathsf{g\text{-}mdim}(\Gamma, \mathcal{WVG}) = m$, if it admits an N-$\mathcal{WVG}$-expression with size m that considers only intersections (or unions), then $\mathsf{dim}(\Gamma) = \mathsf{g\text{-}mdim}(\Gamma, \mathcal{WVG})$ (or $\mathsf{codim}(\Gamma) = \mathsf{g\text{-}mdim}(\Gamma, \mathcal{WVG})$).

Interestingly enough, our next result shows that the generalized multidimension is maintained by duality provided the considered class is closed under duality. The invariance under duality was stated for Boolean dimension by O'Dwyer and Slinko (2017, Proposition 5). Using similar arguments, our next results extends the property to generalized multidimension.

Proposition 2 *Let $\mathcal{G}$ be a class closed under duality. Let Γ be a simple game, then $\Gamma \in \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$ if and only if $\Gamma^* \in \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$. Furthermore, for $\Gamma \in \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$, $\mathsf{g\text{-}mdim}(\Gamma^*, \mathcal{G}) = \mathsf{g\text{-}mdim}(\Gamma, \mathcal{G})$.*

Proof If $\mathcal{G}$ is closed under duality, the dual of $\Gamma \in \mathcal{G}$ belongs also to $\mathcal{G}$. As we have mentioned before, the dual of a weighted voting game is a weighted voting game. In fact,

$[q; w_1, \ldots, w_n]^* = [w(N) - q + 1; w_1, \ldots, w_n]$. Furthermore, we know that $(\Gamma_1 \cap \Gamma_2)^* = \Gamma_1^* \cup \Gamma_2^*$ and $(\Gamma_1 \cup \Gamma_2)^* = \Gamma_1^* \cap \Gamma_2^*$ (Taylor and Zwicker 1999). Combining these statements, we can transform any N-$\mathcal{G}$-expression defining Γ into an N-$\mathcal{G}$-expression defining Γ^* and vice versa, without changing the number of leaves in the tree expression. So, the result follows. □

In Definition 4, we have considered only the two binary operators $\cup$ and $\cap$. In the traditional algebra of simple games, it is usual to consider the dual as a unary operator. From Proposition 2, it is clear that even though Definition 4 allowed the dual operator, the dual of an N-$\mathcal{G}$-expression is another N-$\mathcal{G}$-expression. In this way, we could remove the dual operator obtaining a smaller expression without duality. The thus-obtained generalized multidimension over $\mathcal{WVG}$ of any simple game would be the same that when the dual operator is not allowed.

Below, we mention examples where the generalized multidimension over $\mathcal{WVG}$ and codimension coincide, while the dimension is higher. By duality, in their dual games, the generalized multidimension over $\mathcal{WVG}$ and dimension coincide while the codimension is higher.

Example 4 Let $\Gamma = (N, \mathcal{W}^m)$ where $N = [2k]$ and $\mathcal{W}^m = \{i \in N \mid i \text{ is odd}\} \cup \{j \in N \mid j \text{ is even}\}$. It is clear that $\Gamma = [k; \underbrace{1, 0, \ldots, 1, 0}_{2k}] \cup [k; \underbrace{0, 1, \ldots, 0, 1}_{2k}]$. So, $\mathsf{codim}(\Gamma) = 2$. On the other hand, according to (Faliszewski et al. 2009, Th. 3), $\mathsf{dim}(\Gamma) = n^2/4$ (see also Taylor and Zwicker 1999). Thus, for $n > 2$, $\mathsf{g\text{-}mdim}\,(\Gamma, \mathcal{WVG}) = \mathsf{codim}(\Gamma) = 2$ and $\mathsf{dim}(\Gamma) = n^2/4$.

In the following example, the multidimension and the codimension coincide, but both are exponentially smaller than the dimension.

Example 5 Let $\Gamma = (N, \mathcal{W}^m)$ where $N = [2k]$ and $\mathcal{W}^m = \bigcup_{i \in [k]} \{2i - 1, 2i\}$. According to Freixas and Puente (2001), for $k > 0$, $\mathsf{g\text{-}mdim}\,(\Gamma, \mathcal{WVG}) = \mathsf{codim}(\Gamma) = k$ and $\mathsf{dim}(\Gamma) = 2^{k-1}$.

In Olsen et al. (2016) they provide more examples with exponential dimension and constant codimension. Example 8 will show some simple games with multidimension strictly smaller than the dimension and the codimension.

Taking into account the equivalence among expressions and monotone Boolean formula, the results in Proposition 1 and Corollary 2 of Faliszewski et al. (2009) can be rewritten as follows.

Proposition 3 (Faliszewski et al. 2009) *The total number of games in $\mathcal{SG}_{\mathcal{E}}(\mathcal{WVG})$ with n players and multidimension d is at most $2^{O(dn^2 \log(dn))}$. There are simple games with n players with $\mathsf{mdim} \geq 2^n/n^5$.*

4 Expressiveness: theoretical results

In this section, we study the multidimension of simple games over specific subclasses of weighted voting games. It is well known that intersections (or unions) of games in $\mathcal{WVG}$ generate all $\mathcal{SG}$. Thus, it is obvious that $\mathcal{WVG}$-expressions also generate all $\mathcal{SG}$. However, as we see in Example 3, $\mathcal{G}$-expression for $\mathcal{G}$ being down-$\mathcal{WVG}$ or up-$\mathcal{WVG}$ are not enough to generate all $\mathcal{SG}$, as well as down-$\mathcal{WVG}$ -expressions (or up-$\mathcal{WVG}$ -expressions). Next, we shall define sufficient conditions that a subclass $\mathcal{G} \subseteq \mathcal{SG}$ has to accomplish to verify that for all $\Gamma \in \mathcal{SG}$, $\text{g-mdim}(\Gamma, \mathcal{G}) < \infty$. Before doing that, we provide an upper bound for the generalized multidimension of unions or intersections of games.

Proposition 4 *Let $\mathcal{G} \subseteq \mathcal{SG}$ and let $\Gamma, \Gamma_1, \Gamma_2 \in \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$ with the same set of players N, such that $\Gamma = \Gamma_1 \cup \Gamma_2$ or $\Gamma = \Gamma_1 \cap \Gamma_2$, then $\text{g-mdim}(\Gamma, \mathcal{G}) \leq \text{g-mdim}(\Gamma_1, \mathcal{G}) + \text{g-mdim}(\Gamma_2, \mathcal{G})$.*

Proof Let E_1 be an N-$\mathcal{G}$-expression of minimum size describing Γ_1 and let E_2 be an N-$\mathcal{G}$-expression of minimum size describing Γ_2. It is clear that $E_1 \cup E_2$ and $E_1 \cap E_2$ are N-$\mathcal{G}$-expressions describing Γ and, by definition, their size is the sum of the sizes of E_1 and E_2. As the initial expressions have minimum size, the claim follows. □

To analyze additional properties about the closure under $\mathcal{G}$-expression of subclasses of $\mathcal{WVG}$, we start introducing some notation.

Definition 9 For $j \in N$, the j*-th singleton game* over N players is the simple game where the j-th player is the only minimal winner. We denote those games by $\Gamma_N^{(j)}$. Formally, for $j \in N$, we have that $\Gamma_N^{(j)} = \Gamma(N, \mathcal{W}^m)$ such that $\mathcal{W}^m = \{\{j\}\}$. The class of all singleton games with players N is denoted by $\mathcal{S}_N$-$\mathcal{SG}$, and all singleton games by $\mathcal{S}$-$\mathcal{SG}$.

Note that $\Gamma_N^{(j)}$ is the simple game such that the j-th player is a dictator and the other players are dummies. Next, we prove that using $\mathcal{G}$-expressions, one class can generate all $\mathcal{S}$-$\mathcal{SG}$ if and only if it can generate $\mathcal{SG}$.

Theorem 5 *Let $\mathcal{G} \subseteq \mathcal{SG}$, $\mathcal{SG}_{\mathcal{E}}(\mathcal{G}) = \mathcal{SG}$ if and only if $\mathcal{S}$-$\mathcal{SG} \subseteq \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$. Moreover, for such a subclass $\mathcal{G}$ with $\mathcal{S}$-$\mathcal{SG} \subseteq \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$, we have*

(a) *given $\Gamma = (N, \mathcal{W}^m) \in \mathcal{SG}$, then*

$$\text{g-mdim}(\Gamma, \mathcal{G}) \leq \sum_{S \in \mathcal{W}^m} \sum_{j \in S} \text{g-mdim}\left(\Gamma_N^{(j)}, \mathcal{G}\right),$$

(b) *given $\Gamma = (N, \mathcal{L}^M) \in \mathcal{SG}$, then*

$$\text{g-mdim}(\Gamma, \mathcal{G}) \leq \sum_{T \in \mathcal{L}^M} \sum_{j \in N \setminus T} \text{g-mdim}\left(\Gamma_N^{(j)}, \mathcal{G}\right),$$

up to isomorphism.

Proof Of course, if there is $\Gamma \in \mathcal{S}$-$\mathcal{SG}$ with $\Gamma \notin \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$, then $\mathcal{SG}_{\mathcal{E}}(\mathcal{G}) \subsetneq \mathcal{SG}$. For the other implication, i.e., if $\mathcal{S}$-$\mathcal{SG} \subseteq \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$ then $\mathcal{SG}_{\mathcal{E}}(\mathcal{G}) = \mathcal{SG}$, we prove the two inequalities in two separated statements.

(a) Let be $\Gamma = (N, \mathcal{W}^m) \in \mathcal{SG}$, where $\mathcal{W}^m = \{S_1, \ldots, S_k\}$. It is clear that $\Gamma = \bigcup_{1 \leq i \leq k} \Gamma_i$, where $\Gamma_i = (N, \mathcal{W}_i^m)$ being $\mathcal{W}_i^m = \{S_i\}$, for $1 \leq i \leq k$. Note that $\Gamma_i = \bigcap_{j \in S_i} \Gamma_N^{(j)}$. It is also clear that Γ satisfies

$$\Gamma = \bigcup_{1 \leq i \leq k} \left(\bigcap_{j \in S_i} \Gamma_N^{(j)} \right).$$

Thus, given $\Gamma = (N, \mathcal{W}^m) \in \mathcal{SG}$, from Proposition 4, the result follows

$$\text{g-mdim}(\Gamma, \mathcal{G}) \leq \sum_{S \in \mathcal{W}^m} \sum_{j \in S} \text{g-mdim}\left(\Gamma_N^{(j)}, \mathcal{G}\right).$$

(b) Let be $\Gamma = (N, \mathcal{L}^M) \in \mathcal{SG}$, where $\mathcal{L}^M = \{T_1, \ldots, T_\ell\}$. It results that $\Gamma = \bigcap_{1 \le i \le \ell} \Gamma_i$, where $\Gamma_i = (N, \mathcal{L}^M{}_i)$ being $\mathcal{L}^M{}_i = \{T_i\}$, for $1 \le i \le \ell$. Observe that $\Gamma_i = \bigcup_{j \in N \setminus T_i} \Gamma^{(j)}$.
In this case, we have that

$$\Gamma = \bigcap_{1 \le i \le \ell} \left(\bigcup_{j \in N \setminus T_i} \Gamma^{(j)} \right).$$

So, given $\Gamma = (N, \mathcal{L}^M) \in \mathcal{SG}$, then the inequality results

$$\mathsf{g\text{-}mdim}\,(\Gamma, \mathcal{G}) \le \sum_{T \in \mathcal{L}^M} \sum_{j \in N \setminus T} \mathsf{g\text{-}mdim}\left(\Gamma_N^{(j)}, \mathcal{G}\right).$$

□

Example 6 Let be $\Gamma = (N, \mathcal{W}^m)$, where $N = \{1, 2, 3\}$ and $\mathcal{W}^m = \{\{1, 2\}, \{3\}\})$. Then, $\Gamma_1 = (N, \mathcal{W}_1^m = \{\{1, 2\}\}) = \Gamma_N^{(1)} \cap \Gamma_N^{(2)}$ and $\Gamma_2 = (N, \mathcal{W}_2^m = \{\{3\}\}) = \Gamma_N^{(3)}$ verify $\Gamma = \Gamma_1 \cup \Gamma_2$. On the other hand, let be $\overline{\Gamma} = (N, \mathcal{L}^M)$, where $N = \{1, 2, 3\}$ and $\mathcal{L}^M = \{\{1, 2\}, \{3\}\}$. Then, $\overline{\Gamma}_1 = (N, \mathcal{L}^M{}_1 = \{\{1, 2\}\}) = \Gamma_N^{(1)}$ and $\overline{\Gamma}_2 = (N, \mathcal{L}^M{}_2 = \{\{3\}\}) = \Gamma_N^{(1)} \cup \Gamma_N^{(2)}$ satisfy $\overline{\Gamma} = \overline{\Gamma}_1 \cap \overline{\Gamma}_2$. Hence, $\mathsf{g\text{-}mdim}\,(\Gamma, \mathcal{WVG}) \le 3$.

From the previous theorem, we can establish the following corollary.

Corollary 1 *Given $\Gamma = (N, \mathcal{W}^m) = (N, \mathcal{L}^M) \in \mathcal{SG}$, and a subclass $\mathcal{G} \subseteq \mathcal{SG}$.*

(a) *If $\mathcal{S}$-$\mathcal{SG} \subseteq \mathcal{SG}_{\mathcal{E}}(\mathcal{G})$, then*

$$\textit{g-mdim}\,(\Gamma, \mathcal{G}) \le \min \left\{ \sum_{S \in \mathcal{W}^m} \sum_{j \in S} \textit{g-mdim}\left(\Gamma_N^{(j)}, \mathcal{G}\right), \sum_{T \in \mathcal{L}^M} \sum_{j \in N \setminus T} \textit{g-mdim}\left(\Gamma_N^{(j)}, \mathcal{G}\right) \right\}.$$

(b) *If $\mathcal{S}$-$\mathcal{SG} \subseteq \mathcal{G}$, then*

$$\textit{g-mdim}\,(\Gamma, \mathcal{G}) \le \min \left\{ \sum_{S \in \mathcal{W}^m} |S|, \sum_{T \in \mathcal{L}^M} (|N| - |T|) \right\} \le |N| \min \left\{ |\mathcal{W}^m|, |\mathcal{L}^M| \right\}.$$

A class that does not contain the class $\mathcal{S}$-$\mathcal{SG}$ is p-$\mathcal{WVG}$, as singleton games, for $n > 1$, have dummy players. Our next result shows that p-$\mathcal{WVG}$-expressions cannot generate the family $\mathcal{S}$-$\mathcal{SG}$. Furthermore, we show that singleton games are the unique simple games without finite multidimension over the family of p-$\mathcal{WVG}$.

Theorem 6 $\mathcal{SG}_{\mathcal{E}}(p\text{-}\mathcal{WVG}) = \mathcal{SG} \setminus \mathcal{S}\text{-}\mathcal{SG}$.

Proof The proof consists in two steps.

From Theorem 1 we establish that $\Gamma \notin \mathcal{S}$-$\mathcal{SG}$ with $n > 1$ implies $\mathsf{g\text{-}dim}\,(\Gamma, p\text{-}\mathcal{WVG}) < \infty$ or $\mathsf{g\text{-}codim}\,(\Gamma, p\text{-}\mathcal{WVG}) < \infty$. Thus, in such a case $\Gamma \in \mathcal{SG}_{\mathcal{E}}(p\text{-}\mathcal{WVG})$. Therefore, $\mathcal{SG} \setminus \mathcal{S}\text{-}\mathcal{SG} \subseteq \mathcal{SG}_{\mathcal{E}}(p\text{-}\mathcal{WVG})$.

We prove that singleton games cannot be generated by *reductio ad absurdum*. Without loss of generality, we assume that $\Gamma = \Gamma_N^{(1)} \in \mathcal{S}_N$-$\mathcal{SG}$ and that $\mathsf{g\text{-}mdim}\,(\Gamma, p\text{-}\mathcal{WVG}) < \infty$. Then, we could *remove* the last $n - 2$ players to obtain that $\mathsf{g\text{-}mdim}\left(\Gamma_{[2]}^{(1)}, p\text{-}\mathcal{WVG}\right) < \infty$. Now, by brute force, we check that neither intersection nor union of the two pure weighted voting games with 2 players, $\Gamma_1 = ([2], \mathcal{W}^m = \{\{1\}, \{2\}\})$ and $\Gamma_2 = ([2], \mathcal{W}^m = \{\{1, 2\}\})$, generate $\Gamma_{[2]}^{(1)} = ([2], \mathcal{W}^m = \{\{1\}\})$. □

As a consequence of the previous theorem, we have the following result.

Corollary 2 *Let* $\Gamma \in \mathcal{SG}$ *with* $n > 1$ *players.*

$$\textsf{g-mdim}\,(\Gamma,\, p\text{-}\mathcal{WVG}) \begin{cases} = \infty & \text{if } \Gamma \in \mathcal{S}\text{-}\mathcal{SG}, \\ < \infty & \text{otherwise.} \end{cases}$$

Observe that the difference among the characterization of simple games with finite dimension/codimension over p-$\mathcal{WVG}$ given in Molinero et al. (2023) (see Theorem 1) is obtained by duality. The previous theorem shows that the additional expressiveness provided by multidimension with respect to p-$\mathcal{WVG}$ is obtained by duality. Only those self-dual games that have infinite dimension remain with infinite multidimension.

Theorem 5 shows that $\mathcal{G}$-expressions over singleton games generate all simple games. However, as we will see in Sect. 5, the class $\mathcal{S}$-$\mathcal{SG}$ can not be generated with $\mathcal{G}$-expressions for $\mathcal{G}$ being down-$\mathcal{WVG}$, up-$\mathcal{WVG}$, or down-up-$\mathcal{WVG}$. These classes contain some, but not all singleton games, it remains open to find a characterization of the closure under $\mathcal{G}$-expression for these classes.

Based on Molinero et al. (2023), when there is no game with dimension (codimension) k, it implies that there is no game with dimension (codimension) bigger than k. That is, the dimension values over a set of N players form a contiguous subset of integers. Now, we analyze this property for multidimension.

First, as we have shown in Proposition 4, the multidimension of the union (or intersection) of two simple games is upper bounded by the sum of the multidimension of those considered games. However, our following result shows that a minimum size expression provides a decomposition in which the upper bound is attained with equality.

Proposition 7 *Let* E, E_1 *and* E_2 *be* N-$\mathcal{G}$*-expressions such that* $E = E_1 \cup E_2$ *or* $E = E_1 \cap E_2$. *Let be* $\Gamma = \Gamma(N, E)$, $\Gamma_1 = \Gamma(N, E_1)$ *and* $\Gamma_2 = \Gamma(N, E_2)$. *If* $\textsf{size}(E) = \textsf{g-mdim}\,(\Gamma, \mathcal{G})$*, then* $\textsf{g-mdim}\,(\Gamma, \mathcal{G}) = \textsf{g-mdim}\,(\Gamma_1, \mathcal{G}) + \textsf{g-mdim}\,(\Gamma_2, \mathcal{G})$. *Moreover,* $\textsf{size}(E_1) = \textsf{g-mdim}\,(\Gamma_1, \mathcal{G})$*, and* $\textsf{size}(E_2) = \textsf{g-mdim}\,(\Gamma_2, \mathcal{G})$.

Proof Otherwise, we could decrease $\textsf{g-mdim}\,(\Gamma, \mathcal{G})$ because $E = E_1 \cup E_2$ or $E = E_1 \cap E_2$, which is a contradiction. □

Note that, assuming the same game constructions of Proposition 7, there may be two simple games $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ isomorphic to Γ_1 and Γ_2, respectively (i.e., $\overline{\Gamma}_1 \simeq \Gamma_1$ and $\overline{\Gamma}_2 \simeq \Gamma_2$), such that $\textsf{g-mdim}\left(\overline{\Gamma}_1, \mathcal{G}\right) < \textsf{g-mdim}\,(\Gamma_1, \mathcal{G})$ or $\textsf{g-mdim}\left(\overline{\Gamma}_2, \mathcal{G}\right) < \textsf{g-mdim}\,(\Gamma_2, \mathcal{G})$. The following example verifies such property.

Example 7 Let be $\Gamma_1 = (N = [5], \mathcal{W}_1^m = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{3, 5\}, \{4, 5\}\})$ and $\Gamma_2 = (N = [5], \mathcal{W}_2^m = \{\{2, 3\}, \{1, 4\}, \{1, 2, 5\}, \{3, 5\}, \{4, 5\}\})$. Using the permutation

$$\sigma = (3\ 2\ 5\ 1\ 4),$$

we see that $\Gamma_1 \simeq \Gamma_2$ because $\sigma(\mathcal{W}_1^m) = \mathcal{W}_2^m$. However, we have checked by brute force that

$$\textsf{g-mdim}\,(\text{down-up-}p\text{-}\mathcal{WVG}, \Gamma_1) = 2 < 3 = \textsf{g-mdim}\,(\text{down-up-}p\text{-}\mathcal{WVG}, \Gamma_2)\,.$$

In particular

$$\Gamma_1 = [5; 3, 2, 2, 1, 1] \cup [5; 1, 1, 2, 2, 3]$$

and

$$\Gamma_2 = [5; 1, 1, 2, 2, 3] \cup ([6; 4, 3, 3, 2, 1] \cap [4; 1, 2, 2, 3, 3])\,.$$

The previous results show that, if the multidimension of a game is bigger than one, there exists a decomposition (as either the union or the intersection of two games) whose multidimension coincides with the sum of the multidimension of the two components. This property allows us to establish the following stopping criteria for our experiments.

Theorem 8 *Let $\mathcal{G} \subseteq \mathcal{WVG}$ and let $k > 1$. If no simple game Γ has multidimension g-mdim $(\Gamma, \mathcal{G}) = a$, for integers $0 < \lceil k/2 \rceil \leq a < k$, then no simple game Γ has g-mdim $(\Gamma, \mathcal{G}) \geq k$.*

Proof We prove, by induction on ℓ, that there is no simple game Γ with $\text{g-mdim}\,(\Gamma, \mathcal{G}) = k + \ell$.

For the base case, $\ell = 0$. Suppose that no simple game Γ has $\text{g-mdim}\,(\Gamma, \mathcal{G}) = a$, for $0 < \lceil k/2 \rceil \leq a < k$, and that it exists $\Gamma \in \mathcal{SG}$ such that $\text{g-mdim}\,(\Gamma, \mathcal{G}) = k > 1$. Proposition 7 implies that there exist $\Gamma_1, \Gamma_2 \in \mathcal{SG}$ such that $\text{g-mdim}\,(\Gamma, \mathcal{G}) = \text{g-mdim}\,(\Gamma_1, \mathcal{G}) + \text{g-mdim}\,(\Gamma_2, \mathcal{G})$. Therefore, $a = \max\{\text{g-mdim}\,(\Gamma_1, \mathcal{G}), \text{g-mdim}\,(\Gamma_2, \mathcal{G})\}$ verifies $\lceil k/2 \rceil \leq a < k$, and we get a contradiction.

As induction hypothesis, assume that, for some $\ell \geq 1$, no simple game Γ verifies $\text{g-mdim}\,(\Gamma, \mathcal{G}) = a$, for integers $0 < \lceil k/2 \rceil \leq a < k + \ell$. Now, we can use the previous argument, assuming that it exists $\Gamma \in \mathcal{SG}$ such that $\text{g-mdim}\,(\Gamma, \mathcal{G}) = k + \ell$, there exists $\Gamma_1, \Gamma_2 \in \mathcal{SG}$ such that $\text{g-mdim}\,(\Gamma, \mathcal{G}) = \text{g-mdim}\,(\Gamma_1, \mathcal{G}) + \text{g-mdim}\,(\Gamma_2, \mathcal{G})$, so there is a game with multidimension over $\mathcal{G}$ equal to b with $\lceil \frac{k+\ell}{2} \rceil \leq b < k + \ell$. As $\lceil k/2 \rceil \leq \lceil \frac{k+\ell}{2} \rceil < k + \ell$, this is not possible. □

Note that Theorem 8 gives a stopping criteria when we compute experimentally the multidimension of simple games. For instance, if there are no games with multidimension 5, 6, 7, and 8, it is not possible to generate games with multidimension bigger or equal than 9 because $5 = \lceil 9/2 \rceil \leq 8 < 9$.

Observe that the previous results leave open the possibility that multidimension is not continuous. Our experiments confirm that this is the case. Table 7 gives some examples. When considering the subclass $\mathcal{S}$-$\mathcal{SG}$, for 3 players, no game has multidimension 4, but there is one game with multidimension 5. Analogously, for 5 players, no game has multidimension 13, but there is one game with multidimension 14.

5 Experimental results

This section collects different experimental results of generalized multidimension over subclasses of weighted voting games. We enumerate simple games and pure simple games from $\mathcal{G}$-expressions over specific subclasses of weighted voting games, for up to 6 or 7 players. Our counting results are done *up to isomorphism*, i.e., we count one game provided that at least one isomorphic game has been generated. We do so in order to make the values comparable with usual counting series in which all isomorphic games are counted only once. Nevertheless, it is worth to mention, that for several classes, $\mathcal{G}$-expressions generate only some of the games in an isomorphism class.

All the experiments have been implemented in the C++ programming language. First, we generate all down-p-$\mathcal{WVG}$ according to Molinero et al. (2023). Although our results are up to 6 or 7 players (see Tables 2, 3, 4, 5, 6 and 7), we have been able to compute all down-p-$\mathcal{WVG}$ up to 8 players and up to isomorphism. It was unfeasible for more than 8 players.

From the enumeration of down-p-$\mathcal{WVG}$, we are able to enumerate other subclasses of simple games. The games of up-p-$\mathcal{WVG}$ are obtained by considering the mirror of each game $[q; w_1, \ldots, w_n] \in$ down-p-$\mathcal{WVG}$, i.e., $[q; w_n, \ldots, w_1]$. For instance, given $[5; 3, 2, 2, 1] \in$ down-p-$\mathcal{WVG}$, we consider $[5; 1, 2, 2, 3] \in$ up-p-$\mathcal{WVG}$. On the other hand, the games of down-up-p-$\mathcal{WVG}$ are generated doubling each game $[q; w_1, \ldots, w_n] \in$ down-p-$\mathcal{WVG}$ as a mirror $[q; w_n, \ldots, w_1]$, e.g., given $[5; 3, 2, 2, 1]$, we consider $[5; 3, 2, 2, 1]$ and $[5; 1, 2, 2, 3]$.

For each number of players, we use the list of the games in the considered class (e.g., down-$\mathcal{WVG}$, down-p-$\mathcal{WVG}$ or any class) together with an enumerator of the well-formed expressions over the intersection and union operators to obtain in increasing order of multidimension the new generated games.

The first experiment (see Table 2b) computes the generalized multidimension over p-$\mathcal{WVG}$. First, we generate new simple games with intersection or union of two pure weighted voting games, up to 6 players. Second, we compute new simple games combining intersections and unions of three pure weighted voting games, and, so on. As we said before, all multidimensionality results has been obtained with the stopping criteria described in Theorem 8. Finally, we count the number of generated simple games up to isomorphism.

We follow the same procedure described before for all our experiments, but assuming the corresponding subclass of weighted voting games.

Now, Table 1 presents some known counting results for p-$\mathcal{SG}$, p-$\mathcal{CSG}$, and p-$\mathcal{WVG}$, up to isomorphism. All these results appear in (or can be deduced from) the so-called *The On-Line Encyclopedia of Integer Sequences* (http://oeis.org/).

Table 2 presents the results for the generalized dimension and multidimension over p-$\mathcal{WVG}$ of p-$\mathcal{SG}$ having up to 6 players. From Theorems 1 and 6, we know that not all $\mathcal{SG}$ have finite generalized dimension or multidimension with respect to the class p-$\mathcal{WVG}$. The results over the generalized dimension come from (Molinero et al. 2023). The results over generalized multidimension come from our experiments and thus there are completely new. This table shows us that, with respect to p-$\mathcal{WVG}$, as expected the multidimension of a game can be strictly smaller than its dimension. It is interesting to note that the highest value of the multidimension is smaller than the highest value of the dimension. Even more, the three simple games with 6 players having maximum dimension of 5 have also maximum multidimension of 4. In the next Example, we analyze these three games in more detail.

Example 8 Table 2 shows that there are only three simple games of 6 players with dimension and codimension equal to 5. Reproducing the experiments according to Molinero et al. (2023), these three simple games are

$$\begin{aligned}\Gamma_1 &= ([6], \mathcal{W}_1^m = \{\{1,2,3\},\{1,3,6\},\{1,4,5\},\{1,5,6\},\{2,3,5\},\{2,4,6\},\\ &\qquad \{2,5,6\},\{3,4,5\},\{3,4,6\}\}),\\ \Gamma_2 &= \Gamma_1^*,\end{aligned}$$

and the self-dual

$$\begin{aligned}\Gamma_3 &= ([6], \mathcal{W}_1^m = \{\{1,3,4\},\{2,3,4\},\{1,2,5\},\{2,3,5\},\{1,4,5\},\{1,2,6\},\\ &\qquad \{1,3,6\},\{2,4,6\},\{3,5,6\},\{4,5,6\}\}).\end{aligned}$$

We checked that no other intersection/union with less than 5 p-$\mathcal{WVG}$ gives Γ_i, for $i \in [3]$, i.e., $\textsf{g-dim}\,(p\text{-}\mathcal{WVG}, \Gamma_i) = \textsf{g-codim}\,(p\text{-}\mathcal{WVG}, \Gamma_i) = 5$. In particular, we get the following

Table 1 Counting for $\mathcal{SG}$, p-$\mathcal{SG}$, p-$\mathcal{CSG}$, and p-$\mathcal{WVG}$, up to isomorphism (Molinero et al. 2023)

(Sub)class	Number of players								
	1	2	3	4	5	6	7	8	9
$\mathcal{SG}$ (oeis.org/A003182)	1	3	8	28	208	16,351	490,013,146	1,392,195,548,889,993,356	?
p-$\mathcal{SG}$ (oeis.org/A006602)	1	2	5	20	180	16,143	489,996,795	1,392,195,548,399,980,210	?
p-$\mathcal{CSG}$ (oeis.org/A132183)	1	2	5	17	92	1054	43,142	16,130,875	284,416,554,986
p-$\mathcal{WVG}$ (oeis.org/A000619)	1	2	5	17	92	994	28,262	2,700,791	990,331,318

Table 2 Counting pure simple games with g-dim $(\Gamma, p\text{-}\mathcal{WVG}) = d$ and g-mdim $(\Gamma, p\text{-}\mathcal{WVG}) = m$

(a) Number of games $\Gamma \in p\text{-}\mathcal{SG}$ with g-dim $(\Gamma, p\text{-}\mathcal{WVG}) = d$, for the different values of d, up to isomorphism

d	Number of players (n)					
	1	2	3	4	5	6
1	1	2	5	17	92	994
2	0	0	0	3	86	11,168
3	0	0	0	0	2	3595
4	0	0	0	0	0	383
5	0	0	0	0	0	3
Total	1	2	5	20	180	16,143
p-$\mathcal{SG}$	1	2	5	20	180	16,143

(b) Number of games $\Gamma \in p\text{-}\mathcal{SG}$ with g-mdim $(\Gamma, p\text{-}\mathcal{WVG}) = m$, for the different values of m, up to isomorphism

m	Number of players (n)					
	1	2	3	4	5	6
1	1	2	5	17	92	994
2	0	0	0	3	86	12,755
3	0	0	0	0	2	2388
4	0	0	0	0	0	6
5	0	0	0	0	0	0
Total	1	2	5	20	180	16,143
p-$\mathcal{SG}$	1	2	5	20	180	16,143

expressions:

$$\begin{aligned}
\Gamma_1 &= [4; 2, 2, 2, 1, 1, 1] \cap [4; 2, 1, 1, 2, 2, 1] \cap [5; 2, 3, 1, 3, 1, 2] \cap [8; 1, 2, 5, 3, 4, 3] \cap [8; 3, 2, 3, 1, 4, 5]\\
&= \sigma_1([6; 2, 2, 2, 1, 1, 1] \cup [8; 3, 2, 1, 3, 2, 1] \cup [8; 3, 1, 2, 1, 2, 3] \cup [8; 1, 3, 2, 1, 3, 2] \cup [8; 1, 1, 3, 3, 2, 2]),\\
\Gamma_2 &= \sigma_1([4; 2, 2, 2, 1, 1, 1] \cap [5; 3, 2, 1, 3, 2, 1] \cap [5; 3, 1, 2, 1, 2, 3] \cap [5; 1, 3, 2, 1, 3, 2] \cap [5; 1, 1, 3, 3, 2, 2]\\
&= [6; 2, 2, 2, 1, 1, 1] \cup [6; 2, 1, 1, 2, 2, 1] \cup [8; 2, 3, 1, 3, 1, 2] \cup [11; 1, 2, 5, 3, 4, 3] \cup [11; 3, 2, 3, 1, 4, 5],\\
\Gamma_3 &= [5; 3, 3, 2, 2, 1, 1] \cap [5; 3, 1, 2, 1, 3, 2] \cap [5; 1, 3, 2, 1, 2, 3] \cap [5; 1, 2, 2, 3, 3, 1] \cap [5; 2, 1, 2, 3, 1, 3]\\
&= [8; 3, 3, 2, 2, 1, 1] \cup [8; 3, 1, 2, 1, 3, 2] \cup [8; 1, 3, 2, 1, 2, 3] \cup [8; 1, 2, 2, 3, 3, 1] \cup [8; 2, 1, 2, 3, 1, 3],
\end{aligned}$$

where $\sigma_1 = (6\ 4\ 3\ 2\ 5\ 1)$ is the corresponding permutation among players.

However, our experiments show that g-mdim $(p\text{-}\mathcal{WVG}, \Gamma_1) = 4$. Particular cases with minimum size for Γ_i, being $i \in [3]$, are

$$\begin{aligned}
\Gamma_1 &= \sigma_2([4; 2, 2, 2, 1, 1, 1] \cap ([5; 3, 3, 2, 2, 1, 1] \cap ([8; 4, 3, 3, 2, 2, 1] \cup [11; 5, 4, 3, 3, 2, 1]))),\\
\Gamma_2 &= \sigma_2([6; 2, 2, 2, 1, 1, 1] \cup ([8; 3, 3, 2, 2, 1, 1] \cup ([5; 3, 3, 2, 2, 1, 1] \cap [8; 4, 3, 3, 2, 2, 1]))),\\
\Gamma_3 &= \sigma_3([8; 3, 3, 2, 2, 1, 1] \cup ([10; 6, 5, 4, 3, 2, 1] \cap ([8; 4, 3, 2, 2, 1, 1] \cup [8; 4, 3, 3, 2, 2, 1]))),
\end{aligned}$$

where $\sigma_2 = (2\ 6\ 4\ 3\ 1\ 5)$ and $\sigma_3 = (3\ 4\ 1\ 2\ 6\ 5)$.

It is worth mentioning that only 3 out of 383 games with 6 players and generalized dimension with respect to $p\text{-}\mathcal{WVG}$ equal to 4 keep their generalized multidimension equal to 4. For up to 6 players the maximum value of the generalized dimension is strictly smaller that the maximum value of the generalized multidimension. It remains open to see if this property carries on for any number of players.

Table 3 Counting pure simple games with g-dim $(\Gamma, \text{down-}p\text{-}\mathcal{WVG}) = d$ *and* g-mdim $(\Gamma, \text{down-}p\text{-}\mathcal{WVG}) = m$

(a) Number of games $\Gamma \in p\text{-}\mathcal{SG}$ with g-dim $(\Gamma, \text{down-}p\text{-}\mathcal{WVG}) = d$, for the different values of d, up to isomorphism

d	Number of players						
	1	2	3	4	5	6	7
1	1	2	5	17	92	994	28,262
2	0	0	0	0	0	55	13,808
3	0	0	0	0	0	2	539
4	0	0	0	0	0	0	38
Total	1	2	5	17	92	1051	42,647
$p\text{-}\mathcal{CSG}$	1	2	5	17	92	1054	43,142
$p\text{-}\mathcal{SG}$	1	2	5	20	180	16,143	489,996,795

(b) Number of games $\Gamma \in p\text{-}\mathcal{SG}$ with g-mdim $(\Gamma, \text{down-}p\text{-}\mathcal{WVG}) = m$, for the different values of m, up to isomorphism

m	Number of players						
	1	2	3	4	5	6	7
1	1	2	5	17	92	994	28,262
2	0	0	0	0	0	60	14,880
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
Total	1	2	5	17	92	1054	43,142
$p\text{-}\mathcal{CSG}$	1	2	5	17	92	1054	43,142
$p\text{-}\mathcal{SG}$	1	2	5	20	180	16,143	489,996,795

Experimental results about the generalized dimension and multidimension of $p\text{-}\mathcal{SG}$ with respect to games in down-$p\text{-}\mathcal{WVG}$ are presented in Table 3. As it was mentioned in Molinero et al. (2023), intersections (or unions) of games in down-$p\text{-}\mathcal{WVG}$ belong to $p\text{-}\mathcal{CSG}$. We combined our multidimension enumeration algorithm with a checker for completeness. Our experiments show that, for up to 7 players, all simple games generated by $\mathcal{G}$-expressions over down-$p\text{-}\mathcal{WVG}$ belong to $\mathcal{CSG}$. It will be worth to see whether this property holds for any number of players. In the light of our experiments, we think that it is true, but we have not formally proved it yet.

Another property is that $\mathcal{G}$-expressions over down-$p\text{-}\mathcal{WVG}$ allow us to obtain more games that when using only intersections/unions. This phenomenon appears for 6 and 7 players in our experiments. We want to note that, for 6 and 7 players, down-$p\text{-}\mathcal{WVG}$ generate all games in $\mathcal{CSG}$. Another unexpected property that we can extract from the table is that the maximum value of the multidimension is 2, while the maximum value of the dimension is 4.

The experimental results about generalized multidimension of simple games with respect to the subclasses down-$p\text{-}\mathcal{WVG}$ and down-$\mathcal{WVG}$ appear in Table 4. In this experiment, we removed the filter checking pureness and kept the filter for completeness. All generated games belong to $\mathcal{CSG}$. In particular, from Table 4a, we can see that it is not possible to generate all complete games with n players using down-$p\text{-}\mathcal{WVG}$ -expressions while, up to 7 players, it is possible with down-$\mathcal{WVG}$ -expressions. As we have mentioned before, for up

Table 4 Counting simple games with g-mdim $(\Gamma, \text{down-}p\text{-}\mathcal{WVG}) = m$ and g-mdim $(\Gamma, \text{down-}\mathcal{WVG}) = m$

(a) Number of games $\Gamma \in \mathcal{SG}$ with g-mdim $(\Gamma, \text{down-}p\text{-}\mathcal{WVG}) = m$, for the different values of m, up to isomorphism

m	Number of players						
	1	2	3	4	5	6	7
1	1	2	5	17	92	994	28,262
2	0	0	0	2	15	162	16,030
p-$\mathcal{CSG}$	1	2	5	17	92	1054	43,142
$\mathcal{CSG}\backslash p$-$\mathcal{CSG}$	0	0	0	2	15	2	1150
Total	1	2	5	19	107	1156	44,292
p-$\mathcal{CSG}$	1	2	5	17	92	1054	43,142
$\mathcal{CSG}$	1	3	8	25	117	1171	44,313
p-$\mathcal{SG}$	1	2	5	20	180	16,143	489,996,795

(b) Number of games $\Gamma \in \mathcal{SG}$ with g-mdim $(\Gamma, \text{down-}\mathcal{WVG}) = m$, for the different values of m, up to isomorphism

m	Number of players						
	1	2	3	4	5	6	7
1	1	3	8	25	117	1111	29,373
2	0	0	0	0	0	60	14,940
p-$\mathcal{CSG}$	1	2	5	17	92	1054	43,142
$\mathcal{CSG}\backslash p$-$\mathcal{CSG}$	0	1	3	8	25	117	1171
Total	1	3	8	25	117	1171	44,313
p-$\mathcal{CSG}$	1	2	5	17	92	1054	43,142
$\mathcal{CSG}$	1	3	8	25	117	1171	44,313
p-$\mathcal{SG}$	1	2	5	20	180	16,143	489,996,795

to 7 players, we can generate all p-$\mathcal{CSG}$. Now we can see that some, but not all, games in $\mathcal{CSG}\backslash p$-$\mathcal{CSG}$ can be generated with down-p-$\mathcal{WVG}$ -expressions. For example, for $n = 6$, we obtained only 2 of the 25 complete games that are not pure.

Our results about generalized dimension and multidimension of games in down-up-p-$\mathcal{WVG}$ are presented in Table 5. Molinero et al. (2023) shows that $\Gamma = (N, \mathcal{W}^m)$ with $\mathcal{W}^m = \{\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 3, 5, 6\}, \{2, 3, 5, 6\}, \{1, 4, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}\}$ verifies that g-dim $(\Gamma, p\text{-}\mathcal{WVG}) = \infty$. However, $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = (N, \mathcal{W}_1^m)$ with $\mathcal{W}_1^m = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 6\}, \{1, 2, 4, 6\}, \{1, 3, 4, 6\}, \{2, 3, 4, 6\}, \{1, 2, 5, 6\}, \{1, 3, 5, 6\}, \{2, 3, 5, 6\}, \{1, 4, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}\}$ has a representation $[4; 1, 1, 1, 1, 1, 1]$ and $\Gamma_2 = (N, \mathcal{W}_2^m)$ with $\mathcal{W}_2^m = \{\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 3, 5, 6\}, \{2, 3, 5, 6\}, \{1, 4, 5, 6\}, \{2, 4, 5, 6\}\}$ has a representation $[8; 4, 4, 2, 2, 1, 1]$. Thus, g-mdim $(\Gamma, p\text{-}\mathcal{WVG}) = 2$. That is why Table 5a shows that only 1053 pure complete games for 5 players can be obtained with intersections, but Table 5b shows that all 1054 p-$\mathcal{CSG}$ can be obtained with down-up-p-$\mathcal{WVG}$ -expressions.

Table 6 shows the experimental results about multidimension of $\mathcal{SG}$ with respect to the subclasses down-up-p-$\mathcal{WVG}$ and down-up-$\mathcal{WVG}$. As expected, we can see that neither down-up-p-$\mathcal{WVG}$ -expressions nor down-up-$\mathcal{WVG}$ -expressions can generate all simple games. Comparing Tables 6a and 4a, it is clear that even down-up-p-$\mathcal{WVG}$ -expression generate a much bigger quantity of games than down-p-$\mathcal{WVG}$ -expression.

Table 5 Counting pure simplegames with g-dim $(\Gamma, \text{down-up-}p\text{-}\mathcal{WVG}) = d$ *and* g-mdim $(\Gamma, \text{down-}p\text{-}\mathcal{WVG}) = m$

(a) Number of games $\Gamma \in p\text{-}\mathcal{SG}$ with g-dim $(\Gamma, \text{down-up-}p\text{-}\mathcal{WVG}) = d$, for the different values of d, up to isomorphism

d	Number of players					
	1	2	3	4	5	6
1	1	2	5	17	92	994
2	0	0	0	3	66	3403
3	0	0	0	0	2	118
4	0	0	0	0	0	0
$p\text{-}\mathcal{CSG}$	1	2	5	17	92	1053
$p\text{-}\mathcal{SG}\setminus p\text{-}\mathcal{CSG}$	0	0	0	3	68	3462
Total	1	2	5	20	160	4515
$p\text{-}\mathcal{CSG}$	1	2	5	17	92	1054
$p\text{-}\mathcal{SG}$	1	2	5	20	180	16,143

(b) Number of games $\Gamma \in p\text{-}\mathcal{SG}$ with g-mdim $(\Gamma, \text{down-up-}p\text{-}\mathcal{WVG}) = m$, for the different values of m, up to isomorphism

m	Number of players					
	1	2	3	4	5	6
1	1	2	5	17	92	994
2	0	0	0	3	76	5342
3	0	0	0	0	10	6237
4	0	0	0	0	2	3273
5	0	0	0	0	0	163
6	0	0	0	0	0	69
7...12	0	0	0	0	0	0
$p\text{-}\mathcal{CSG}$	1	2	5	17	92	1054
$p\text{-}\mathcal{SG}\setminus p\text{-}\mathcal{CSG}$	0	0	0	3	88	15,024
Total	1	2	5	20	180	16,078
$p\text{-}\mathcal{CSG}$	1	2	5	17	92	1054
$p\text{-}\mathcal{SG}$	1	2	5	20	180	16,143

Note that the two games with 5 players and mutidimension 4 that appear in Table 5b also appear in Table 6b, up to isomorphism. Moreover, they both are dual one each other. On the one hand, our experiments of Table 5b give us $\Gamma_1 = ([5], \mathcal{W}_1^m = \{\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,4,5\}\})$ and $\Gamma_2 = ([5], \mathcal{W}_2^m = \{\{1,2\},\{2,3\},\{1,4\},\{3,5\},\{4,5\}\})$, where

$$\Gamma_1 = [4;2,2,1,1,1] \cap ([5;1,1,2,2,3] \cap ([8;4,3,3,2,1] \cup [8;1,2,2,3,3]))$$
and
$$\Gamma_2 = [4;2,2,1,1,1] \cup ([5;1,1,2,2,3] \cup ([6;4,3,3,2,1] \cap [4;1,2,2,3,3])).$$

On the other hand, our experiments of Table 6b generate $\overline{\Gamma}_1 = ([5], \mathcal{W}_1^m = \{\{1,2,4\},\{2,3,4\},\{1,3,5\},\{2,3,5\},\{1,4,5\}\})$ and $\overline{\Gamma}_2 = ([5], \mathcal{W}_2^m = \{\{1,2\},\{1,3\},\{3,4\},\{2,5\},$

Table 6 Counting simple games with g-mdim (Γ, down-up-p-$\mathcal{WVG}$) $=m$ *and* g-mdim (Γ, down-up-$\mathcal{WVG}$) $=m$

(a) Number of games $\Gamma \in \mathcal{SG}$ with g-mdim (Γ, down-up-p-$\mathcal{WVG}$) $= m$, for the different values of m, up to isomorphism

m	Number of players					
	1	2	3	4	5	6
1	1	2	5	17	92	994
2	0	0	2	9	99	5462
3	0	0	0	1	13	6266
4	0	0	0	0	3	3306
5	0	0	0	0	0	173
6	0	0	0	0	0	75
7...12	0	0	0	0	0	0
Total	1	2	7	27	207	16,276
$\mathcal{CSG}$	1	3	8	25	117	1171
p-$\mathcal{CSG}$	1	2	5	17	92	1054
$\mathcal{SG}$	1	3	8	28	208	16,351
p-$\mathcal{SG}$	1	2	5	20	180	16,143

(b) Number of games $\Gamma \in \mathcal{SG}$ with g-mdim (Γ, down-up-$\mathcal{WVG}$) $= m$, for the different values of m, up to isomorphism

m	Number of players					
	1	2	3	4	5	6
1	1	3	8	25	117	1111
2	0	0	0	3	82	5566
3	0	0	0	0	6	6053
4	0	0	0	0	3	3299
5	0	0	0	0	0	173
6	0	0	0	0	0	75
7...12	0	0	0	0	0	0
Total	1	3	8	28	208	16,277
$\mathcal{CSG}$	1	3	8	25	117	1171
p-$\mathcal{CSG}$	1	2	5	17	92	1054
$\mathcal{SG}$	1	3	8	28	208	16,351
p-$\mathcal{SG}$	1	2	5	20	180	16,143

{4, 5}}), where

$$\overline{\Gamma}_1 = [1; 0, 0, 0, 1, 1] \cap ([5; 3, 2, 2, 1, 1] \cap ([3; 0, 0, 1, 1, 2] \cup [5; 0, 3, 2, 2, 1)]))$$

and

$$\overline{\Gamma}_2 = [2; 0, 0, 0, 1, 1] \cup ([5; 3, 2, 2, 1, 1] \cup ([2; 0, 0, 1, 1, 2] \cap [4; 0, 3, 2, 2, 1)])).$$

Observe that $\Gamma_1 = \sigma(\overline{\Gamma}_1)$ and $\Gamma_2 = \sigma(\overline{\Gamma}_2)$, where $\sigma = (1\ 4\ 2\ 3\ 5)$.

Our last experiment analyzes the generalized multidimension over $\mathcal{S}$-$\mathcal{SG}$. Remind that Theorem 5 shows us $\mathcal{SG}_{\mathcal{E}}(\mathcal{S}$-$\mathcal{SG})$ is the set of all simple games. Now, Table 7 enumerates the generalized multidimension of all simple games from unions and intersections of $\mathcal{S}$-$\mathcal{SG}$, up to 6 players.

Table 7 Number of games $\Gamma \in p\text{-}\mathcal{SG}$ with $\text{g-mdim}\left(\Gamma, \bigcup_{j\in N} \Gamma^{(j)}\right) = m$, for values of m, up to isomorphism

m	Number of players					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	2	2	2	2	2
3	0	0	4	4	4	4
4	0	0	0	10	10	10
5	0	0	1	2	26	26
6	0	0	0	6	16	82
7	0	0	0	1	42	107
8	0	0	0	2	35	326
9	0	0	0	0	44	613
10	0	0	0	0	18	1258
11	0	0	0	0	3	2078
12	0	0	0	0	6	2902
13	0	0	0	0	0	3285
14	0	0	0	0	1	2878
15	0	0	0	0	0	1780
16	0	0	0	0	0	786
17	0	0	0	0	0	172
18	0	0	0	0	0	38
19	0	0	0	0	0	3
Total	1	3	8	28	208	16,351
$\mathcal{CSG}$	1	3	8	25	117	1171
$p\text{-}\mathcal{CSG}$	1	2	5	17	92	1054
$\mathcal{SG}$	1	3	8	28	208	16,351
$p\text{-}\mathcal{SG}$	1	2	5	20	180	16,143

Our new experiments for generalized multidimension, and the reproduced experiments according to Molinero et al. (2023) for generalized dimension and generalized codimension show us that, even for 3 players, there exists $\Gamma \in \mathcal{SG}$ such that

$$\text{g-mdim}\left(\Gamma, \bigcup_{j\in N} \Gamma_N^{(j)}\right) < \min\left\{\text{g-dim}\left(\Gamma, \bigcup_{j\in N} \Gamma_N^{(j)}\right), \text{g-codim}\left(\Gamma, \bigcup_{j\in N} \Gamma_N^{(j)}\right)\right\}.$$

Example 9 For 3 players, $\Gamma = ([1; 1, 0, 0] \cap [1; 0, 1, 0]) \cup ([1; 0, 0, 1] \cap ([1; 1, 0, 0] \cup [1; 0, 1, 0]))$ really verifies $\text{g-mdim}\left(\Gamma, \bigcup_{j\in N} \Gamma_N^{(j)}\right) = 5$, but $\text{g-dim}\left(\Gamma, \bigcup_{j\in N} \Gamma_N^{(j)}\right) = \text{g-codim}\left(\Gamma, \bigcup_{j\in N} \Gamma_N^{(j)}\right) = 6$.

On the other hand, for $N = [6]$ players,

$$\begin{aligned}\Gamma' = &\left(\left(\Gamma_N^{(1)} \cup \Gamma_N^{(3)}\right) \cap \left(\Gamma_N^{(2)} \cup \Gamma_N^{(6)}\right) \cap \left(\Gamma_N^{(4)} \cup \Gamma_N^{(5)}\right)\right) \cup \\ &\left(\left(\Gamma_N^{(1)} \cup \Gamma_N^{(5)}\right) \cap \left(\Gamma_N^{(2)} \cup \Gamma_N^{(3)}\right) \cap \left(\Gamma_N^{(4)} \cup \Gamma_N^{(6)}\right)\right) \cup \\ &\left(\left(\Gamma_N^{(1)} \cup \Gamma_N^{(6)}\right) \cap \left(\Gamma_N^{(2)} \cup \Gamma_N^{(5)}\right) \cap \left(\Gamma_N^{(3)} \cup \Gamma_N^{(4)}\right)\right)\end{aligned}$$

verifies g-mdim $\left(\Gamma', \bigcup_{j\in N} \Gamma_N^{(j)}\right) = 18$. To give a brief idea how our experiments work, we show the output of our program with all specific information for Γ' using a structure that indents the level of the expression:

```
               (1; 1 0 0 0 0 0) 1
          (UNION) 1 3
               (1; 0 0 1 0 0 0) 3
     (INTERSECTION) 124 234 125 235 146 346 156 356
                    (1; 0 0 0 0 1 0) 5
               (UNION) 4 5
                    (1; 0 0 0 1 0 0) 4
          (INTERSECTION) 24 25 46 56
                    (1; 0 0 0 0 0 1) 6
               (UNION) 2 6
                    (1; 0 1 0 0 0 0) 2
(UNION) 123 124 134 234 125 135 235 14  5 245 345 126 136 236 146 246 346 156 256 356 456
                    (1; 1 0 0 0 0 0) 1
               (UNION) 1 5
                    (1; 0 0 0 0 1 0) 5
          (INTERSECTION) 124 134 245 345 126 136 256 356
                         (1; 0 0 0 0 0 1) 6
                    (UNION) 4 6
                         (1; 0 0 0 1 0 0) 4
               (INTERSECTION) 24 34 26 36
                         (1; 0 1 0 0 0 0) 2
                    (UNION) 2 3
                         (1; 0 0 1 0 0 0) 3
     (UNION) 123 124 134 135 145 245 345 126 136 236 246 256 356 456
                    (1; 1 0 0 0 0 0) 1
               (UNION) 1 6
                    (1; 0 0 0 0 0 1) 6
          (INTERSECTION) 123 124 135 145 236 246 356 456
                         (1; 0 0 0 0 1 0) 5
                    (UNION) 2 5
                         (1; 0 1 0 0 0 0) 2
               (INTERSECTION) 23 24 35 45
                         (1; 0 0 0 1 0 0) 4
                    (UNION) 3 4
                         (1; 0 0 1 0 0 0) 3
```

Note that the elements to the right of each row give the minimal wining coalitions of the game. For instance, `24 25 46 56` at the 8th row indicates that the set of minimal winning coalitions of $[1; 0, 0, 0, 0, 1, 0] \cap [1; 0, 0, 0, 0, 0, 1]$ is $\mathcal{W}^m = \{\{2, 4\}, \{2, 5\}, \{4, 6\}, \{5, 6\}\}$.

Reproducing the experiments of Molinero et al. (2023), we obtain that, even though g-mdim $\left(\Gamma', \bigcup_{j\in N} \Gamma_N^{(j)}\right) = 18$, g-dim $\left(\Gamma', \bigcup_{j\in N} \Gamma_N^{(j)}\right) =$ g-codim $\left(\Gamma', \bigcup_{j\in N} \Gamma_N^{(j)}\right) = 60$.

In Appendix A, we show the output of some of our experiments. There we give the games with maximum generalized multidimension over $\mathcal{S}$-$\mathcal{SG}$ and the corresponding $\mathcal{G}$-expression certifying this fact, from three to six players.

6 Conclusions and future work

In this work, we consider subclasses of simple games generated from $\mathcal{G}$-expressions, using the operators intersection and union, over restricted subclasses of weighted voting games. Using those expressions, we have introduced the multidimension as an extension of the notions of dimension and codimension of simple games. We have generalized this notion with respect to a subfamily of $\mathcal{WVG}$, namely, the generalized multidimension. Most of the considered subclasses of $\mathcal{WVG}$ are formed by games without dummy players (pure weighted voting games) and are obtained by selecting particular types of representations of $\mathcal{WVG}$. Thus, our work extends the theoretical and experimental results of Molinero et al. (2023).

A theoretically relevant type of game is the singleton game, made up of exactly one minimal winning coalition formed by a single player. Singleton games are the unique games having one dictator while all the other players are dummies. According to Theorem 5, a class able to generate all the singleton games, through union and intersection operations, generates all the simple games. Furthermore, if we only have pure weighted voting games, that is, games without dummies, the only games we cannot generate (through union and/or intersection operations) are the singleton games (see Theorem 6). Therefore, any other game than a singleton game will have finite generalized multidimension with respect to pure weighted voting games. From the point of view of voting systems, the above means that dictators can only emerge to the extent that we assume the existence of dummies. Comparing our characterization of the simple games that can be generated with expressions on p-$\mathcal{WVG}$ with respect to the ones having finite generalized dimension or codimension provided in Molinero et al. (2023), we can observe that the games that cannot be expressed in this way are the self-dual games that do not have finite generalized dimension/codimension. Thus, singleton games are the only self-dual games that do not have finite generalized dimension/codimension with respect to p-$\mathcal{WVG}$.

Besides the above, we have proved that the multidimension has properties quite different from the dimension or the codimension. We have shown that the generalized dimension, with respect to a class closed under duality, is the same for dual games. Surprisingly, we have demonstrated the existence of gaps in the attained multidimension values. The latter has to be seen in contraposition with the continuity of the dimension/codimension values proved in Molinero et al. (2023).

Although the multidimension values are not contiguous, we have proved that having a big enough interval without games of these multidimension ensures that no game has higher multidimension. This result is the key ingredient in our enumeration algorithm. According to Theorem 8, we need a gap of half the size of the value. The gap size is relevant to programming the correct termination criterion in the enumeration algorithm. It remains an open question, both theoretically and practically, to see if this gap size can be reduced.

Thus, experimentally, we were able to calculate the multidimensions for all simple games with respect to some subclasses of p-$\mathcal{WVG}$ up to 6 players and, in some cases, up to 7 players. Our results show how generalized multidimension allows representing more simple games than considering only intersections or only unions. Moreover, the representations tend to be more compact, even for a small number of players and for subclasses with a relatively small number of games.

We can state several problems of interest. It remains open finding (if there exist) examples where the multidimension is linear on the number of players, but the dimension and the codimension are exponential on the number of players. In all our enumerations, the maximum generalized multidimension attained has been smaller than the maximum generalized dimension/codimension, except, as expected, when the base set of games is $\mathcal{S}$-$\mathcal{SG}$. It remains open to see if this property is maintained for any number of players.

Another line of future research is to explore other subclasses of simple games as, for example, proper, strong, or decisive (self-dual) weighted voting games. Simple games where weights are Fibonacci numbers (i.e., weighted voting games such that $w_{i+2} = w_{i+1} + w_i$ for $3 \leq i \leq n$ being $w_1 = a$ and $w_2 = b$) are also an interesting class to consider.

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Declarations

Conflict of interest The authors acknowledge that there is no conflict of interest in this work.

Ethical standard This article does not contain any information with human participants or animals performed by any of the authors.

Appendix A The games with maximum generalized multidimension

This appendix presents the output of our experiments related to all simple games with maximum generalized multidimension over $\mathcal{S}$-$\mathcal{SG}$, from three to six players. Note that this output gives us the corresponding $\mathcal{G}$-expression.

```
3 players
--------------------
          (1; 1 0 0) 1
     (INTERSECCIO) 12
          (1; 0 1 0) 2
(UNIO) 12 13 23
          (1; 0 0 1) 3
     (INTERSECCIO) 13 23
               (1; 1 0 0) 1
          (UNIO) 1 2
               (1; 0 1 0) 2
1 with m = 5
```

```
4 players
--------------------
          (1; 1 0 0 0) 1
     (INTERSECCIO) 12
          (1; 0 1 0 0) 2
(UNIO) 12 13 23 14 24 34
               (1; 0 0 1 0) 3
          (INTERSECCIO) 34
               (1; 0 0 0 1) 4
     (UNIO) 13 23 14 24 34
                    (1; 0 0 1 0) 3
               (UNIO) 3 4
                    (1; 0 0 0 1) 4
          (INTERSECCIO) 13 23 14 24
                    (1; 1 0 0 0) 1
               (UNIO) 1 2
                    (1; 0 1 0 0) 2
****
          (1; 1 0 0 0) 1
     (UNIO) 1 2
          (1; 0 1 0 0) 2
(INTERSECCIO) 123 124 134 234
               (1; 0 0 1 0) 3
          (INTERSECCIO) 34
               (1; 0 0 0 1) 4
     (UNIO) 123 124 34
               (1; 1 0 0 0) 1
          (INTERSECCIO) 123 124
                    (1; 0 1 0 0) 2
               (INTERSECCIO) 23 24
                         (1; 0 0 1 0) 3
                    (UNIO) 3 4
                         (1; 0 0 0 1) 4
2 with m = 8
```

```
5 players
--------------------
          (1; 1 0 0 0 0) 1
     (INTERSECCIO) 123 145
                    (1; 0 1 0 0 0) 2
               (INTERSECCIO) 23
                    (1; 0 0 1 0 0) 3
          (UNIO) 23 45
                    (1; 0 0 0 1 0) 4
               (INTERSECCIO) 45
                    (1; 0 0 0 0 1) 5
(UNIO) 123 124 134 234 125 135 235 145 245 345
               (1; 0 1 0 0 0) 2
          (UNIO) 2 3
               (1; 0 0 1 0 0) 3
     (INTERSECCIO) 124 134 234 125 135 235 245 345
                    (1; 0 0 0 1 0) 4
               (INTERSECCIO) 45
                    (1; 0 0 0 0 1) 5
          (UNIO) 14 234 15 235 45
                         (1; 0 0 0 1 0) 4
                    (UNIO) 4 5
                         (1; 0 0 0 0 1) 5
               (INTERSECCIO) 14 234 15 235
                         (1; 1 0 0 0 0) 1
                    (UNIO) 1 23
                              (1; 0 1 0 0 0) 2
                         (INTERSECCIO) 23
                              (1; 0 0 1 0 0) 3
1 with m = 14
```

```
6 players
--------------------
```

```
          (1; 1 0 0 0 0 0) 1
     (INTERSECCIO) 123
               (1; 0 1 0 0 0 0) 2
          (INTERSECCIO) 23
               (1; 0 0 1 0 0 0) 3
(UNIO) 123 1245 1345 2345 1246 1346 2346 1256 1356 2356 1456 2456 3456
               (1; 0 0 0 1 0 0) 4
          (UNIO) 4 5
               (1; 0 0 0 0 1 0) 5
     (INTERSECCIO) 1245 1345 2345 1246 1346 2346 1256 1356 2356 1456 2456 3456
                    (1; 0 0 0 0 0 1) 6
               (UNIO) 45 6
                         (1; 0 0 0 1 0 0) 4
                    (INTERSECCIO) 45
                         (1; 0 0 0 0 1 0) 5
          (INTERSECCIO) 1245 1345 2345 126 136 236 1456 2456 3456
                         (1; 1 0 0 0 0 0) 1
                    (INTERSECCIO) 12
                         (1; 0 1 0 0 0 0) 2
               (UNIO) 12 13 23 1456 2456 3456
                              (1; 0 0 1 0 0 0) 3
                         (INTERSECCIO) 13 23
                                   (1; 1 0 0 0 0 0) 1
                              (UNIO) 1 2
                                   (1; 0 1 0 0 0 0) 2
                    (UNIO) 13 23 1456 2456 3456
                              (1; 0 0 0 0 0 1) 6
                         (INTERSECCIO) 1456 2456 3456
                                   (1; 0 0 0 1 0 0) 4
                              (INTERSECCIO) 145 245 345
                                        (1; 0 0 0 0 1 0) 5
                                   (INTERSECCIO) 15 25 35
                                             (1; 0 0 1 0 0 0) 3
                                        (UNIO) 1 2 3
                                                  (1; 1 0 0 0 0 0) 1
                                             (UNIO) 1 2
                                                  (1; 0 1 0 0 0 0) 2

****
          (1; 1 0 0 0 0 0) 1
     (INTERSECCIO) 123
               (1; 0 1 0 0 0 0) 2
          (INTERSECCIO) 23
               (1; 0 0 1 0 0 0) 3
(UNIO) 123 124 134 234 125 135 235 145 245 345 126 136 236 146 246 346 156 256 356
               (1; 1 0 0 0 0 0) 1
          (UNIO) 1 2 3
                    (1; 0 1 0 0 0 0) 2
               (UNIO) 2 3
                    (1; 0 0 1 0 0 0) 3
     (INTERSECCIO) 124 134 234 125 135 235 145 245 345 126 136 236 146 246 346 156 256 356
                    (1; 0 0 0 1 0 0) 4
               (INTERSECCIO) 45
                    (1; 0 0 0 0 1 0) 5
          (UNIO) 124 134 234 125 135 235 45 126 136 236 46 56
                         (1; 0 0 0 0 0 1) 6
                    (INTERSECCIO) 46 56
                              (1; 0 0 0 1 0 0) 4
                         (UNIO) 4 5
                              (1; 0 0 0 0 1 0) 5
               (UNIO) 124 134 234 125 135 235 126 136 236 46 56
                              (1; 1 0 0 0 0 0) 1
                         (UNIO) 1 2
                              (1; 0 1 0 0 0 0) 2
                    (INTERSECCIO) 124 134 234 125 135 235 126 136 236
                                   (1; 0 0 1 0 0 0) 3
                              (UNIO) 12 3
                                        (1; 1 0 0 0 0 0) 1
```

```
                                   (INTERSECCIO) 12
                                        (1; 0 1 0 0 0 0) 2
                         (INTERSECCIO) 124 34 125 35 126 36
                                   (1; 0 0 0 0 0 1) 6
                              (UNIO) 4 5 6
                                        (1; 0 0 0 1 0 0) 4
                                   (UNIO) 4 5
                                        (1; 0 0 0 0 1 0) 5

****
          (1; 1 0 0 0 0 0) 1
     (UNIO) 1 2 3 45
               (1; 0 1 0 0 0 0) 2
          (UNIO) 2 3 45
                    (1; 0 0 1 0 0 0) 3
               (UNIO) 3 45
                         (1; 0 0 0 1 0 0) 4
                    (INTERSECCIO) 45
                         (1; 0 0 0 0 1 0) 5
(INTERSECCIO) 1234 1235 1245 1345 2345 126 136 236 146 246 346 156 256 356 456
                    (1; 0 0 0 1 0 0) 4
               (UNIO) 4 5
                    (1; 0 0 0 0 1 0) 5
          (INTERSECCIO) 1234 1235 46 56
                    (1; 0 0 0 0 0 1) 6
               (UNIO) 123 6
                         (1; 1 0 0 0 0 0) 1
                    (INTERSECCIO) 123
                              (1; 0 1 0 0 0 0) 2
                         (INTERSECCIO) 23
                              (1; 0 0 1 0 0 0) 3
     (UNIO) 1234 1235 1245 1345 2345 126 136 236 46 56
                    (1; 1 0 0 0 0 0) 1
               (UNIO) 1 2
                    (1; 0 1 0 0 0 0) 2
          (INTERSECCIO) 1245 1345 2345 126 136 236
                         (1; 0 0 1 0 0 0) 3
                    (UNIO) 12 3
                              (1; 1 0 0 0 0 0) 1
                         (INTERSECCIO) 12
                              (1; 0 1 0 0 0 0) 2
               (INTERSECCIO) 1245 345 126 36
                         (1; 0 0 0 0 0 1) 6
                    (UNIO) 45 6
                              (1; 0 0 0 1 0 0) 4
                         (INTERSECCIO) 45
                              (1; 0 0 0 0 1 0) 5
3 with m = 19
```

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