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The *M*-matrix group inverse problem for distance-biregular graphs

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Abstract
In this work, we obtain the group inverse of the combinatorial Laplacian matrix of distance-biregular graphs. This expression can be obtained trough the so-called equilibrium measures for sets obtained by deleting a vertex. Moreover, we show that the two equilibrium arrays characterizing distance-biregular graphs can be expressed in terms of the mentioned equilibrium measures. As a consequence of the minimum principle, we provide a characterization of when the group inverse of the combinatorial Laplacian matrix of a distance-biregular graph is an M-matrix.

Keywords Distance-biregular graph · Equilibrium measure · Group inverse · Combinatorial Laplacian · M-matrix

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1 Introduction

One problem with the theory of distance-regular graphs is that it does not apply directly to the graphs of generalised polygons. Godsil and Shawe-Taylor (1987), overcame this difficulty by introducing the class of distance-regularised graphs, a natural common generalisation. These graphs are shown to either be distance-regular or distance-biregular. This family includes the generalised polygons and other interesting graphs. Distance-biregular graphs, which were introduced by Delorme et al. (1983) in 1983, can be viewed as a bipartite variant of distance-regular graph: the graphs are bipartite and for each vertex there exists an intersection array depending on the stable component of the vertex. Thus such graphs are to distance-regular graphs as bipartite regular graphs are to regular graphs. They also are to non-symmetric association schemes as distance-regular graphs are to symmetric association schemes. Since their introduction, distance-biregular graphs have received quite some attention, see (van den Akker 1990; Curtin 1999a, b; Delorme 1994; Fiol 2013; Howlader and Panigrahi 2022; Mohar and Shawe-Taylor 1985) or (Brouwer et al. 1989, Chapter 4) for an overview.

In the first part of this paper we compute the group inverse of the combinatorial Laplacian matrix of distance-biregular graphs. The group inverse matrix can be seen in the framework of discrete potential theory as the Green's functions associated with the Laplacian operator and it can be used to deal with diffusion-type problems on graphs, such as chip-firing, load balancing, and discrete Markov chains. For some graph classes, the group inverse is known. Instances of it are the work of Urakawa (1997), Bendito et al. (2000, 2010) or more recently the study of the Green function for forests by Chung and Zeng (yyy). Other generalized inverses, such as the Moore–Penrose inverse, have been studied. For instance, the Moore–Penrose inverse of the incidence matrix of several graphs has been investigated by Azimi and Bapat (2019, 2018) and Azimi et al. (2019). Nevertheless, the problem of computing group inverses still remains wide open for most graph classes. In the first part of this paper we obtain an explicit expression for the group inverse of the combinatorial Laplacian matrix of a distance-biregular graph in terms of its intersection numbers. This result, together with the group inverse of a distance-regular graph found by Bendito et al. (2010), and independently, by Chung and Yau (2000), completes the investigation for distance-regularised graphs.

In matrix theory, the Laplacian matrix is known to be a symmetric M-matrix (a symmetric positive semi-definite matrix with non-positive off-diagonal elements). Nonnegative matrices and M-matrices have become a staple in contemporary linear algebra, and they arise frequently in its applications. Such matrices are encountered not only in matrix analysis, but also in stochastic processes, graph theory, electrical networks, and demographic models (Kirkland and Neumann 1995). A fundamental problem related with M-matrices is the so-called inverse M-matrix problem, that consists in characterizing all nonnegative matrices whose inverses are M-matrices. For singular matrices, the inverse problem was originally posed by Neumann, Poole and Werner as follows.

Question 1 (Deutsch and Neumann 1984; Neumann et al. 1982; Kirkland and Neumann 2013, Question 3.3.8) *Characterize all singular and irreducible M-matrices for which its group inverse is also an M-matrix.*

This is an open problem that has been solved for some few family of matrices. In the graph setting, this question has been answered for weighted trees by Kirkland and Neumann (1998), and for distance-regular graphs by Bendito et al. (2012). In a more general setting, Question 1 has been investigated for nonnegative matrices having few eigenvalues by Kirkland and Neumann (1995), for periodic and nonperiodic Jacobi matrices by Chen et al. (1995) and for general symmetric M-matrices whose underlying graphs are paths by Bendito et al. (2012)

and Carmona et al. (2013). Recently, matrices whose group inverses are M-matrices were investigated by Kalauch et al. (2021).

We answer Question 1 for distance-biregular graphs, completing, together with the known results for distance-regular graphs (Bendito et al. 2012), the characterization of when the group inverse of the combinatorial Laplacian matrix of a distance-regularised graph is an M-matrix.

2 Preliminaries

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops or multiple edges, with vertex set V, whose cardinality equals $n \geq 2$, and edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. The conductance can be considered as a symmetric function $c\colon V \times V \longrightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and moreover, $x \sim y$, that is vertex x is adjacent to vertex y, iff $c(x, y) > 0$. We define the *degree function* k as

$$k(x) = \sum_{y \in V} c(x, y)$$

for each $x \in V$. The usual distance from vertex x to vertex y is denoted by $d(x, y)$ and $D = \max\{d(x, y) : x, y \in V\}$ stands for the *diameter* of Γ. We denote as $\Gamma_i(x)$ the set of vertices at distance i from vertex x, $\Gamma_i(x) = \{y : d(x, y) = i\}$, $0 \leq i \leq D$ and define $k_i(x) = \big|\Gamma_i(x)\big|$. Then,

$$B_i(x) = \sum_{j=0}^{i} k_j(x)$$

is the cardinal of the i-ball centered at x. The *complement* of Γ is defined as the graph $\overline{\Gamma}$ on the same vertices such that two vertices are adjacent iff they are not adjacent in Γ; that is $x \sim y$ in $\overline{\Gamma}$ iff $c(x, y) = 0$. More generally, for any $i = 1, \ldots, D$, we denote by Γ_i the graph whose vertices are those of Γ and in which two vertices are adjacent iff they are at distance i in Γ. Therefore for any $x \in V$, $\Gamma_i(x)$ is the set of adjacent vertices to x in Γ_i. Clearly Γ_1 is the graph subjacent to the network Γ and $\Gamma_2 = \overline{\Gamma}$ when $D = 2$.

The set of real-valued functions on V is denoted by $\mathcal{C}(V)$. When necessary, we identify the functions in $\mathcal{C}(V)$ with vectors in $\mathbb{R}^{|V|}$ and the endomorphisms of $\mathcal{C}(V)$ with $|V|$-order square matrices.

The *combinatorial Laplacian* or simply the *Laplacian* of the graph Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y)\Big(u(x) - u(y)\Big) = k(x)u(x) - \sum_{y \in V} c(x, y)\, u(y), \quad x \in V. \tag{1}$$

It is well-known that $\mathcal{L}$ is a positive semi-definite self-adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. So, $\mathcal{L}$ can be interpreted as an irreducible, symmetric, diagonally dominant and singular M-matrix, that in the sequel will be denoted as L. Therefore, the Poisson equation $\mathcal{L}(u) = f$ on V has solution iff

$$\sum_{x \in V} f(x) = 0$$

and, when this happens, there exists a unique solution $u \in \mathcal{C}(V)$ such that $\sum\limits_{x \in V} u(x) = 0$, see (Bendito et al. 2000).

The *Green operator* is the linear operator $\mathcal{G} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to any $f \in \mathcal{C}(V)$ the unique solution of the Poisson equation $\mathcal{L}(u) = f - \frac{1}{n} \sum\limits_{x \in V} f(x)$ such that $\sum\limits_{x \in V} u(x) = 0$. It is easy to prove that $\mathcal{G}$ is a positive semi-definite self-adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. Moreover, if $\mathcal{P}$ denotes the projection on the subspace of constant functions then,

$$\mathcal{L} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{L} = \mathcal{I} - \mathcal{P}.$$

In addition, we define the *Green function* as $G : V \times V \longrightarrow \mathbb{R}$ given by $G(x, y) = \mathcal{G}(\varepsilon_y)(x)$, where ε_y stands for the Dirac function at y. Therefore, interpreting $\mathcal{G}$, or G, as a matrix it is nothing else but $L^{\#}$ the group inverse inverse of L, that coincides with its Moore–Penrose inverse. In consequence, $L^{\#}$ is a *M-matrix* iff $L^{\#}(x, y) \leq 0$ for any $x, y \in V$ with $x \neq y$ and then $L^{\#}$ can be identified with the combinatorial Laplacian matrix of a new connected network with the same vertex set, that we denote by $\Gamma^{\#}$.

From now on we will say that a network Γ has the *M-property* iff $L^{\#}$ is an M-matrix; that is, if L provides an answer to Question 1.

In Bendito et al. (2000) it was proved that for any $y \in V$, there exists a unique $\nu^y \in \mathcal{C}(V)$ such that $\nu^y(y) = 0$, $\nu^y(x) > 0$ for any $x \neq y$ and satisfying

$$\mathcal{L}(\nu^y) = 1 - n\varepsilon_y \quad \text{on } V. \tag{2}$$

We call ν^y the *equilibrium measure of* $V \setminus \{y\}$ and then we define *capacity* as the function $cap \in \mathcal{C}(V)$ given by $cap(y) = \sum\limits_{x \in V} \nu^y(x)$.

Following the ideas in Bendito et al. (2000, 2012) and Urakawa (1997), we define, for any $y \in V$, the *equilibrium array for* y as the set $\{\nu^y(x) : x \in V\}$ of different values taken by the equilibrium measure of y, and we consider the *length of the equilibrium array* to be $\ell(y) = \left|\left\{\nu^y(x) : x \in V\backslash\{y\}\right\}\right|$. Since Γ is connected and $n \geq 2$, we obtain that $\ell(y) \geq 1$ for any $y \in V$. On the other hand, since $0 = \nu^y(y)$ we obtain that $\{\nu^y(x) : x \in V\} = \{q_i(y) : i = 0, \dots, \ell(y)\}$, where $0 = q_0(y) < q_1(y) < \cdots < q_{\ell(y)}(y)$. In addition, given $y \in V$ for any $i = 0, \dots, \ell(y)$, we define $m_i(y) = \left|\{x \in V : \nu^y(x) = q_i(y)\}\right|$. Clearly, for any $y \in V$ we have that

$$m_0(y) = 1, \quad n = \sum_{i=0}^{\ell(y)} m_i(y), \quad \text{and } cap(y) = \sum_{i=1}^{\ell(y)} m_i(y) q_i(y).$$

In (Bendito et al. 2000, Proposition 3.12) it was shown that, for any $y \in V$, the equilibrium measure (and hence the equilibrium array) reflects the graph depth from y, since

$$\nu^y(x) = q_i(y) \Longrightarrow d(x, y) \leq i \tag{3}$$

and hence

$$\sum_{j=0}^{i} m_j(y) \leq B_i(y)$$

for any $0 \leq i \leq D$. In particular, (3) implies that if $\nu^y(x) = q_1(y)$ then $x \sim y$; that is, that the minimum values of the equilibrium measure for y are attained at vertices adjacent to y (in fact this a formulation of the so-called *minimum principle*).

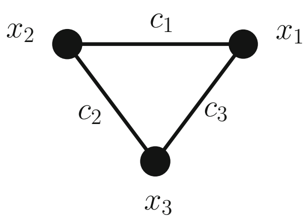

Fig. 1 Complete graph K_3

In general, when $d(x, y) = i$, Property (3) only assures that $\nu^y(x) \geq q_i(y)$, but the inequality can be strict. In particular the length of some equilibrium arrays could be greater than D.

Example 2 To illustrate the above statements, consider the complete graph K_3 with vertex set $V = \{x_1, x_2, x_3\}$ and conductances $c_1 = c(x_1, x_2)$, $c_2 = c(x_2, x_3)$ and $c_3 = c(x_3, x_1)$, see Fig. 1.

Then, keeping in mind that the Laplacian matrix is

$$L = \begin{pmatrix} c_1 + c_3 & -c_1 & -c_3 \\ -c_1 & c_1 + c_2 & -c_2 \\ -c_3 & -c_2 & c_2 + c_3 \end{pmatrix} \tag{4}$$

we find that

$$\begin{aligned}
\nu^{x_1}(x_2) &= \frac{2c_2 + c_3}{c_1c_2 + c_2c_3 + c_3c_1} & \nu^{x_1}(x_3) &= \frac{2c_2 + c_1}{c_1c_2 + c_2c_3 + c_3c_1} & cap(x_1) &= \frac{4c_2 + c_1 + c_3}{c_1c_2 + c_2c_3 + c_3c_1} \\
\nu^{x_2}(x_1) &= \frac{2c_3 + c_2}{c_1c_2 + c_2c_3 + c_3c_1} & \nu^{x_2}(x_3) &= \frac{2c_3 + c_1}{c_1c_2 + c_2c_3 + c_3c_1} & cap(x_2) &= \frac{4c_3 + c_1 + c_2}{c_1c_2 + c_2c_3 + c_3c_1} \\
\nu^{x_3}(x_1) &= \frac{2c_1 + c_2}{c_1c_2 + c_2c_3 + c_3c_1} & \nu^{x_3}(x_2) &= \frac{2c_1 + c_3}{c_1c_2 + c_2c_3 + c_3c_1} & cap(x_3) &= \frac{4c_1 + c_2 + c_3}{c_1c_2 + c_2c_3 + c_3c_1}.
\end{aligned}$$

So, $D = 1$, but $\ell(x_1) = 1$ iff $c_1 = c_3$, $\ell(x_2) = 1$ iff $c_1 = c_2$ and $\ell(x_3) = 1$ iff $c_2 = c_3$.

The group inverse of the Laplacian matrix and the equilibrium measures provide an equivalent information about the network structure, since the expression of $L^\#$ can be obtained from equilibrium measures and conversely. Specifically, see (Bendito et al. 2000, Proposition 3.9), the group inverse $L^\#$ is given by

$$L^\#(x, y) = \frac{1}{n^2}\big(cap(y) - n\,\nu^y(x)\big) \tag{5}$$

and this equality also implies that $cap(y) = n^2 L^\#(y, y)$ and that

$$\nu^y(x) = n\left(L^\#(y, y) - L^\#(x, y)\right), \quad x, y \in V. \tag{6}$$

Applying the above expressions for our small example, we will get that

$$L^\# = \frac{1}{9(c_1c_2 + c_2c_3 + c_3c_1)} \begin{pmatrix} 4c_3 + c_1 + c_2 & -2(c_2 + c_3) + c_1 & -2(c_1 + c_2) + c_3 \\ -2(c_2 + c_3) + c_1 & 4c_2 + c_1 + c_3 & -2(c_1 + c_3) + c_2 \\ -2(c_1 + c_2) + c_3 & -2(c_1 + c_3) + c_2 & 4c_1 + c_2 + c_3 \end{pmatrix}.$$

In addition, the symmetry of the group inverse leads to the following relation for the equilibrium measures

$$\nu^y(x) - \nu^x(y) = \frac{1}{n}\big(cap(y) - cap(x)\big) = n\left(L^\#(y, y) - L^\#(x, x)\right), \quad x, y \in V. \tag{7}$$

From (5) the minimum principle states that a network Γ has the M-property iff for any $y \in V$

$$cap(y) \leq n\nu^y(x) \quad \text{for any } x \sim y, \tag{8}$$

see (Bendito et al. 2012, Theorem 1). In this case, $\overline{\Gamma}$ is a subgraph of the subjacent graph of $\Gamma^{\#}$. In fact, to achieve the M-property it is sufficient to satisfy that

$$\sum_{i=1}^{\ell(y)} m_i(y) q_i(y) \leq n q_1(y)$$

for any $y \in V$. Since this inequality trivially holds when $\ell(y) = 1$, and assuming the common agreement that empty sum equals 0, we have that Γ has the M-property iff

$$\sum_{i=2}^{\ell(y)} m_i(y)\big(q_i(y) - q_1(y)\big) \leq q_1(y) \tag{9}$$

for any $y \in V$. Therefore, when $\ell(y) = 1$ for any $y \in V$, then Γ is a complete network and moreover satisfies the M-property. As Example 2 shows, a complete network does not necessarily satisfy the M-property: K_3 has the M-property if and only if $3\max\{c_1, c_2, c_3\} \leq 2(c_1 + c_2 + c_3)$. In particular, if $c_1 = c_2 = c_3$, then K_3 has the M-property, but if for instance $c_3 > 2(c_1 + c_2)$, then K_3 does not satisfy the M-property.

3 Group inverse for distance-biregular graphs

We say that the graph $\Gamma = (V, E)$ is *semiregular* if Γ is bipartite with $V = V_0 \cup V_1$, where the degree of each vertex in V_0 and the degree of each vertex in V_1 are (possibly different) constants. Hereinafter, we denote these two constants as the numbers k_0 and k_1 such that each vertex in V_0 has k_0 neighbors and each vertex in V_1 has k_1 neighbors. In this case, we define $D_\ell = \max\{d(x, y) : \ y \in V, x \in V_\ell\}$, $\ell = 0, 1$. Moreover, for $\ell = 0, 1$, we denote by $\bar{\ell} = 1 - \ell$. In the sequel without loss of generality we always suppose that $1 \leq D_0 \leq D_1$.

A connected graph Γ is a *distance-biregular graph* if Γ is semiregular and for any two vertices x and y at distance i, the numbers $|\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ and $|\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ only depend on i and on the stable set where x is. Remember that a stable set is a subset S of V containing no edges of Γ, so in this case these sets are V_0 and V_1.

Examples of distance-biregular graphs are the subdivision graph of minimal (k, g)-cages. In particular, the subdivision graph of the Petersen graph is a distance-biregular graph, see Fig. 2. Also, any bipartite distance-regular graph is a distance-biregular graph with $k_0 = k_1$.

For $x \in V_\ell$, $\ell = 0, 1$, we define the *intersection numbers* by $c_{\ell,i} = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ and $b_{\ell,i} = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$, $i = 0, \ldots, D_\ell$, with the usual agreement $c_{\ell,0} = b_{\ell,D_\ell} = 0$. Clearly, for $\ell = 0, 1$ it is satisfied that $b_{\ell,0} = k_\ell$, $c_{\ell,1} = 1$, $b_{\ell,1} = k_{\bar{\ell}} - 1$, $\ell = 0, 1$ and more generally for any $i \in \{0, \ldots, D_\ell\}$ the following holds

$$c_{\ell,i} + b_{\ell,i} = \begin{cases} k_\ell & \text{if } i \text{ is even,} \\ k_{\bar{\ell}} & \text{if } i \text{ is odd.} \end{cases}$$

Therefore, a distance-biregular graph has a double intersection array which will be denoted by

$$\{k_\ell; c_{\ell,1}, \ldots c_{\ell,D_\ell}\}, \quad \ell = 0, 1.$$

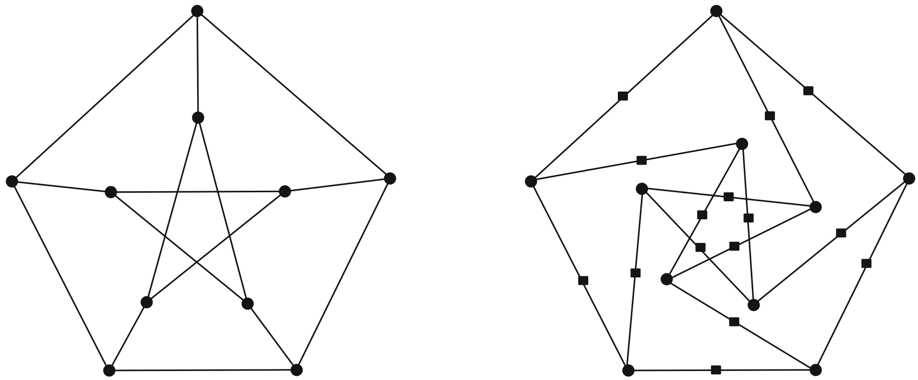

Fig. 2 Petersen graph and its subdivision graph

If, for $i \in \{0, \ldots, D_\ell\}$ and $x \in V_\ell$ we define $k_{\ell,i} = |\Gamma_i(x)|$ and $B_{\ell,i} = \sum\limits_{j=0}^{i} k_{\ell,j}$. Then, $k_{\ell,0} = 1$, $k_{\ell,1} = k_\ell$ and $n = B_{\ell,D_\ell}$, $\ell = 0, 1$ and moreover the following relationships hold, see the Lemmas 2.1−2.8 in (van den Akker 1990, Section 2.1) and the references therein.

Lemma 3 *If* Γ *is a distance-biregular graphs with intersection arrays* $\left\{k_\ell; c_{\ell,1}, \ldots c_{\ell,D_\ell}\right\}$, $\ell = 0, 1$. *Then,*

(i) $0 \le D_1 - D_0 \le 1$ *and when* $D_1 = D_0 + 1$*, then* D_0 *is odd.*
(ii) *For* $\ell = 0, 1$,

$$k_{\ell,i} = \prod_{j=0}^{i-1} \frac{b_{\ell,j}}{c_{\ell,j+1}}, \quad i = 0, \ldots, D_\ell$$

and hence,

$$k_{\ell,i} b_{\ell,i} = k_{\ell,i+1} c_{\ell,i+1}, \quad i = 0, \ldots, D_\ell.$$

(iii) $k_0 k_{1,2i+1} = k_1 k_{0,2i+1}$*, for any* $i = 0, \ldots, \lfloor \frac{D_0-1}{2} \rfloor$.
(iv) $c_{0,2i} c_{0,2i+1} = c_{1,2i} c_{1,2i+1}$ *and* $b_{0,2i-1} b_{0,2i} = b_{1,2i-1} b_{1,2i}$ *for any* $i = 1, \ldots, \lfloor \frac{D_0-1}{2} \rfloor$.
(v) *For* $\ell = 0, 1$, $1 \le c_{\ell,i} \le c_{\bar\ell,i+1}$ *and* $b_{\ell,i} \ge b_{\bar\ell,i+1}$ *for any* $i = 0, \ldots, D_{\bar\ell} - 1$. *Moreover,* $b_{\ell,i} \ge c_{\bar\ell,i+1}, \quad i = 1, \ldots, D_{\bar\ell} - 2$.
(vi) *For* $\ell = 0, 1$, $c_{\ell,2} \le \binom{c_{\ell,3}-1}{c_{\bar\ell,2}-1}$.
(vii) *For* $\ell = 0, 1$*, if* $i + j$ *is even and* $i + j \le D_\ell$*, then* $c_{\ell,i} \le b_{\ell,j}$.
(viii) *For* $\ell = 0, 1$*, if* $i + j$ *is odd and* $i + j \le D_0$*, then* $c_{\ell,i} \le b_{\bar\ell,j}$ *and* $c_{\bar\ell,i} \le b_{\ell,j}$.

The properties (ii), (iii) and (iv) imply that for $\ell = 0, 1$, the intersection numbers $\{c_{\ell,i}, b_{\ell,i}\}$ are determined by the intersection numbers $\{c_{\bar\ell,i}, b_{\bar\ell,i}\}$. In particular both sequences are the same iff $k_0 = k_1$ and in this case Γ is a (bipartite) distance-regular graph.

Lemma 4 (van den Akker 1990, Corollary 2.11) *Let* Γ *be a distance-bipartite regular graph. We can assume w.l.o.g. that one of the following holds*

1. $D_0 = D_1$ *and* $k_0 = k_1$*; so* Γ *is a bipartite distance-regular graph.*
2. $D_0 = D_1 - 1$ *is odd and* $k_0 > k_1$.
3. $D_0 = D_1$ *is even and* $k_0 > k_1$.

We display some preliminary results about the intersection parameters of a distance-biregular graphs, whose proofs are omitted since they follow trivially from (van den Akker 1990).

Lemma 5 *If* $k_0 > k_1$*, then*

$$\frac{b_{1,i}}{b_{0,i}} < \frac{k_1}{k_0} < \frac{c_{1,i}}{c_{0,i}}, \quad \text{if } i \text{ is even},$$
$$\frac{b_{0,i}}{b_{1,i}} < \frac{k_1}{k_0} < \frac{c_{0,i}}{c_{1,i}}, \quad \text{if } i \text{ is odd}.$$

The result provides an explicit expression of the equilibrium measure for sets $V \setminus \{y\}$, $\forall y \in V$ of distance-biregular graphs.

Proposition 6 *Let* Γ *be a distance-biregular graph with* $V = V_0 \cup V_1$*. Then, for any* $\ell = 0, 1$*, there exists an array* q_ℓ *of length* D_ℓ *such that if* $x \in V_\ell$*, for any* $y \in V$ *it holds*

$$\nu^x(y) = q_{\ell,m} \iff d(x, y) = m, \ m = 0, \ldots, D_\ell.$$

Moreover,

$$q_{\ell,m} = \sum_{j=0}^{m-1} \frac{n - B_{\ell,j}}{k_{\ell,j} b_{\ell,j}} = \sum_{j=1}^{m} \frac{n - B_{\ell,j-1}}{k_{\ell,j} c_{\ell,j}}.$$

In particular, $q_{\ell,1} = \dfrac{n-1}{k_\ell}$ *and* $q_{\ell,2} = \dfrac{n-1}{k_\ell} + \dfrac{n-1-k_\ell}{k_\ell(k_{\bar{\ell}} - 1)}$.

Proof Take $x \in V_\ell$ with $\ell = 0, 1$. Assume that the value $\nu^x(y)$ depends only on the distance from x to y, that is, there exists $q_{\ell,i}$, $i = 1, \ldots, D_\ell$ such that $\nu^x(y) = q_{\ell,i} \iff d(x, y) = i$. Moreover, we define $q_{\ell,D_\ell+1} = 0$. Note that, since the equilibrium system $\mathcal{L}\nu^x(y) = 1$ for all $y \in V \setminus \{x\}$ has a unique solution, then if with our hypothesis we can solve the system, such solution must correspond to the equilibrium measure $\nu^x(y) = q_{\ell,i}$.

In our case, $\mathcal{L}\nu^x(y) = 1$ for all $y \in V \setminus \{x\}$ is equivalent to the system

$$(b_{\ell,i} + c_{\ell,i})q_{\ell,i} - c_{\ell,i}q_{\ell,i-1} - b_{\ell,i}q_{\ell,i+1} = 1, \quad i = 1, \ldots, D_\ell$$

where $\ell = 0, 1$. Multiplying by $k_{\ell,i}$, we obtain

$$k_{\ell,i}c_{\ell,i}(q_{\ell,i} - q_{\ell,i-1}) - k_{\ell,i}b_{\ell,i}(q_{\ell,i+1} - q_{\ell,i}) = k_{\ell,i}, \quad i = 1, \ldots, D_\ell.$$

Since $k_{\ell,i}c_{\ell,i} = k_{\ell,i-1}b_{\ell,i-1}$ and denoting $\gamma_{\ell,i} = k_{\ell,i}b_{\ell,i}(q_{\ell,i+1} - q_{\ell,i})$, then

$$\gamma_{\ell,i-1} - \gamma_{\ell,i} = k_{\ell,i}, \qquad \text{for } i = 1, \ldots, D_\ell.$$

Observing that $\gamma_{\ell,D_\ell} = 0$, then summing up

$$n - B_{\ell,j} = \sum_{i=j+1}^{D_\ell} k_{\ell,i} = \gamma_{\ell,j} - \gamma_{\ell,D_\ell} = \gamma_{\ell,j}$$

for $j = 0, \ldots, D_\ell - 1$, it follows

$$q_{\ell,i+1} - q_{\ell,i} = \frac{n - B_{\ell,i}}{k_{\ell,i}b_{\ell,i}}, \qquad \text{for } i = 0, \ldots, D_\ell - 1.$$

Finally, since $q_{\ell,0} = 0$, it follows

$$q_{\ell,m} = \sum_{j=0}^{m-1} \frac{n - B_{\ell,j}}{k_{\ell,j} b_{\ell,j}}, \qquad \text{for } m = 0, \ldots, D_\ell.$$

The expression for $q_{\ell,m}$ in terms of $c_{\ell,i}$ follows from Lemma 3 (ii). □

The above proposition motives the definition of equilibrium arrays for a distance-biregular graph. If Γ is a distance-biregular graph, we call *equlibrium arrays* to the values $q_{\ell,i}$, $\ell = 0, 1$ and $i = 0, \ldots, D_\ell$. Denote $m_{\ell,i} = \big|\{y \in V : \nu^x(y) = q_{\ell,i}\}\big|$.

Corollary 7 *Let Γ be a distance-biregular graph with $y \in V_\ell$ and $x \in V_{\hat{\ell}}$, $\ell, \hat{\ell} = 0, 1$. Then,*

$$q_{\ell,d(x,y)} = q_{\hat{\ell},d(x,y)} + (n-1)\left(\frac{1}{k_\ell} - \frac{1}{k_{\hat{\ell}}}\right).$$

Proof From (7) we know that $\nu^y(x) - \nu^x(y) = \frac{1}{n}\big(cap(y) - cap(x)\big)$. On the other hand, $cap(y) = cap(z)$ for any $z \in V_\ell$ and $cap(x) = cap(w)$ for any $w \in V_{\hat{\ell}}$. So, for $\ell \neq \hat{\ell}$, we can choose $z \in V_\ell$ and $w \in V_{\hat{\ell}}$ such that $d(z, w) = 1$, then

$$\frac{1}{n}\big(cap(y) - cap(x)\big) = \frac{1}{n}\big(cap(z) - cap(w)\big) = q_{\ell,1} - q_{\hat{\ell},1} = (n-1)\left(\frac{1}{k_\ell} - \frac{1}{k_{\hat{\ell}}}\right),$$

which implies that

$$q_{\ell,d(x,y)} = q_{\hat{\ell},d(x,y)} + (n-1)\left(\frac{1}{k_\ell} - \frac{1}{k_{\hat{\ell}}}\right).$$

If $\ell = \hat{\ell}$, the result trivially holds. □

As an straightforward application of Proposition 6 we can find the intersection array of a distance-biregular graph in terms of the equilibrium arrays, analogously as was done in (Bendito et al. 2000, Proposition 4.5) for distance-regular graphs.

Proposition 8 *Let Γ be a distance-biregular graph with equilibrium arrays $q_{\ell,i}$ for $\ell = 0, 1$ and $i = 0, \ldots, D_\ell$. Then, for any $i = 0, \ldots, D_\ell - 1$, it holds*

$$k_{\ell,i} = m_{\ell,i},$$

$$b_{\ell,i} = \frac{1}{m_{\ell,i}(q_{\ell,i+1} - q_{\ell,i})} \sum_{j=i+1}^{D_\ell} m_{\ell,j},$$

$$c_{\ell,i+1} = \frac{1}{m_{\ell,i+1}(q_{\ell,i+1} - q_{\ell,i})} \sum_{j=i+1}^{D_\ell} m_{\ell,j}.$$

The computation of the equilibrium measure is usually done using linear programming (Bendito et al. 2000). In this regard, Proposition 8 provides a tool to calculate the intersection arrays of a distance-biregular graph solving one linear system.

Another application of the equilibrium measure concerns the estimation of the effective resistance of a resistive electrical network as well as the Kirchhoff Index, a well-known parameter in the context of organic chemistry. As a consequence of Proposition 6, we determine the effective resistance of distance-biregular graphs.

Corollary 9 *Let Γ be a distance-biregular graph with $y \in V_\ell$ and $x \in V_{\hat{\ell}}$, $\ell, \hat{\ell} = 0, 1$. Then, the effective resistance bewteen x, y is*

$$R(x, y) = \frac{2}{n} q_{\ell, d(x,y)} + \frac{(n-1)}{n} \left(\frac{1}{k_{\hat{\ell}}} - \frac{1}{k_\ell} \right).$$

Moreover,

$$K(\Gamma) = cap(y) + (n-1)|V_{\bar{\ell}}| \left(\frac{1}{k_{\bar{\ell}}} - \frac{1}{k_\ell} \right).$$

Proof The result follows from Corollary 7 taking into account that $R(x, y) = \dfrac{\nu^x(y) + \nu^y(x)}{n}$ and from the fact that

$$K(\Gamma) = \frac{1}{2} \sum_{u,v \in V} R(u, v) = \frac{1}{n} \Big(|V_{\bar{\ell}}| cap(u) + |V_\ell| cap(v) \Big),$$

see (Bendito et al. 2003). □

The next results shows the group inverse of the Laplacian of a distance-biregular graphs in terms of the intersection arrays.

Theorem 10 *Let Γ be a distance-biregular graph. Then, for each $y \in V_\ell$ with $\ell = 0, 1$, the group inverse of L is given by*

$$L^\#(x, y) = \frac{1}{n} \sum_{j=d(x,y)+1}^{D_\ell} \frac{n - B_{\ell, j-1}}{k_{\ell, j} c_{\ell, j}} - \frac{1}{n^2} \sum_{j=1}^{D_\ell} \frac{B_{\ell, j-1}(n - B_{\ell, j-1})}{k_{\ell, j} c_{\ell, j}}.$$

Proof From (5), we know that $L^\#(x, y) = \frac{1}{n^2}(cap(y) - n\nu^y(x))$. Take $y \in V_\ell$ with $\ell = 0, 1$. Now, using Proposition 6,

$$\begin{aligned} cap(y) &= \sum_{x \in V} \nu^y(x) = \sum_{m=0}^{D_\ell} k_{\ell,m} q_{\ell,m} = \sum_{m=0}^{D_\ell} k_{\ell,m} \sum_{j=1}^{m} \frac{n - B_{\ell,j-1}}{k_{\ell,j} c_{\ell,j}} \\ &= \sum_{j=1}^{D_\ell} \sum_{m=j}^{D_\ell} \frac{k_{\ell,m}(n - B_{\ell,j-1})}{k_{\ell,j} c_{\ell,j}} = \sum_{j=1}^{D_\ell} \frac{(n - B_{\ell,j-1})^2}{k_{\ell,j} c_{\ell,j}}. \end{aligned}$$

Therefore,

$$\begin{aligned} L^\#(x, y) &= \sum_{j=1}^{D_\ell} \frac{(n - B_{\ell,j-1})^2}{n^2 k_{\ell,j} c_{\ell,j}} - \sum_{j=1}^{d(x,y)} \frac{n - B_{\ell,j-1}}{n k_{\ell,j} c_{\ell,j}} \\ &= \sum_{j=d(x,y)+1}^{D_\ell} \frac{n - B_{\ell,j-1}}{n k_{\ell,j} c_{\ell,j}} - \sum_{j=1}^{D_\ell} \frac{B_{\ell,j-1}(n - B_{\ell,j-1})}{n^2 k_{\ell,j} c_{\ell,j}}. \end{aligned}$$

□

Remark 11 Observe that, since the intersection numbers of a distance-biregular graph are related, from Theorem 10, the expression of the group inverse is equivalent to

$$L^\#(x, y) = \frac{1}{n} \sum_{j=d(x,y)}^{D_\ell - 1} \frac{n - B_{\ell,j}}{k_{\ell,j} b_{\ell,j}} - \frac{1}{n^2} \sum_{j=0}^{D_\ell - 1} \frac{B_{\ell,j}(n - B_{\ell,j})}{k_{\ell,j} b_{\ell,j}}.$$

Example 12 As an application of Theorem 10 we obtain the group inverse of the Laplacian of a complete bipartite graph using its parameters:

$$\left.\begin{array}{l} D_0 = 2, k_0, c_{0,1} = 1, c_{0,2} = k_0, b_{0,0} = k_0, b_{0,1} = k_1 - 1, \\ D_1 = 2, k_1, c_{1,1} = 1, c_{1,2} = k_1, b_{1,0} = k_1, b_{1,1} = k_0 - 1, \end{array}\right\} \Longrightarrow n = k_0 + k_1.$$

Take $x, \hat{x} \in V_0$, $\hat{x} \neq x$ and $y, \hat{y} \in V_1$, $\hat{y} \neq y$. Then,

$$\begin{aligned} L^{\#}(x, x) &= \frac{1}{n^2}\left[\frac{(k_0 + k_1 - 1)^2}{k_0} + \frac{k_1 - 1}{k_0}\right] = \frac{(n-1)^2 + n - k_0 - 1}{n^2 k_0} = \frac{n^2 - n - k_0}{k_0 n^2}, \\ L^{\#}(y, y) &= \frac{1}{n^2}\left[\frac{(k_0 + k_1 - 1)^2}{k_1} + \frac{k_0 - 1}{k_1}\right] = \frac{n^2 - n - k_1}{k_1 n^2}, \\ L^{\#}(y, x) &= \frac{n^2 - n - k_0}{k_0 n^2} - \frac{n(n-1)}{k_0 n^2} = -\frac{1}{n^2} = L^{\#}(x, y), \\ L^{\#}(\hat{x}, x) &= \frac{n^2 - n - k_0}{k_0 n^2} - \frac{n(n-1)}{k_0 n^2} - \frac{n}{k_0 n^2} = -\frac{(n + k_0)}{k_0 n^2}, \\ L^{\#}(\hat{y}, y) &= -\frac{(n + k_1)}{k_1 n^2}. \end{aligned}$$

Observe that for a complete bipartite graph, it holds that $L^{\#}$ is always an M-matrix. The above expression is valid when $D_0 = D_1 = 1$, and $D_0 = 1$ and $D_1 = 2$; that is, for the star graph.

4 Distance-biregular graphs with the *M*-property

In this section, we answer Question 1 for distance-biregular graphs, completing, together with the known results for distance-regular graphs (Bendito et al. 2012), the characterization of when the group inverse of the combinatorial Laplacian matrix of a distance-regularised graph is an M-matrix.

Proposition 13 *Let Γ be a distance-biregular graph. Then, Γ has the M-property if and only if, it holds*

$$\sum_{j=1}^{D_0-1} \frac{1}{k_{0,j} b_{0,j}} \Big(\sum_{i=j+1}^{D_0} k_{0,i} \Big)^2 \leq \frac{(n-1)}{k_0}.$$

Proof We know that a graph Γ satisfies the M-property if and only if the entries of $L^{\#}(x, y) \leq 0$ for all $x \sim y$. The result follows from using that

$$k_{0,0} = 1, \;\; b_{0,0} = k_0, \;\; \text{and} \;\; \sum_{i=1}^{D_0} k_{0,i} = n - 1.$$

□

Remark 14 The condition from Proposition 13 is equivalent to

$$\sum_{j=1}^{D_1-1} \frac{1}{k_{1,j} b_{1,j}} \Big(\sum_{i=j+1}^{D_1} k_{1,i} \Big)^2 \leq \frac{(n-1)}{k_1}. \tag{10}$$

Using Proposition 13 we can also obtain the following necessary condition for a distance-biregular graph having the M-property.

Corollary 15 *If* Γ *is a distance-biregular graph with the* M*-property and* $D_0 \geq 2$*, then*

$$n < 2k_1 + k_0.$$

Proof Since $D_1 \geq D_0 \geq 2$, from Proposition 13 we obtain that

$$\frac{1}{k_{1,1}b_{1,1}}\Big(\sum_{i=2}^{D_1} k_{1,i}\Big)^2 \leq \sum_{j=1}^{D_1-1} \frac{1}{k_{1,j}b_{1,j}}\Big(\sum_{i=j+1}^{D_1} k_{1,i}\Big)^2 \leq \frac{(n-1)}{k_1}.$$

Now, observing that

$$\frac{1}{k_{1,1}b_{1,1}}\Big(\sum_{i=2}^{D_1} k_{1,i}\Big)^2 = \frac{(n-k_1-1)^2}{k_1 b_{1,1}}$$

we get

$$\begin{aligned}
&(n-k_1-1)^2 \leq (n-1)b_{1,1} = (n-1)(k_0-1) \iff \\
&(n-1)^2 - 2k_1(n-1) + k_1^2 \leq (n-1)(k_0-1) \iff \\
&n - 2k_1 + \frac{k_1^2}{n-1} - k_0 \leq 0,
\end{aligned}$$

and since $\frac{k_1^2}{n-1} > 0$, the result follows. □

Note that the inequality $n < 2k_1 + k_0$ turns out to be a strong restriction for a distance-biregular graph to have the M-property. Observe that such condition implies that the distance-biregular graph needs to be quite dense.

The following result generalizes the above observation by showing that only distance-biregular graphs with small D_ℓ can satisfy the M-property. A related result appeared in (Bendito et al. 2012, Proposition 5), where it was shown that the diameter of a distance-regular graphs with the M-property must be at most 3.

Proposition 16 *If* Γ *is a distance-biregular graph with the* M*-property, then* $D_1 \leq 4$ *and* $D_0 \leq 3$.

Proof By means of a contradiction, assume $D_0, D_1 \geq 4$. Then,

$$1 + k_\ell + k_{\ell,2} + k_{\ell,3} < 1 + k_\ell + k_{\ell,2} + k_{\ell,3} + k_{\ell,4} \leq n.$$

We can assume that $k_0 > k_1$, since otherwise Γ is a bipartite distance-regular graphs and hence $D_0 = D_1 \leq 3$, see (Bendito et al. 2012, Proposition 5). Then,

$$k_{0,2} = k_0 \frac{b_{0,1}}{c_{0,2}} \geq k_0 \frac{c_{1,2}}{c_{0,2}} > k_0 \frac{k_1}{k_0} = k_1.$$

On the other hand, since $b_{0,1} \geq c_{0,3}$ and $b_{0,2} \geq c_{0,2}$, we obtain that

$$k_{0,3} = k_0 \frac{b_{0,1}b_{0,2}}{c_{0,2}c_{0,3}} \geq k_0.$$

Finally, from Corollary 15 it follows that $n + 1 < 1 + 2k_0 + k_1 \leq n$, a contradiction. □

As an application of Proposition 16, we classify distance-biregular graphs having the M-property. We follow the notation from (van den Akker 1990).

Case 1: $D_0 = D_1 = 1$. This case corresponds to a digon that is a distance-regular and has the M-property.

Case 2: $D_0 = 1, D_1 = 2$. This case corresponds to star graphs, which are known to have the M-property, see (Carmona et al. 2014; Kirkland and Neumann 1998) or Example 12.

Case 3: $D_0 = D_1 = 2$. This case corresponds to a complete bipartite graph (see Example 12).

Case 4: $D_0 = D_1 = 3$. This case corresponds to a bipartite distance-regular graph (van den Akker 1990, Section 5.1), and thus it was already studied in Bendito et al. (2012). In this case, the intersection array is $\{k, k-1, k-\mu; 1, \mu, k\}$, where $1 \le \mu \le k-1$ and μ divides $k(k-1)$. They are antipodal iff $\mu = k-1$. Otherwise, they are the incidence graphs of nontrivial square $2-\left(\frac{n}{2}, k, \mu\right)$ designs. Therefore, $k-\mu$ must be a square, see (Brouwer et al. 1989, Th. 1.10.4).

Proposition 17 (Bendito et al. 2012, Proposition 13) *A bipartite distance-regular graph with $D = 3$ satisfies the M-property if and only if*

$$\frac{4k}{5} \le \mu \le k-1$$

and these inequalities imply that $k \ge 5$. In particular, if $1 \le \mu < k-1$, then either Γ or Γ_3 has the M-property, except when $k-1 < 5\mu < 4k$, in which case none of them has the M-property.

Case 5: $D_0 = 3, D_1 = 4$ with $k_0 > k_1$. In (van den Akker 1990, Proposition 5.3) it is shown that Γ is the point-line incidence graph of a quasi-symmetric design with $x = 0$ if and only if Γ is a distance-biregular graph with $D_0 = 3, D_1 = 4$ and intersection array

$$\left\{\begin{matrix} r;\ 1,\ \lambda,\ k \\ k;\ 1,\ y,\ \frac{k\lambda}{y},\ k \end{matrix}\right\}.$$

Recall that a 2-(v, k, λ) *quasi-symmetric design* is a design with two intersection numbers, and we are interested in those having $x = 0 < y < k$. Moreover, $k\lambda$ needs to be a multiple of y, that is, $k\lambda = \alpha y$, $\alpha \in \mathbb{N}$. Also, recall that $r > \lambda$. Moreover, since $B_{0,D_0} = B_{1,D_1}$, it holds that $(y-1)(r-1) = (k-1)(\lambda-1)$, see also (Baartmans and Shrikhande 1982) for a proof based on design techniques.

Next, using the condition in Proposition 13 with $r = k_0$, $k = k_1$, we obtain a necessary and sufficient condition for a distance-biregular graph with $D_0 = 3$, $D_1 = 4$ to have the M-property.

Proposition 18 *A distance-biregular graph with diameters $D_0 = 3, D_1 = 4$ has the M-property if and only if*

$$(k-1)(r-\lambda)\Big((k+r)^2 - \lambda k\Big) \le k^2\lambda^2.$$

Proof We use Proposition 13 to obtain:

$$\frac{1}{k_{0,1}b_{0,1}}\Big(k_{0,2} + k_{0,3}\Big)^2 + \frac{1}{k_{0,2}b_{0,2}}\Big(k_{0,3}\Big)^2 \le \frac{(n-1)}{k_0}.$$

Keeping in mind that

$$\begin{aligned}&k_0 = r, \quad b_{0,1} = k-1,\ b_{0,2} = r-\lambda,\\ &k_{0,0} = 1,\ k_{0,1} = r, \qquad k_{0,2} = r\frac{(k-1)}{\lambda},\ \ k_{0,3} = r\frac{(k-1)}{\lambda}\frac{(r-\lambda)}{k},\end{aligned}$$

we get that

$$\frac{n-1}{r} = 1 + \frac{(k-1)}{\lambda k}\big(k+r-\lambda\big).$$

After performing some simplifications on the first inequality, the desired result follows. □

Finally, we study the M-property for some classes of distance-biregular graphs with $D_0 = 3$ and $D_1 = 4$.

Example 19 Consider the point-line incident graph of the affine plane $\mathcal{A}(2,n)$ of order n, whose intersection array is

$$\left\{\begin{matrix} n+1; & 1,\ 1,\ n & \\ n; & 1,\ 1,\ n,\ n \end{matrix}\right\}.$$

It is easy to check that it does not verify the inequality in Proposition 13 and hence it does not verify the M-property.

Example 20 Let $\Gamma = S(K_{r+1})$ be the subdivision graph of the the complete graph K_{r+1} of order $r+1$. This provides a class of distance-biregular graphs with diameters $D_0 = 4$, $D_1 = 3$ and parameters

$$\begin{aligned}&D_0 = 3,\ k_0 = r,\ c_{0,1} = c_{0,2} = 1, c_{0,3} = 2, b_{0,0} = r, b_{0,1} = 1, b_{0,2} = r-1;\\ &D_1 = 4,\ k_1 = 2,\ c_{1,1} = c_{1,2} = 1, c_{1,3} = c_{1,4} = 2, b_{1,0} = 2, b_{1,1} = r-1, b_{1,2} = 1,\\ &\quad b_{1,3} = r-2;\end{aligned}$$

which does not hold the condition from Proposition 13, and thus does not have the M-property. In fact, we can find the group inverse. We denote $L^{\#}_{\ell,j} = L^{\#}(x,y)$ when $y \in V_\ell$ and $d(x,y) = j$. Then,

$$\begin{aligned}&L^{\#}_{0,0} = \frac{r(r+3)(2r+3)}{(r+1)^2(r+2)^2},\quad L^{\#}_{0,1} = \frac{(r+3)(r^2-2)}{(r+1)^2(r+2)^2},\ L^{\#}_{0,2} = -\frac{r^2+7r+8}{(r+1)^2(r+2)^2},\\ &L^{\#}_{0,3} = -\frac{2(r^2+5r+5)}{(r+1)^2(r+2)^2},\end{aligned}$$

and

$$\begin{aligned}&L^{\#}_{1,0} = \frac{(r+3)(r^3+5r^2+2r-4)}{2(r+1)^2(r+2)^2},\ L^{\#}_{1,1} = \frac{(r+3)(r^2-2)}{(r+1)^2(r+2)^2},\quad L^{\#}_{1,2} = \frac{r^3-r^2-18r-20}{2(r+1)^2(r+2)^2},\\ &L^{\#}_{1,3} = -\frac{2(r^2+5r+5)}{(r+1)^2(r+2)^2},\qquad L^{\#}_{1,4} = -\frac{(r+3)(3r+4)}{(r+1)^2(r+2)^2}.\end{aligned}$$

For the classification of existing quasi-symmetric 2-designs, see (Shrikhande 2007, Table 48.25). We should note that for the existing quasi-symmetric 2-designs with $x = 0$, none passes the condition from Proposition 18.

The above discussion extends the results in Bendito et al. (2012) and completes the classification of distance-regularised graphs that have the M-property. There, it was shown that if there are distance-regular graphs with valency $k \geq 3$ and diameter $D \geq 2$ having the M-property, then they have at most $3k$ vertices and $D \leq 3$. Also in Bendito et al. (2012), it was conjectured that there is no primitive distance-regular graph with diameter 3 having the M-property. This conjecture was shown to be true except possibly for finitely many primitive distance-regular graphs (Koolen and Park 2013, Theorem 1).

In view of the above results, we conclude this paper with the following conjecture.

Conjecture 21 *There are no point-line incidence graphs of a* 2*-quasi-symmetric design with* $x = 0$ *that have the* M*-property.*

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