A domain decomposition method for the Navier–Stokes equations with stochastic input

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Abstract

The paper is committed to studying the domain decomposition method for the incompressible Navier–Stokes equations (NSEs) with stochastic input. The stochastic input is represented spectrally by employing orthogonal polynomial functionals from the Askey scheme as trial basis to represent the random space, and the stochastic NSEs system are transformed into deterministic ones via the polynomial chaos expansion. The corresponding deterministic equations are transformed into the constrained optimization problem by minimizing the cost function on the common interface after the whole domain decomposed into two sub-domains. The constrained optimization problems are transformed into unconstrained problems by the Lagrange multiplier rule. A gradient method-based approach to the solutions of domain decomposition problem is proposed to solve the unconstrained optimality system. Finally, one numerical simulation experiment for square cavity flow problem with the stochastic boundary conditions are performed to demonstrate the feasibility and applicability of the gradient method.

Keywords Navier–Stokes equations · Polynomial chaos · Domain decomposition method

Mathematics Subject Classification 60H35 · 60H15

1 Introduction

The incompressible NSE is a well accepted model for atmospheric and ocean dynamics. The stochastic NSE has a long history [e.g., see (Chandrasekhar 1989; Bensoussan and Temam 1973) for two of the earlier studies] as a model to understand external random forces in aeronautical applications random forcing of the NSE models structural vibrations and in atmospheric dynamics, unknown external forces such as sun heating and industrial pollution can be represented as random forces. The cost for dealing with a stochastic model is obviously

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an increase in the complexity of the problem. Thus, efficient simulation and design of these uncertainties become the primary motivation behind the present effort.

Usually, uncertainties can be processed by using the probabilistic methods. The probabilistic methods in engineering can be broadly classified into two major categories: methods using a statistical approach and methods using a non-statistical approach. One methodology of the non-statistical type is to discretize directly the random field, which has drawn attentions in recent years (e.g. see Xiu and Karniadakis 2002; Wan et al. 2004). Ghanem and Spanos pioneered a polynomial chaos expansion method (see Ghanem and Spanos 1991), the random variables directly discrete as a polynomial chaos expansion. It allows high-order representation and promises fast convergence, coupled with Karhunen Loeve (KL) decomposition for the input and Galerkin projection in random space, it results in computationally tractable algorithms for large engineering systems (Ghanem and Red-Horse 1999). The classical polynomial chaos expansion is based on the Hermite polynomials in terms of Gaussian random variables (Schoutens 2000; Wiener 1938). Although in theory, it converges to any $\mathcal{L}^2$ functionals in the random space (Cameron and Martin 1947), it achieves optimal convergence rate only for Gaussian and near Gaussian random fields (Xiu and Karniadakis 2002), and does not readily apply to the random fields with discrete distribution. A more general framework, called the generalized polynomial chaos or the Askey-chaos, was proposed in Xiu and Karniadakis (2002). Here the polynomials are chosen from the hypergeometric polynomials of the Askey scheme (Askey and Wilson 1985), and the underlying random variables are not restricted to Gaussian random variables. Instead, the type of random variables are chosen according to the stochastic input and the weighting function of these random variables determines the type of orthogonal polynomials to be used as the basis in the random space. The convergence properties of different bases were studied in Xiu and Karniadakis (2002) and exponential convergence rate was demonstrated for model problems. In summary, the stochastic input will be expanded of generalized polynomial chaos in the paper, and finally the NSEs with stochastic input are formulated by the set of equations consists of the system of 'NS-like' equations.

The NSEs obviously becomes larger in scale after the above treatment, effective simulation algorithms for the set of equations must be developed. Numerical method based on domain decomposition is a powerful technique to compute solutions of partial differential equations (Kong et al. 2009; Haslinger et al. 2014). The method is especially good when a computer's memory is not large enough for the complete problem or when a domain has an irregular shape, which often happens in practical applications. Due to the obvious implication for parallel processing, domain decomposition methods have attracted many researchers' great attention. Some large scale international conferences are held on this subject every year, and the processing of these and other conferences provide both a summary and a history of developments in the field, see e.g., (Chan et al. 1989, 1990, 1992; Glowinski et al. 1988, 1991). In Gunzburger and Lee (2000), Bresch and Koko (2006), Koko and Sassi (2016), Qiang 2001, various domain decomposition algorithms are obtained for partial differential equations with different energy functionals and solution strategies. Kong et al. (2009) only restrict the interest to the velocity vector for the domain decomposition method and construct the optimization problems by using the sensitivity derivatives method and the Lagrange multiplier rule (Malczyk and Czek 2015), the constrained optimization problems are transformed into unconstrained problems, then, a gradient method-based approach to the solution of domain decomposition problem is proposed to solve the unconstrained optimality system.

The main objective of the paper is to give a broad algorithmic framework to solve NSEs with stochastic input based on the generalized polynomial chaos expansion, so, the plan of the paper is as follows. The NS equations with stochastic input are transformed into deterministic

ones via the polynomial chaos expansion in Sect. 2. The optimization-based domain decomposition method for the system of 'NS-like' equations is proposed in Sect. 3. The constrained optimization problems are transformed into unconstrained problems by the Lagrange multiplier rule and a gradient-type approach to the solutions of domain decomposition problem is proposed in Sects. 3.1 and 3.2. One numerical simulation experiment for square cavity flow problem with the stochastic boundary conditions are performed to demonstrate the feasibility and applicability of the gradient method in Sect. 4.

2 Setting of the problem based on the generalized polynomial chaos expansion

Let D denote a bounded open set in $\mathcal{R}^N$ with boundary Γ . Consider the incompressible NSEs in D ,

$$\begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \Pi = \mathbf{f}, & \text{in } D \\ \operatorname{div} \mathbf{u} = 0, & \text{in } D \\ \mathbf{u} = \mathbf{g}, & \text{on } \Gamma \end{cases} \quad (1)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}; \omega)$ is the velocity field, $\Pi = \Pi(\mathbf{x}; \omega)$ is the pressure, $\mathbf{f} = \mathbf{f}(\mathbf{x}; \omega)$ is the body force, $\mathbf{g} = \mathbf{g}(\mathbf{x}; \omega)$ is Dirichlet boundary condition and ν is the viscosity of the flow.

This can be considered as a model with stochastic input subject to external (source term $\mathbf{f}$ and/or Dirichlet boundary condition $\mathbf{g}$) uncertainties. The existence and uniqueness of solutions for 2-D stochastic NSEs were obtained in Flandoli and Gatarek (1995) and Sritharan and Sundar (2006). In this paper, we solve the equation (1) by generalized polynomial chaos expansion, where the uncertainties can be introduced through $\mathbf{f}$, or $\mathbf{g}$, or some combinations. As follows, we present the detailed method for the application of the generalized polynomial chaos expansion to Eq. (1). By applying the chaos expansion, we expand the variables as

$$\begin{aligned} \mathbf{u}(\mathbf{x}; \omega) &= \sum_{i=0}^P \mathbf{u}_i(\mathbf{x}) \Phi_i(\xi); \quad \Pi(\mathbf{x}; \omega) = \sum_{i=0}^P \Pi_i(\mathbf{x}) \Phi_i(\xi); \\ \mathbf{f}(\mathbf{x}; \omega) &= \sum_{i=0}^P \mathbf{f}_i(\mathbf{x}) \Phi_i(\xi); \quad \mathbf{g}(\mathbf{x}; \omega) = \sum_{i=0}^P \mathbf{g}_i(\mathbf{x}) \Phi_i(\xi), \end{aligned} \quad (2)$$

where we have replaced the infinite summation of ξ in infinite dimensions by a truncated finite-term summation of Φ in the finite dimensions of $\xi = (\xi_1, \xi_2, \dots, \xi_n)$. The dimensionality n of ξ is determined by the random inputs. The random parameter ω is absorbed into the polynomial basis $\Phi(\xi)$, thus the expansion coefficients $\mathbf{u}_i$, Π_i , $\mathbf{f}_i$ and $\mathbf{g}_i$ are deterministic. The total number of expansion terms, $P + 1$, depends on the number of random dimensions n of ξ and the highest order p of the polynomials Φ_i ,

$$P + 1 = \frac{(n + p)!}{n! p!}. \quad (3)$$

By substituting the expansion into governing equation (1), we obtain the following equations,

$$\left\{ \begin{array}{ll} -\nu \sum_{i=0}^P \Delta \mathbf{u}_i \Phi_i + \sum_{i=0}^P \nabla \Pi_i \Phi_i + \sum_{i=0}^P \sum_{j=0}^P (\mathbf{u}_i \cdot \nabla) \mathbf{u}_j \Phi_i \Phi_j = \sum_{i=0}^P \mathbf{f}_i \Phi_i, & \text{in } D \\ \sum_{i=0}^P \operatorname{div} \mathbf{u}_i \Phi_i = 0, & \text{in } D \\ \sum_{i=0}^P \mathbf{u}_i \Phi_i = \sum_{i=0}^P \mathbf{g}_i \Phi_i, & \\ k = 0, \dots, P, & \text{on } \Gamma. \end{array} \right. \quad (4)$$

Multiplying (4) by Φ_k and taking the expected value, we obtain

$$\left\{ \begin{array}{ll} -\nu \sum_{i=0}^P \Delta \mathbf{u}_i \langle \Phi_i \Phi_k \rangle + \sum_{i=0}^P \nabla \Pi_i \langle \Phi_i \Phi_k \rangle + \sum_{i=0}^P \sum_{j=0}^P (\mathbf{u}_i \cdot \nabla) \mathbf{u}_j \langle \Phi_i \Phi_j \Phi_k \rangle \\ = \sum_{i=0}^P \mathbf{f}_i \langle \Phi_i \Phi_k \rangle, & \text{in } D \\ \sum_{i=0}^P \operatorname{div} \mathbf{u}_i \langle \Phi_i \Phi_k \rangle = 0, & \text{in } D \\ \sum_{i=0}^P \mathbf{u}_i \langle \Phi_i \Phi_k \rangle = \sum_{i=0}^P \mathbf{g}_i \langle \Phi_i \Phi_k \rangle, \quad k = 0, \dots, P, & \text{on } \Gamma. \end{array} \right. \quad (5)$$

By virtue of the orthogonality of the polynomial chaos, (5) reduces to

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{u}_k \langle \Phi_k \Phi_k \rangle + \nabla \Pi_k \langle \Phi_k \Phi_k \rangle + \sum_{i=0}^P \sum_{j=0}^P (\mathbf{u}_i \cdot \nabla) \mathbf{u}_j \langle \Phi_i \Phi_j \Phi_k \rangle \\ = \mathbf{f}_k \langle \Phi_k \Phi_k \rangle, & \text{in } D \\ \operatorname{div} \mathbf{u}_k \langle \Phi_k \Phi_k \rangle = 0, & \text{in } D \\ \mathbf{u}_i \langle \Phi_k \Phi_k \rangle = \mathbf{g}_i \langle \Phi_k \Phi_k \rangle, \quad k = 0, \dots, P, & \text{on } \Gamma. \end{array} \right. \quad (6)$$

Dividing by $\langle \Phi_k \Phi_k \rangle$, we obtain

$$\left\{ \begin{array}{ll} -\nu \Delta \mathbf{u}_k + \nabla \Pi_k + \frac{1}{\langle \Phi_k^2 \rangle} \sum_{i=0}^P \sum_{j=0}^P e_{ijk} (\mathbf{u}_i \cdot \nabla) \mathbf{u}_j = \mathbf{f}_k, & \text{in } D \\ \operatorname{div} \mathbf{u}_k = 0, & \text{in } D \\ \mathbf{u}_k = \mathbf{g}_k, \quad k = 0, \dots, P, & \text{on } \Gamma. \end{array} \right. \quad (7)$$

where $e_{ijk} = \langle \Phi_i \Phi_j \Phi_k \rangle$. Together with $\langle \Phi_k^2 \rangle$, the coefficients e_{ijk} can be evaluated analytically from the definition $\langle \cdot, \cdot \rangle = \int \cdot \cdot f(x) dx$, where $f(x)$ is the probability density function on ξ . The set of equations consists of $P + 1$ system of 'N-S-like' equations for each mode coupled through the convective terms.

3 The problems solving

3.1 The domain decomposition method

The main objective of this section is to give a broad explanation of the domain decomposition method, the following problem is explained with D partitioned into two simply connected non-overlapping sub-domains D_1 and D_2 without loss of generality.

Fig. 1 The domain D divided into two non-overlapping sub-domains

As in Fig. 1, D is partitioned into two simply connected non-overlapping sub-domains D_1 and D_2 with the common interface $\Gamma_0 = \overline{D}_1 \cap \overline{D}_2$.

In the sequel, we shall consider homogeneous boundary conditions on Γ_0 . This choice is made to simplify the presentation, but the method exposed extends straightly to the situation where $\mathbf{g}$ is non null. Please, notice that this simplification concerns only exterior boundaries Γ_1 and Γ_2 , but it does not concern the internal boundary Γ_0 as written in the sequel, the conditions on Γ_0 will be Robin conditions.

Consider the following pair of NSEs with the mixed boundary conditions,

$$\begin{cases} -\nu \Delta \mathbf{u}_{k1} + \nabla \Pi_{k1} + \frac{1}{\langle \Phi_k^2 \rangle} \sum_{i=0}^P \sum_{j=0}^P e_{ijk} (\mathbf{u}_{i1} \cdot \nabla) \mathbf{u}_{j1} = \mathbf{f}_{k1}, & \text{in } D_1 \\ \operatorname{div} \mathbf{u}_{k1} = 0, & \text{in } D_1 \\ \mathbf{u}_{k1} = 0, & \text{on } \Gamma_1 \\ -\Pi_{k1} \mathbf{n}_1 + \nu \nabla \mathbf{u}_{k1} \cdot \mathbf{n}_1 + r \mathbf{u}_{k1} = \mathbf{h}_k, \quad k = 0, 1, \dots, P & \text{on } \Gamma_0 \end{cases} \quad (8)$$

and

$$\begin{cases} -\nu \Delta \mathbf{u}_{k2} + \nabla \Pi_{k2} + \frac{1}{\langle \Phi_k^2 \rangle} \sum_{i=0}^P \sum_{j=0}^P e_{ijk} (\mathbf{u}_{i2} \cdot \nabla) \mathbf{u}_{j2} = \mathbf{f}_{k2}, & \text{in } D_2 \\ \operatorname{div} \mathbf{u}_{k2} = 0, & \text{in } D_2 \\ \mathbf{u}_{k2} = 0, & \text{on } \Gamma_2 \\ -\Pi_{k2} \mathbf{n}_2 + \nu \nabla \mathbf{u}_{k2} \cdot \mathbf{n}_2 - r \mathbf{u}_{k2} = -\mathbf{h}_k, & \text{on } \Gamma_0, \text{ for } k = 0, 1, \dots, P, \end{cases} \quad (9)$$

Where, $\Gamma_1 = \overline{D}_1 \cap \Gamma$ and $\Gamma_2 = \overline{D}_2 \cap \Gamma$, $\mathbf{n}_1$ and $\mathbf{n}_2$ denote the outward unit normal vectors to D_1 and D_2 , respectively. $\mathbf{h}_k = -\Pi_k \mathbf{n}_1 + \nu \nabla \mathbf{u}_k \cdot \mathbf{n}_1 + r \mathbf{u}_k$ is the Robin boundary conditions.

For the convenience of calculation, the weak formulations for the Navier–Stokes equations is given. Let $H^r(D)$ denote the Sobolev space of order r with respect to a domain D , equipped with the standard norm $\|\cdot\|_{r,D}$. Define the subspaces

$$H_0^r(D) = \{\mathbf{v} \in H^r(D) : \mathbf{v} = 0 \text{ on } \Gamma\}$$

and

$$H_{\Gamma_i}^r(D) = \{\mathbf{v} \in H^r(D_i) : \mathbf{v} = 0 \text{ on } \Gamma_i\}$$

for $i = 1$ and 2 .

Define

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_{\Gamma_i} &= \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{v} \, d\Gamma_i \\ (\Pi, q)_{\Gamma_i} &= \int_{\Gamma_i} \Pi \cdot q \, d\Gamma_i, \end{aligned}$$

for $i = 1, 2$ and whenever $\mathbf{u} \cdot \mathbf{v}, \Pi q \in L^2(\Gamma_i)$

Define

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_{D_i} &= \int_{D_i} \mathbf{u} \cdot \mathbf{v} \, dD_i \\ (\Pi, q)_{D_i} &= \int_{D_i} \Pi \cdot q \, dD_i, \end{aligned}$$

for $i = 1, 2$ and whenever $\mathbf{u} \cdot \mathbf{v}, \Pi q \in L^2(D_i)$

The bilinear and trilinear forms on domain D are defined as

$$\begin{aligned} a(\mathbf{u}_k, \mathbf{v}) &= \int_D \nabla \mathbf{u}_k \cdot \nabla \mathbf{v} \, dD, & \forall \mathbf{u}_k, \mathbf{v} \in H_0^1(D), \\ b(\mathbf{u}_k, \mathbf{w}_k, \mathbf{v}) &= \int_D (\mathbf{u}_k \cdot \nabla) \mathbf{w}_k \cdot \mathbf{v} \, dD, & \forall \mathbf{u}_k, \mathbf{w}_k, \mathbf{v} \in H_0^1(D), \\ c(\mathbf{u}_k, \Pi_k) &= \int_D \Pi_k \operatorname{div} \mathbf{u}_k \, dD, & \forall \mathbf{u}_k \in H_0^1(D), \forall \Pi_k \in L^2(D) \end{aligned}$$

Analogously, the bilinear and trilinear forms on domain D_i , $i = 1, 2$ are defined as

$$\begin{aligned} a_i(\mathbf{u}_{ki}, \mathbf{v}) &= \int_{D_i} \nabla \mathbf{u}_{ki} \cdot \nabla \mathbf{v} \, dD_i, & \forall \mathbf{u}_{ki}, \mathbf{v} \in H_0^1(D_i), \\ b_i(\mathbf{u}_{ki}, \mathbf{w}_{ki}, \mathbf{v}) &= \int_{D_i} (\mathbf{u}_{ki} \cdot \nabla) \mathbf{w}_{ki} \cdot \mathbf{v} \, dD_i, & \forall \mathbf{u}_{ki}, \mathbf{w}_{ki}, \mathbf{v} \in H_0^1(D_i), \\ c_i(\mathbf{u}_{ki}, \Pi_{ki}) &= \int_{D_i} \Pi_{ki} \operatorname{div} \mathbf{u}_{ki} \, dD_i, & \forall \mathbf{u}_{ki} \in H_0^1(D_i), \forall \Pi_{ki} \in L^2(D_i) \end{aligned}$$

Using the above definitions, the weak formulations corresponding to Eq.(1) are given by

$$\begin{cases} \nu a(\mathbf{u}_k, \mathbf{v}) + \frac{1}{\langle \Phi_k^2 \rangle} \sum_{i=0}^P \sum_{j=0}^P e_{ijk} b(\mathbf{u}_i, \mathbf{u}_j, \mathbf{v}) - c(\mathbf{v}, \Pi_k) = (\mathbf{f}_k, \mathbf{v})_D, & \forall \mathbf{v} \in H_{\Gamma_i}^1(D) \\ -c(\mathbf{u}_k, q) = 0, & \forall q \in L^2(D) \end{cases} \quad (10)$$

Similarly, the weak formulations corresponding to (8) and (9) are given by

$$\begin{cases} \nu a_i(\mathbf{u}_{ki}, \mathbf{v}) + \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\mathbf{u}_{mi}, \mathbf{u}_{ni}, \mathbf{v}) - c_i(\mathbf{v}, \Pi_{ki}) \\ \quad + (-1)^{i+1} r(\mathbf{u}_{ki}, \mathbf{v})_{\Gamma_0} \\ \quad = (\mathbf{f}_{ki}, \mathbf{v})_{D_i} + (-1)^{i+1} (\mathbf{h}_k, \mathbf{v})_{\Gamma_0}, & \forall \mathbf{v} \in H_{\Gamma_i}^1(D_i), \\ -c_i(\mathbf{u}_{ki}, q) = 0, & \forall q \in L^2(D_i) \end{cases} \quad (11)$$

for $\mathbf{u}_{ki} \in H_{\Gamma_i}^1(D_i)$, $\Pi_{ki} \in L^2(D_i)$ and $i = 1, 2$.

The remainder of this section gives a statement of the optimization problem for the decomposition method for Eq.(11). We try to find $\mathbf{h}_k$ that minimizes the $\mathcal{L}^2(\Gamma_0)$ norm of $\mathbf{u}_{k1} - \mathbf{u}_{k2}$ on the common interface Γ_0 . Hence, the optimization problem to minimize the functional is

$$\mathcal{J}_k(\mathbf{u}_{k1}(\mathbf{h}_k), \mathbf{u}_{k2}(\mathbf{h}_k)) = \frac{1}{2} \int_{\Gamma_0} (\mathbf{u}_{k1} - \mathbf{u}_{k2})^2 \, d\Gamma_0. \quad (12)$$

To improve the convergence rate, we minimize the penalized functional

$$\tilde{\mathcal{J}}_k(\mathbf{u}_{k1}(\mathbf{h}_k), \mathbf{u}_{k2}(\mathbf{h}_k), \mathbf{h}_k) = \frac{\alpha}{2} \int_{\Gamma_0} (\mathbf{u}_{k1} - \mathbf{u}_{k2})^2 \, d\Gamma_0 + \frac{\beta}{2} \int_{\Gamma_0} \mathbf{h}_k^2 \, d\Gamma_0. \quad (13)$$

with the admissibility set $\mathcal{U}_{\text{ad}}$ is defined by

$$\mathcal{U}_{\text{adk}} = \{\mathbf{h}_k \in L^2(\Gamma_0) : \text{there exist } (\mathbf{u}_{ki}, \Pi_{ki}) \in H_{\Gamma_i}^1(D_i) \times L^2(D_i) \text{ for } i = 1, 2 \\ \text{such that (11) is satisfied and } \tilde{\mathcal{J}}_k(\mathbf{u}_{k1}(\mathbf{h}_k), \mathbf{u}_{k2}(\mathbf{h}_k), \mathbf{h}_k) < \infty.\}$$

Where α, β are positive constants that can chosen to change the relative importance of the two terms and adjust the convergence.

So, the domain decomposition method for NSEs is transformed into the following optimization problem,

$$\min_{\mathbf{h}_k \in \mathcal{U}_{\text{adk}}} \tilde{\mathcal{J}}_k(\mathbf{u}_{k1}(\mathbf{h}_k), \mathbf{u}_{k2}(\mathbf{h}_k), \mathbf{h}_k). \quad (14)$$

3.2 The Lagrange multiplier rule and a gradient method

The Lagrange multiplier rule is used to reduce the constrained minimization problem (14) to an unconstrained problem in the section. Define the Lagrangian functional,

$$\begin{aligned} \mathcal{L}_k(\mathbf{u}_{k1}, \Pi_{k1}, \mathbf{u}_{k2}, \Pi_{k2}, \mathbf{h}_k; \boldsymbol{\varphi}_{k1}, \mu_{k1}, \boldsymbol{\varphi}_{k2}, \mu_{k2}) \\ = \tilde{\mathcal{J}}_k(\mathbf{u}_{k1}, \mathbf{u}_{k2}, \mathbf{h}_k) + \sum_{i=1}^2 \left[\nu a_i(\mathbf{u}_{ki}, \boldsymbol{\varphi}_{ki}) \right. \\ + \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\mathbf{u}_{mi}, \mathbf{u}_{ni}, \boldsymbol{\varphi}_{ki}) \\ - c_i(\boldsymbol{\varphi}_{ki}, \Pi_{ki}) + (-1)^{i+1} r(\mathbf{u}_{ki}, \boldsymbol{\varphi}_{ki})_{\Gamma_0} - (\mathbf{f}, \boldsymbol{\varphi}_{ki})_{D_i} \\ \left. + (-1)^i (\mathbf{h}_k, \boldsymbol{\varphi}_{ki})_{\Gamma_0} - c_i(\mathbf{u}_{ki}, \mu_{ki}) \right], \end{aligned} \quad (15)$$

where $\boldsymbol{\varphi}_{k1}$, μ_{k1} , $\boldsymbol{\varphi}_{k2}$, and μ_{k2} are Lagrange multipliers. So, the constrained problem (14) can be recast as the unconstrained problem of finding stationary points of (15) with the Eulerian derivatives of the Lagrangian functional $\mathcal{L}_k$ at $\boldsymbol{\varphi}_{ki}$ in the direction $\delta\boldsymbol{\varphi}_{ki} \in H_{\Gamma_i}^1(D_i)$, $i = 1, 2$, which is denoted by $\frac{\partial \mathcal{L}_k}{\partial \boldsymbol{\varphi}_{ki}} \delta\boldsymbol{\varphi}_{ki}$, $i = 1, 2$. Similarly, the Eulerian derivatives of the Lagrangian functional $\mathcal{L}_k$ at μ_{k1} , $\mathbf{u}_{ki}$, Π_{ki} , $\mathbf{h}_k$ are denoted by $\frac{\partial \mathcal{L}_k}{\partial \mu_{ki}} \delta\mu_{ki}$, $\frac{\partial \mathcal{L}_k}{\partial \mathbf{u}_{ki}} \delta\mathbf{u}_{ki}$, $\frac{\partial \mathcal{L}_k}{\partial \Pi_{ki}} \delta\Pi_{ki}$, $\frac{\partial \mathcal{L}_k}{\partial \mathbf{h}_k} \delta\mathbf{h}_k$, $i = 1, 2$ respectively.

The Eulerian derivatives of the Lagrangian functional $\mathcal{L}_k$ are given by

$$\left\{ \begin{aligned} \frac{\partial \mathcal{L}_k}{\partial \boldsymbol{\varphi}_{ki}} \delta\boldsymbol{\varphi}_{ki} &= \nu a_i(\mathbf{u}_{ki}, \delta\boldsymbol{\varphi}_{ki}) + \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\mathbf{u}_{mi}, \mathbf{u}_{ni}, \delta\boldsymbol{\varphi}_{ki}) \\ &\quad - c_i(\delta\boldsymbol{\varphi}_{ki}, \Pi_{ki}) + (-1)^{i+1} r(\mathbf{u}_{ki}, \delta\boldsymbol{\varphi}_{ki})_{\Gamma_0} - (\mathbf{f}_k, \delta\boldsymbol{\varphi}_{ki}) \\ &\quad + (-1)^i (\mathbf{h}_k, \delta\boldsymbol{\varphi}_{ki})_{\Gamma_0} \\ \frac{\partial \mathcal{L}_k}{\partial \mu_{ki}} \delta\mu_{ki} &= -c_i(\mathbf{u}_{ki}, \delta\mu_{ki}) \\ \frac{\partial \mathcal{L}_k}{\partial \mathbf{u}_{ki}} \delta\mathbf{u}_{ki} &= (-1)^{i+1} \alpha((\mathbf{u}_{k1} - \mathbf{u}_{k2}), \delta\mathbf{u}_{ki})_{\Gamma_0} + \nu a_i(\delta\mathbf{u}_{ki}, \boldsymbol{\varphi}_{ki}) \\ &\quad + \mathbf{I}_{\{m=k\}} \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\delta\mathbf{u}_{mi}, \mathbf{u}_{ni}, \boldsymbol{\varphi}_{ki}) \\ &\quad + \mathbf{I}_{\{n=k\}} \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\mathbf{u}_{mi}, \delta\mathbf{u}_{ni}, \boldsymbol{\varphi}_{ki}) - c_i(\delta\mathbf{u}_{ki}, \mu_{ki}) \\ &\quad + (-1)^{i+1} r(\delta\mathbf{u}_{ki}, \boldsymbol{\varphi}_{ki})_{\Gamma_0} \\ \frac{\partial \mathcal{L}_k}{\partial \Pi_{ki}} \delta\Pi_{ki} &= -c_i(\boldsymbol{\varphi}_{ki}, \delta\Pi_{ki}) \\ \frac{\partial \mathcal{L}_k}{\partial \mathbf{h}_k} \delta\mathbf{h}_k &= \beta(\mathbf{h}_k, \delta\mathbf{h}_k)_{\Gamma_0} - (\boldsymbol{\varphi}_{k1} - \boldsymbol{\varphi}_{k2}, \delta\mathbf{h}_k)_{\Gamma_0}, \quad i = 1, 2. \end{aligned} \right. \quad (16)$$

Setting $\frac{\partial \mathcal{L}_k}{\partial \mathbf{u}_{ki}} \delta\mathbf{u}_{ki} = 0$ and $\frac{\partial \mathcal{L}_k}{\partial \Pi_{ki}} \delta\Pi_{ki} = 0$ for $i = 1, 2$, we obtain the following adjoint equations,

$$\left\{ \begin{aligned} &\nu a_i(\delta\mathbf{u}_{ki}, \boldsymbol{\varphi}_{ki}) + \mathbf{I}_{\{m=k\}} \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\delta\mathbf{u}_{mi}, \mathbf{u}_{ni}, \boldsymbol{\varphi}_{ki}) \\ &\quad + \mathbf{I}_{\{n=k\}} \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\mathbf{u}_{mi}, \delta\mathbf{u}_{ni}, \boldsymbol{\varphi}_{ki}) \\ &\quad - c_i(\delta\mathbf{u}_{ki}, \mu_{ki}) + (-1)^{i+1} r(\delta\mathbf{u}_{ki}, \boldsymbol{\varphi}_{ki})_{\Gamma_0} \\ &= (-1)^i \alpha((\mathbf{u}_{k1} - \mathbf{u}_{k2}), \delta\mathbf{u}_{ki})_{\Gamma_0} - c_i(\boldsymbol{\varphi}_{ki}, \delta\Pi_{ki}) \\ &= 0, \quad \forall \delta\mathbf{u}_{ki} \in H_{\Gamma_i}^1(D_i), \quad \forall \delta\Pi_{ki} \in L^2(D_i). \end{aligned} \right. \quad (17)$$

Finally, setting $\frac{\partial \mathcal{L}_k}{\partial \mathbf{h}_k} \delta\mathbf{h}_k = 0$, we can obtain the optimality condition. Now, we must at least decouple the subdomain problems. To achieve a parallel algorithm and make the individual subdomain problems tractable, we should uncouple the state and adjoint systems as well

by using a gradient-type method iteration. So, we need define $\frac{d\tilde{\mathcal{J}}(\mathbf{h}_k)}{d\mathbf{h}_k}$ through its action on variations $\tilde{\mathbf{h}}_k$ by

$$\langle \frac{d\tilde{\mathcal{J}}(\mathbf{h}_k)}{d\mathbf{h}_k}, \tilde{\mathbf{h}}_k \rangle = \alpha(\mathbf{u}_{k1} - \mathbf{u}_{k2}, \tilde{\mathbf{u}}_{k1} - \tilde{\mathbf{u}}_{k2})_{\Gamma_0} + \beta(\mathbf{h}_k, \tilde{\mathbf{h}}_k)_{\Gamma_0} \quad (18)$$

for $i = 1$ and 2 , where $\tilde{\mathbf{u}}_{ki} \in H_{\Gamma_i}^1(D_i)$ are solutions of the sensitivity equations

$$\begin{cases} \nu a_i(\tilde{\mathbf{u}}_{ki}, \mathbf{v}) + \mathbf{I}_{\{m=k\}} \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\tilde{\mathbf{u}}_{mi}, \mathbf{u}_{ni}, \mathbf{v}) \\ + \mathbf{I}_{\{n=k\}} \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\mathbf{u}_{mi}, \tilde{\mathbf{u}}_{ni}, \mathbf{v}) - c_i(\mathbf{v}, \tilde{\Pi}_{ki}) \\ + (-1)^{i+1} r(\tilde{\mathbf{u}}_{ki}, \mathbf{v})_{\Gamma_0} = (-1)^{i+1} (\tilde{\mathbf{h}}_k, \mathbf{v})_{\Gamma_0}, \forall \mathbf{v} \in H_{\Gamma_i}^1(D_i) \\ -c_i(\tilde{\mathbf{u}}_{ki}, q) = 0, \forall q \in L^2(D_i), \end{cases} \quad (19)$$

So, $\tilde{\mathbf{u}}_{ki}$ viewed as sensitivities give the changes in $\mathbf{u}_{ki}$ that result from the changes $\tilde{\mathbf{h}}_k$ in $\mathbf{h}_k$ for $i = 1, 2$. Set $\mathbf{v} = \boldsymbol{\varphi}_{ki}$, $q = \mu_{ki}$ in (19) and $\delta \mathbf{u}_{ki} = \tilde{\mathbf{u}}_{ki}$, $\delta \Pi_{ki} = \tilde{\Pi}_{ki}$ in (17), combining the results yields that

$$(\tilde{\mathbf{h}}_k, \boldsymbol{\varphi}_{k1} - \boldsymbol{\varphi}_{k2})_{\Gamma_0} = -\alpha(\mathbf{u}_{k1} - \mathbf{u}_{k2}, \tilde{\mathbf{u}}_{k1} - \tilde{\mathbf{u}}_{k2})_{\Gamma_0}.$$

From (18), we obtain that

$$\langle \frac{d\tilde{\mathcal{J}}(\mathbf{h}_k)}{d\mathbf{h}_k}, \tilde{\mathbf{h}}_k \rangle = \beta(\mathbf{h}_k, \tilde{\mathbf{h}}_k)_{\Gamma_0} - (\tilde{\mathbf{h}}_k, \boldsymbol{\varphi}_{k1} - \boldsymbol{\varphi}_{k2})_{\Gamma_0}. \quad (20)$$

Let $\frac{d\tilde{\mathcal{J}}(\mathbf{h}_k)}{d\mathbf{h}_k} = 0$, yields that

$$\beta(\mathbf{h}_k, \tilde{\mathbf{h}}_k)_{\Gamma_0} = (\tilde{\mathbf{h}}_k, \boldsymbol{\varphi}_{k1} - \boldsymbol{\varphi}_{k2})_{\Gamma_0}, \quad \forall \tilde{\mathbf{h}}_k \in L^2(\Gamma_0) \quad (21)$$

Note that (21) yields an explicit formula for the gradient of $\tilde{\mathcal{J}}_k$, i.e.,

$$\frac{d\tilde{\mathcal{J}}_k(\mathbf{h}_k)}{d\mathbf{h}_k} = \beta \mathbf{h}_k - (\boldsymbol{\varphi}_{k1}(\mathbf{h}) - \boldsymbol{\varphi}_{k2}(\mathbf{h}))|_{\Gamma_0}, \quad (22)$$

where $\boldsymbol{\varphi}_{ki}(\mathbf{h}_k)$ are determined from $\mathbf{h}_k$ through (11) and (17) for $i = 1, 2$. Then, by using a gradient method, e.g., a method that requires $\tilde{\mathcal{J}}(\mathbf{h}_k)$ and $\frac{d\tilde{\mathcal{J}}(\mathbf{h}_k)}{d\mathbf{h}_k}$ for a given approximation of $\mathbf{h}_k$ to solve our optimization problem (14).

The optimality system of (11), (17) and (22) is a coupled system whose solutions yield solutions of the optimization problem (14). So, we can use a parallelizable algorithm to solve the coupled system. Here, we choose a gradient method iteration to accomplish the goal. Given a starting guess $\mathbf{h}_k^{(0)}$, let

$$\mathbf{h}_k^{(n+1)} = \mathbf{h}_k^{(n)} - \frac{\gamma}{\beta} \frac{d\tilde{\mathcal{J}}_k(\mathbf{h}_k^{(n)})}{d\mathbf{h}_k}, \quad \text{for } n = 1, 2, \dots, \quad (23)$$

where γ/β is a step size. Combining with (22) yields

$$\mathbf{h}_k^{(n+1)} = (1 - \gamma) \mathbf{h}_k^{(n)} + \frac{\gamma}{\beta} (\boldsymbol{\varphi}_{k1}^{(n)} - \boldsymbol{\varphi}_{k2}^{(n)}), \quad \text{for } k = 0, 1, 2.$$

The main algorithm is given as follows.

The Main Algorithm:

- (1) Choose a $\mathbf{h}_k^{(0)}$.
- (2) For $j = 1, 2, \dots$,

Table 1 The performances of the individual algorithms

α	Iteration number	CPU time (s)	e relative error
50	49	1545.86ss	6.6026e-009
80	37	837.05	4.91416e-009
100	28	744.578	4.84719e-009
150	37	932.656	4.8518e-009
180	Divergence	—	—

A. For $i = 1, 2$, compute $\mathbf{u}_{ki}^{(j)}, \Pi_{ki}^{(j)}$ by

$$\begin{cases} va_i(\mathbf{u}_{ki}^{(j)}, \mathbf{v}) + \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\mathbf{u}_{mi}^{(j)}, \mathbf{u}_{ni}^{(n)}, \mathbf{v}) - c_i(\mathbf{v}, \Pi_{ki}^{(j)}) \\ \quad + (-1)^{i+1} r(\mathbf{u}_{ki}^{(j)}, \mathbf{v})_{\Gamma_0} \\ = (\mathbf{f}_k, \mathbf{v})_{D_i} + (-1)^{i+1} (\mathbf{h}_k^{(j)}, \mathbf{v})_{\Gamma_0}, \quad \forall \mathbf{v} \in \mathbf{H}_{\Gamma_i}^1(D_i), \\ -c_i(\mathbf{u}_{ki}^{(j)}, q) = 0, \quad \forall q \in L^2(D_i) \end{cases}$$

B. Compute $\boldsymbol{\varphi}_{ki}^{(j)}, \mu_{ki}^{(j)}$ by

$$\begin{cases} va_i(\delta \mathbf{u}_{ki}, \boldsymbol{\varphi}_{ki}^{(j)}) + \mathbf{I}_{\{m=k\}} \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\delta \mathbf{u}_{mi}, \mathbf{u}_{ni}^{(j)}, \boldsymbol{\varphi}_{ki}^{(j)}) \\ \quad + \mathbf{I}_{\{n=k\}} \frac{1}{\langle \Phi_k^2 \rangle} \sum_{m=0}^P \sum_{n=0}^P e_{mnk} b_i(\mathbf{u}_{mi}^{(j)}, \delta \mathbf{u}_{ni}, \boldsymbol{\varphi}_{ki}^{(j)}) - c_i(\delta \mathbf{u}_{ki}, \mu_{ki}^{(j)}) \\ \quad + (-1)^{i+1} r(\delta \mathbf{u}_{ki}, \boldsymbol{\varphi}_{ki}^{(j)})_{\Gamma_0} \\ = (-1)^i \alpha(\mathbf{u}_{k1}^{(j)} - \mathbf{u}_{k2}^{(j)}, \delta \mathbf{u}_{ki})_{\Gamma_0}, \quad \forall \delta \mathbf{u}_{ki} \in \mathbf{H}_{\Gamma_i}^1(D_i) \\ -c_i(\boldsymbol{\varphi}_{ki}^{(j)}, \delta \Pi_{ki}) = 0, \quad \forall \delta \Pi_{ki} \in L^2(D_i). \end{cases}$$

C. Compute $\mathbf{h}_k^{(j+1)}$ by

$$\mathbf{h}_k^{(j+1)} = (1 - \gamma) \mathbf{h}_k^{(j)} + \frac{\gamma}{\beta} (\boldsymbol{\varphi}_{k1}^{(j)} - \boldsymbol{\varphi}_{k2}^{(j)}) \text{ on } \Gamma_0.$$

Remark 1 Compared with Kong et al. (2009), the method is used to solve the $P + 1 = \frac{(n+p)!}{n!p!}$ equations instead of single equation with constantly adjusting $\mathbf{h}_k$ in the solution process.

4 Numerical experiments: square cavity flow problem

One model is illustrated by using the optimization-based domain decomposition method for the incompressible NS flows with stochastic boundary conditions in the section. The computations have been carried out on a home PC with AMD Athlon(tm) 64 X2 Dual Core Processor 2.12 GHz and 1GB memory.

Consider a square cavity flow model with the top velocity $\mathbf{u} = (1 + \sigma \xi, 0)$ on unit square domain $D = (0, 1) \times (0, 1) \in \mathbb{R}^2$, where $\sigma = 0.2$ is the standard deviation of ξ . For the problem, the first condition is that D is divided into two subdomains $D_1 = (0, 1) \times (0.5, 1)$ and $D_2 = (0, 1) \times (0, 0.5)$ with the interface $\Gamma_0 = (0, 1) \times \{0.5\}$ (see Fig. 2).

In the process of computing, assume that ξ is one dimensional Gaussian variable and the order of Hermite polynomial chaos is 2. Similar to the treatment of field variables, a "weak formulation" approach is adopted for the boundary conditions. For brevity, we only illustrate this approach for the top wall condition. Expanding the $\mathbf{u}$ in terms of the polynomial

Fig. 2 The mesh **Fig. 3** Velocity field, left: the numerical solution in the whole domain D ; right: the numerical solution in subdomains D_1 and D_2

chaos system, we have $\mathbf{u}_0(\mathbf{x}) = (1, 0)$, $\mathbf{u}_1(\mathbf{x}) = (\sigma, 0)$, $\mathbf{u}_2(\mathbf{x}) = (0, 0)$, $\mathbf{x} \in (0, 1) \times 1$. The convergence criterion is defined as,

$$\frac{\|\mathbf{u}_k^{j+1} - \mathbf{u}_k^j\|}{\|\mathbf{u}_k^1 - \mathbf{u}_k^0\|} \leq \varepsilon, \quad \forall k \in \{0, 1, 2\}.$$

The penalty parameter $\beta = 10^{-8}$, γ is chosen to be 2×10^{-8} and the mean-square error between the numerical solution from finite element method and the one from optimization-based domain decomposition method is computed,

$$e = E \left[\frac{\|\mathbf{u}_o - \mathbf{u}\|_{D_1} + \|\mathbf{u}_o - \mathbf{u}\|_{D_2}}{\|\mathbf{u}_o\|_D} \right]. \quad (24)$$

where E denotes the 'expectation' operator, $\mathbf{u}_o$ and $\mathbf{u}$ is the numerical solution from finite element method and optimization-based domain decomposition method respectively. For the mesh with 10×10 elements (see Fig. 2), Table 1 lists the experiment results for different

Fig. 4 The horizontal component of the velocity, left: the numerical solution in the whole domain D ; right: the numerical solution in subdomains D_1 and D_2

Fig. 5 The vertical component of the velocity, left: the numerical solution in the whole domain D ; right: the numerical solution in subdomains D_1 and D_2

α , and Figs. 3 and 5 picture the velocity field and the contours of the mean u and v velocity with $\alpha = 50$. Figures 6 and 7 give the contours of the variance of the u and v velocity with $\alpha = 50$.

From Fig. 6, we see that the numerical variance 0.039 is much accord with the assumption $\sigma^2 = 0.04$.

5 Conclusion

In this paper, the NSEs system with stochastic input are transformed into deterministic ones via the polynomial chaos expansion, the optimization-based domain decomposition method for the deterministic NSEs system is used to numerical simulation of the NS-like equations. The Lagrange multiplier rule transforms the constrained optimization problem into an uncon-

Fig. 6 The horizontal component of the variance. Left: the numerical solution in the whole domain D ; right: the numerical solution in subdomains D_1 and D_2

Fig. 7 The vertical component of the variance. Left: the numerical solution in the whole domain D ; right: the numerical solution of the second condition in subdomains D_1 and D_2

strained one, "sensitivity" derivatives is applied to obtain the optimality condition, a gradient method-based approach to the solution of domain decomposition problem is constructed.

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