Some improvements on the Ky Fan theorem for tensors

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Abstract

The improvements of Ky Fan theorem are given for tensors. First, based on Brauer-type eigenvalue inclusion sets, we obtain some new Ky Fan-type theorems for tensors. Second, by characterizing the ratio of the smallest and largest values of a Perron vector, we improve the existing results. Third, some new eigenvalue localization sets for tensors are given and proved to be tighter than those presented by Li and Ng (Numer Math 130(2):315–335, 2015) and Wang et al. (Linear Multilinear Algebra 68(9):1817–1834, 2020). Finally, numerical examples are given to validate the efficiency of our new bounds.

Keywords Ky Fan theorem · Tensor eigenvalue · Eigenvalue inclusion set

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1 Introduction

Let $\mathbb{C}(\mathbb{R})$ be the set of all complex (real) numbers, and $\mathbb{C}^n(\mathbb{R}^n)$ be the set of all dimension n complex (real) vectors. An order m dimension n complex (real) tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$, denoted by $\mathcal{A} \in \mathbb{C}^{[m, n]}$ ($\mathcal{A} \in \mathbb{R}^{[m, n]}$, respectively), consists of n^m entries:

$$a_{i_1 i_2 \dots i_m} \in \mathbb{C}(\mathbb{R}), \quad \forall i_j = 1, \dots, n, \quad j = 1, \dots, m.$$

A tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m, n]}$ is called nonnegative (positive) if:

$$a_{i_1 i_2 \dots i_m} \geq 0 \quad (a_{i_1 i_2 \dots i_m} > 0), \quad \forall i_j = 1, \dots, n, \quad j = 1, \dots, m.$$

Tensors have many similarities with matrices and many related results of matrices such as eigenvector and eigenvalue can be extended to higher order tensors (e.g., see Qi and Luo 2017; Wei and Ding 2017).

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For a vector $x \in \mathbb{C}^n$, we use x_i to denote its components and $x^{[m-1]}$ to denote a vector in $\mathbb{C}^n$, such that:

$$x_i^{[m-1]} = x_i^{m-1} \quad \text{for all } i.$$

$\mathcal{A}x^{m-1}$ denotes a vector in $\mathbb{C}^n$, whose i th component is:

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, i_3, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m}.$$

A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenpair (eigenvalue–eigenvector pair) of $\mathcal{A}$ if they satisfy:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

Specifically, (λ, x) is called an H-eigenpair if $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$.

With the introduction of tensor eigenvalue, it is of interest to investigate eigenvalue inclusion regions, i.e., to find the regions including all eigenvalues in the complex plane. The first work owes to Liqun Qi. Qi (2005) gave an eigenvalue inclusion set for real symmetric tensors, which is a generalization of the well-known Gershgorin set of matrices (Horn and Johnson 2012). Subsequently, due to its fundamental applications in various fields, many researchers are interested in investigating eigenvalue inclusion regions for tensors, e.g., Bu et al. (2017), Li and Li (2016a), Li et al. (2014, 2020), Xu et al. (2019), etc.

In 1958, Fan (1958) obtained the famous Ky Fan theorem that plays an important role in the study of the nonnegative eigenvalue problem. Li and Huang (2005) presented an improvement on the Ky Fan theorem for any weakly irreducible $n \times n$ complex matrix A . Yang and Yang (2010) generalized the Ky Fan-type theorem from matrices to tensors. Later, Li and Ng (2015) improved it, and recently, Wang et al. (2020) proved some new Ky Fan-type theorems for weakly irreducible nonnegative tensors with Brualdi-type eigenvalue inclusion set. In this paper, we provide some new improvements on the Ky Fan theorem, which improves the existing bounds.

2 Preliminaries

We begin this section with some definitions and statements that are needed for the main results of our work which are taken from Bu et al. (2017), Li and Li (2016a, b), Li et al. (2014), Qi and Luo (2017), and Yang and Yang (2010).

Definition 1 Let $\mathcal{A}$ be a tensor of order m dimension n .

- (i) We call $\sigma(\mathcal{A})$ as the set of all eigenvalues of $\mathcal{A}$ and the spectral radius of $\mathcal{A}$ is denoted by:

$$\rho(\mathcal{A}) = \max \{ |\lambda| : \lambda \in \sigma(\mathcal{A}) \}.$$

- (ii) We call a tensor $\mathcal{A}$ reducible if there exists a nonempty proper index subset $I \subset \langle n \rangle := \{1, 2, \dots, n\}$, such that:

$$a_{i_1 i_2 \dots i_m} = 0, \quad \forall i_1 \in I, \quad i_2, \dots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

- (iii) We call a tensor $\mathcal{A}$ nonnegative weakly irreducible, if, for any nonempty proper index subset $I \subset \langle n \rangle$, there is at least an entry $a_{i_1 i_2 \dots i_m} > 0$, where $i_1 \in I$, and at least an $i_j \notin I$, $j = 2, \dots, m$.
- (iv) We denote by $\delta_{i_1 i_2 \dots i_m}$, the Kronecker symbol for the case of m indices, that is:

$$\delta_{i_1 i_2 \dots i_m} = \begin{cases} 1 & i_1 = i_2 = \dots = i_m \\ 0 & \text{otherwise.} \end{cases}$$

Let us recall the Perron–Frobenius theorem for irreducible nonnegative tensors given in Yang and Yang (2010).

Theorem 1 (see Theorem 2.2 of Yang and Yang 2010) *Suppose that $\mathcal{A}$ is an irreducible nonnegative tensor of order m dimension n . Then, $\rho(\mathcal{A}) > 0$ is an eigenvalue of $\mathcal{A}$ with a positive eigenvector x corresponding to it.*

Remark 1 It is noted that the spectral radius $\rho(\mathcal{A})$ is the largest H-eigenvalue for the non-negative tensor Yang and Yang (2010).

Note that $\rho(\mathcal{A})$ and x in Theorem 1 are called the Perron root and the Perron vector of $\mathcal{A}$, respectively, and $(\rho(\mathcal{A}), x)$ is regarded as a Perron eigenpair.

Definition 2 If $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ is a real tensor of order m dimension n and D is a matrix, then the product of $\mathcal{A}$ and D is defined as follows:

$$\mathcal{A}_D = \mathcal{A} \cdot D^{-(m-1)} \cdot \overbrace{D \dots D}^{m-1},$$

where

$$(\mathcal{A}_D)_{i_1 i_2 \dots i_m} = \sum_{j_1, j_2, \dots, j_m=1}^n a_{j_1 \dots j_m} D_{i_1, j_1}^{-(m-1)} D_{i_2, j_2} \dots D_{i_m, j_m}.$$

If D is a diagonal matrix, then:

$$(\mathcal{A}_D)_{i_1 i_2 \dots i_m} = a_{i_1 \dots i_m} D_{i_1, i_1}^{-(m-1)} D_{i_2, i_2} \dots D_{i_m, i_m}.$$

Remark 2 (Yang and Yang 2010) In fact, if:

$$\mathcal{A}_D = \mathcal{A} \cdot D^{-(m-1)} \cdot \overbrace{D \dots D}^{m-1},$$

where D is a diagonal nonsingular matrix, and then, $\mathcal{A}_D$ and $\mathcal{A}$ have the same eigenvalues. If λ is an eigenvalue of $\mathcal{A}_D$ with corresponding eigenvector x , then λ is also an eigenvalue of $\mathcal{A}$ with corresponding eigenvector Dx ; if μ is an eigenvalue of $\mathcal{A}$ with corresponding eigenvector y , then μ is an eigenvalue of $\mathcal{A}_D$ with corresponding eigenvector $D^{-1}y$.

In recent years, the spectral theory of tensors has attracted much attention. In 2005, Qi and Luo (2017) gave an eigenvalue localization set for real symmetric tensors, which is a generalization of the well-known Gershgorin's eigenvalue localization theorem of matrices (Horn and Johnson 2012; Varga 2004). This result has also been generalized to general tensors (Li et al. 2014; Yang and Yang 2010).

Theorem 2 (Li et al. 2014, Theorem 1.1) *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be a tensor of order m dimension n . Then:*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in \langle n \rangle} \Gamma_i(\mathcal{A}),$$

where $\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{ii \dots i}| \leq r_i(\mathcal{A})\}$, and $r_i(\mathcal{A}) = \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{i i_2 \dots i_m}=0}}^n |a_{i i_2 \dots i_m}|$.

The well-known Brauer's eigenvalue inclusion set of matrices was given in Brauer (1947), which is always contained in the Gershgorin set. Recently, Li et al. (2014) extended the Brauer's eigenvalue inclusion set of matrices to tensors, and gave the following Brauer-type eigenvalue inclusion set for tensors.

Theorem 3 (Li et al. 2014, Theorem 2.1) *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be a tensor of order m dimension n , and then:*

$$\sigma(\mathcal{A}) \subseteq \mathcal{F}(\mathcal{A}) = \bigcup_{i, j \in \langle n \rangle, j \neq i} \mathcal{F}_{i,j}(\mathcal{A}),$$

where

$$\mathcal{F}_{i,j} = \left\{ z \in \mathbb{C} : \left(|z - a_{ii \dots i}| - r_i^j(\mathcal{A}) \right) |z - a_{jj \dots j}| \leq |a_{ij \dots j}| r_j(\mathcal{A}) \right\},$$

and

$$r_i^j(\mathcal{A}) := \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{i i_2 \dots i_m}=0 \\ \delta_{j i_2 \dots i_m}=0}}^n |a_{i i_2 \dots i_m}| = r_i(\mathcal{A}) - |a_{ij \dots j}|. \quad (1)$$

Very recently, Bu et al. (2017) gave another Brauer-type eigenvalue localization sets of tensors as follows, which is proved to be tighter than the sets $\Gamma(\mathcal{A})$ and $\mathcal{F}(\mathcal{A})$.

Theorem 4 (Bu et al. 2017, Theorem 3.1) *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be a tensor of order m dimension n . Then, every eigenvalue of $\mathcal{A}$ lies in the region:*

$$\mathfrak{B}(\mathcal{A}) = \bigcup_{\substack{i, j \in \langle n \rangle \\ i \neq j}} \left\{ z \in \mathbb{C} : |z - a_{ii \dots i}|^{m-1} |z - a_{jj \dots j}| \leq r_i(\mathcal{A})^{m-1} r_j(\mathcal{A}) \right\}. \quad (2)$$

Theorem 5 (Bu et al. 2017, Theorem 3.3) *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be a tensor of order m dimension n , and $r_i(\mathcal{A}) \neq 0$, $i = 1, \dots, n$. Then, every eigenvalue of $\mathcal{A}$ lies in the region:*

$$\mathcal{Z}(\mathcal{A}) = \bigcup_{\substack{a_{i_1 i_2 \dots i_m} \neq 0 \\ \delta_{i_1 i_2 \dots i_m} = 0}} \left\{ z \in \mathbb{C} : \prod_{j=1}^m |z - a_{i_j i_j \dots i_j}| \leq \prod_{j=1}^m r_{i_j}(\mathcal{A}) \right\}.$$

To find such relationships between other Brauer-type eigenvalue localization sets, see (Li and Li 2016b; Xu et al. 2019).

3 Improvements of Ky Fan theorem

The Ky Fan theorem (e.g., see Theorem 8.2.12 of Horn and Johnson 2012) plays an important role in estimating the eigenvalues of matrices. Yang and Yang (2010, Theorem 4.1), gave the Ky Fan-type inequality for tensors as follows:

Theorem 6 Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \dots i_m})$ be complex tensors of order m dimension n with $|\mathcal{A}| \leq \mathcal{B}$. If

$$\mathcal{K}^{(i)}(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{ii \dots i}| \leq \rho(\mathcal{B}) - b_{ii \dots i}\}, \quad i = 1, \dots, n,$$

denotes the i th Ky Fan disk for $\mathcal{A}$, then, the union of these n Ky Fan disks contains all eigenvalues of $\mathcal{A}$, that is:

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) := \bigcup_{i=1}^n \mathcal{K}^{(i)}(\mathcal{A}).$$

The Ky Fan-type theorem naturally extends from matrices to tensors (see Yang and Yang 2010). Later, Li and Ng (2015) improved it by applying the ratio of the entries in the Perron vector, and recently, Wang et al. (2020) proved some new Ky Fan-type theorems for weakly irreducible nonnegative tensors with Brualdi-type eigenvalue inclusion sets. In the following sections, we will present some improvements of Ky Fan theorem. Also, we will give examples showing that our results are sharp.

3.1 Improvements of Ky Fan theorem based on Brualdi-type sets

In this section, an improvement of Ky Fan theorem is suggested by Brauer-type eigenvalue inclusion set $\mathcal{Z}(\mathcal{A})$, as defined in Theorem 5.

Theorem 7 Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \dots i_m})$ be complex tensors of order m dimension n . If $|\mathcal{A}| \leq \mathcal{B}$, and $r_i(\mathcal{A}) \neq 0$, $i = 1, \dots, n$, then:

$$\sigma(\mathcal{A}) \subseteq \bigcup_{\substack{a_{i_1 i_2 \dots i_m} \neq 0 \\ \delta_{i_1 i_2 \dots i_m} = 0}} \left\{ z \in \mathbb{C} : \prod_{j=1}^m |z - a_{i_j i_j \dots i_j}| \leq \prod_{j=1}^m (\rho(\mathcal{B}) - b_{i_j i_j \dots i_j}) \right\} := \mathcal{Z}_1(\mathcal{A}).$$

Proof We may assume that $\mathcal{B} > 0$ (if some entries of $\mathcal{B}$ are zero, we may consider $\mathcal{B}_\varepsilon \equiv (b_{i_1 i_2 \dots i_m} + \varepsilon)$, for $\varepsilon > 0$; $\mathcal{B}_\varepsilon > |\mathcal{A}|$ and $\rho(\mathcal{B}_\varepsilon) - (b_{ii \dots i} + \varepsilon) \rightarrow \rho(\mathcal{B}) - b_{ii \dots i}$ when $\varepsilon \rightarrow 0$). By Perron–Frobenius theorem (Chang et al. 2008), there exists a positive vector $x = (x_1, x_2, \dots, x_n)^T$, such that:

$$\mathcal{B}x^{m-1} = \rho(\mathcal{B})x^{[m-1]}. \quad (3)$$

Let $X := \text{diag}(x_1, x_2, \dots, x_n)$, and $\mathcal{A}_X := \mathcal{A}X^{-(m-1)} \overbrace{X \dots X}^{m-1}$. Then, by Remark 2 $\sigma(\mathcal{A}) = \sigma(\mathcal{A}_X)$ and $|(\mathcal{A}_X)_{ii \dots i}| = |a_{ii \dots i}| < b_{ii \dots i}$, for all $i \in \langle n \rangle$.

By Theorem 5 for any eigenvalue λ of $\mathcal{A}_X$, we have:

$$\prod_{j=1}^m |\lambda - a_{i_j i_j \dots i_j}| \leq \prod_{j=1}^m r_{i_j}(\mathcal{A}_X) = \prod_{j=1}^m \left(\sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{i_j i_2 \dots i_m}=0}}^n \frac{|a_{i_j i_2 \dots i_m}| x_{i_2} \dots x_{i_m}}{x_{i_j}^{m-1}} \right)$$

$$\begin{aligned}
&\leq \prod_{j=1}^m \left(\sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{ij i_2 \dots i_m}=0}}^n \frac{b_{ij i_2 \dots i_m} x_{i_2} \dots x_{i_m}}{x_{ij}^{m-1}} \right) \\
&= \prod_{j=1}^m \left(\sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{ij i_2 \dots i_m}=0}}^n \frac{b_{ij i_2 \dots i_m} x_{i_2} \dots x_{i_m}}{x_{ij}^{m-1}} + b_{ij i_j \dots i_j} - b_{ij i_j \dots i_j} \right) \\
&= \prod_{j=1}^m \left(\sum_{i_2, \dots, i_m=1}^n \frac{b_{ij i_2 \dots i_m} x_{i_2} \dots x_{i_m}}{x_{ij}^{m-1}} - b_{ij i_j \dots i_j} \right) \\
&= \prod_{j=1}^m (\rho(\mathcal{B}) - b_{ij i_j \dots i_j}).
\end{aligned}$$

Thus, we have:

$$\sigma(\mathcal{A}) \subseteq \bigcup_{\substack{a_{i_1 i_2 \dots i_m} \neq 0 \\ \delta_{i_1 i_2 \dots i_m} = 0}} \left\{ z \in \mathbb{C} : \prod_{j=1}^m |z - a_{ij i_j \dots i_j}| \leq \prod_{j=1}^m (\rho(\mathcal{B}) - b_{ij i_j \dots i_j}) \right\} := \mathcal{Z}_1(\mathcal{A}).$$

The theorem is proved. $\square$

Remark 3 Note that in Theorem 7, $\sigma(\mathcal{A}) \subseteq \mathcal{Z}_1(\mathcal{A})$ is also obtained by removing the condition $r_i(\mathcal{A}) \neq 0$, $i = 1, \dots, n$.

Similarly, using Theorem 4 and the method of proof employed in Theorem 7, we have the following:

Corollary 1 Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \dots i_m})$ be complex tensors of order m dimension n with $|\mathcal{A}| \leq \mathcal{B}$. Then, every eigenvalue of $\mathcal{A}$ lies in the region:

$$\begin{aligned}
&\bigcup_{\substack{i, j=1 \\ i \neq j}}^n \left\{ z \in \mathbb{C} : |z - a_{ii \dots i}|^{m-1} \cdot |z - a_{jj \dots j}| \leq (\rho(\mathcal{B}) - b_{ii \dots i})^{m-1} (\rho(\mathcal{B}) - b_{jj \dots j}) \right\} \\
&:= \mathfrak{B}_1(\mathcal{A}).
\end{aligned}$$

3.2 Improvements of Ky Fan theorem based on the ratio of the entries in the Perron vector

Li and Ng (2015) considered lower bound the problem of estimating the ratio $\frac{x_{\min}}{x_{\max}}$ for an irreducible nonnegative tensor $\mathcal{A}$ as follows:

$$\kappa(\mathcal{A}) \leq \frac{x_{\min}}{x_{\max}}, \quad (4)$$

where $x_{\min} = \min_{1 \leq i \leq n} x_i$ and $x_{\max} = \max_{1 \leq i \leq n} x_i$, and:

$$\kappa(\mathcal{A}) = \max_{2 \leq k, k' \leq m} \min_{\substack{1 \leq i_1, i'_1 \leq n \\ 1 \leq i_k = i'_k \leq n}} \frac{\sum_{\substack{i_2, \dots, i_m=1 \\ \text{except } i_k}}^n a_{i_1, \dots, i_m}}{\sum_{\substack{i'_2, \dots, i'_m=1 \\ \text{except } i'_k}}^n a_{i'_1, \dots, i'_m}}. \quad (5)$$

Using similar technique, the lower bound of (4) can be further improved as below (see Lemma 4.1 in Wang et al. 2020):

$$\iota(\mathcal{A}) \leq \left(\frac{x_{\min}}{x_{\max}} \right)^2, \quad (6)$$

where

$$\iota(\mathcal{A}) = \max_{2 \leq k, k', l, l' \leq m} \min_{\substack{1 \leq i_1, i'_1 \leq n \\ 1 \leq i_k = i'_k \leq n \\ 1 \leq i_l = i'_l \leq n}} \frac{\sum_{\substack{i_2, \dots, i_m=1 \\ \text{except } i_k, i_l}}^n a_{i_1, \dots, i_m}}{\sum_{\substack{i'_2, \dots, i'_m=1 \\ \text{except } i'_k, i'_l}}^n a_{i'_1, \dots, i'_m}}. \quad (7)$$

Also, using similar technique as presented in Wang et al. (2020, Lemma 4.1) (or Li and Ng 2015, Lemma 3.2), we obtain the lower bound for the ratio of the smallest and largest entries in a Perron vector.

Lemma 1 Let $\mathcal{A}$ be a weakly irreducible nonnegative tensor of order m dimension n with maximal eigenvector x . Then:

$$\tau(\mathcal{A}) = \max_{1 \leq k \leq m-1} \tau^{(k)}(\mathcal{A}) \leq \frac{x_{\min}}{x_{\max}}, \quad (8)$$

where

$$\tau^{(k)}(\mathcal{A}) = \max_{\substack{\{j_1, \dots, j_k\} \subseteq \{2, \dots, m\} \\ \{j'_1, \dots, j'_k\} \subseteq \{2, \dots, m\}}} \min_{\substack{1 \leq i_1, i'_1 \leq n \\ 1 \leq i_{j_t} = i'_{j'_t} \leq n \\ 1 \leq t \leq k}} \left(\frac{\sum_{\substack{i_2, \dots, i_m=1 \\ \text{except } i_{j_1}, \dots, i_{j_k}}}^n a_{i_1, \dots, i_m}}{\sum_{\substack{i'_2, \dots, i'_m=1 \\ \text{except } i'_{j'_1}, \dots, i'_{j'_k}}}^n a_{i'_1, \dots, i'_m}} \right)^{\frac{1}{k}}. \quad (9)$$

Proof Let $x_\alpha = \min x_i$ and $x_\beta = \max x_i$, Then, by (3), we have:

$$\rho(\mathcal{A})(x_\alpha)^{m-1} = \sum_{i_2, \dots, i_m=1}^n a_{\alpha, i_2, \dots, i_m} x_{i_2} \dots x_{i_m}$$

$$\geq \left(\sum_{i_2, \dots, i_m=1}^n a_{\alpha, i_2, \dots, i_m} x_{i_{j_1}} \dots x_{i_{j_k}} \right) x_{\alpha}^{m-1-k},$$

where $\{j_1, \dots, j_k\} \subseteq \{2, \dots, m\}$. It implies that:

$$\rho(\mathcal{A})(x_{\alpha})^k \geq \sum_{i_2, \dots, i_m=1}^n a_{\alpha, i_2, \dots, i_m} x_{i_{j_1}} \dots x_{i_{j_k}}.$$

Similarly, we get:

$$\rho(\mathcal{A})(x_{\beta})^k \leq \sum_{i_2, \dots, i_m=1}^n a_{\beta, i_2, \dots, i_m} x_{i_{j_1}} \dots x_{i_{j_k}}.$$

Therefore:

$$\begin{aligned} \left(\frac{x_{\alpha}}{x_{\beta}} \right)^k &\geq \frac{\sum_{i_2, \dots, i_m=1}^n a_{\alpha, i_2, \dots, i_m} x_{i_{j_1}} \dots x_{i_{j_k}}}{\sum_{i'_2, \dots, i'_m=1}^n a_{\beta, i'_2, \dots, i'_m} x_{i'_{j_1}} \dots x_{i'_{j_k}}} \\ &= \frac{\sum_{i_{j_1}, \dots, i_{j_k}=1}^n \underbrace{\sum_{i_2, \dots, i_m=1}^n a_{\alpha, i_2, \dots, i_m} x_{i_{j_1}} \dots x_{i_{j_k}}}_{\text{except } i_{j_1}, \dots, i_{j_k}}}{\sum_{i'_{j_1}, \dots, i'_{j_k}=1}^n \underbrace{\sum_{i'_2, \dots, i'_m=1}^n a_{\beta, i'_2, \dots, i'_m} x_{i'_{j_1}} \dots x_{i'_{j_k}}}_{\text{except } i'_{j_1}, \dots, i'_{j_k}}} \\ &\geq \min_{\substack{1 \leq i_{j_t} = i'_{j_t} \leq n \\ 1 \leq t \leq k}} \frac{\sum_{\substack{i_2, \dots, i_m=1 \\ \text{except } i_{j_1}, \dots, i_{j_k}}}^n a_{\alpha, i_2, \dots, i_m}}{\sum_{\substack{i'_2, \dots, i'_m=1 \\ \text{except } i'_{j_1}, \dots, i'_{j_k}}}^n a_{\beta, i'_2, \dots, i'_m}} \\ &\geq \min_{1 \leq i_1 = i'_1 \leq n} \min_{\substack{1 \leq i_{j_t} = i'_{j_t} \leq n \\ 1 \leq t \leq k}} \frac{\sum_{\substack{i_2, \dots, i_m=1 \\ \text{except } i_{j_1}, \dots, i_{j_k}}}^n a_{i_1, i_2, \dots, i_m}}{\sum_{\substack{i'_2, \dots, i'_m=1 \\ \text{except } i'_{j_1}, \dots, i'_{j_k}}}^n a_{i'_1, i'_2, \dots, i'_m}}, \end{aligned}$$

which proves the lemma. $\square$

Based on (4), Li and Ng (2015, Theorem 3.3) proposed the following theorem.

Theorem 8 Let $\mathcal{A}$ and $\mathcal{B}$ be nonnegative tensors of order m dimension n . If $\mathcal{B}$ is irreducible with $|\mathcal{A}| \leq \mathcal{B}$. Then, for any eigenvalue λ of $\mathcal{A}$, there exists i , such that:

$$|\lambda - a_{i\dots i}| \leq \rho(\mathcal{B}) - b_{ii\dots i} - (\kappa(\mathcal{B}))^{m-1} r_i(\mathcal{E}), \quad (10)$$

where $\mathcal{E} = \mathcal{B} - |\mathcal{A}|$ and $\kappa(\mathcal{B})$ is given in (5).

Recently, Wang et al. (2020, Theorem 4.6) improved the eigenvalue inclusion set (10), using (6) and $v(\mathcal{A}) = \frac{l}{\rho(\mathcal{A}) - R_{\min}(\mathcal{A}) + l}$, which $l = \min_{\delta(i_1, \dots, i_m)=0} a_{i_1 \dots i_m}$ and $R_i(\mathcal{A}) = \sum_{i_2, \dots, i_m=1}^n |a_{i, i_2, \dots, i_m}|$.

Theorem 9 Let $\mathcal{A}$ be a weakly irreducible tensors of order m dimensional n , and $\mathcal{B}$ be a tensors of order m dimensional n with $|\mathcal{A}| \leq \mathcal{B}$. Then, for any eigenvalue λ of $\mathcal{A}$, there exists $\gamma \in C(\mathcal{A})$, such that:

$$\prod_{i \in \gamma} |\lambda - a_{i\dots i}| \leq \prod_{i \in \gamma} (\rho(\mathcal{B}) - b_{ii\dots i}) - P^{|\gamma|} \prod_{i \in \gamma} r_i(\mathcal{E}), \quad (11)$$

where $P = \max\{(\kappa(\mathcal{B}))^{m-1}, (\iota(\mathcal{B}))^{(m-1)/2}, v(\mathcal{B})\}$, $\mathcal{E} = \mathcal{B} - |\mathcal{A}|$ and $C(\mathcal{A})$ is the set of circuits of digraph $\mathcal{A}$ and $|\gamma|$ denotes the length of circuit γ .

Based on Brauer-type eigenvalue inclusion sets (Theorem 5) and estimating the ratio of the smallest and largest values of a Perron vector (Lemma 1), we can obtain the following Ky Fan-type theorem.

Theorem 10 Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \dots i_m})$ be complex tensors of order m dimension n with $|\mathcal{A}| \leq \mathcal{B}$. Then, every eigenvalue of $\mathcal{A}$ lies in the region:

$$\bigcup_{\substack{a_{i_1 i_2 \dots i_m} \neq 0 \\ \delta_{i_1 i_2 \dots i_m} = 0}} \left\{ z \in \mathbb{C} : \prod_{j=1}^m |z - a_{i_j i_j \dots i_j}| \leq \prod_{j=1}^m (\rho(\mathcal{B}) - b_{i_j i_j \dots i_j} - (\tau(\mathcal{B}))^{m-1} r_{i_j}(\mathcal{E})) \right\} := \mathcal{Z}_2(\mathcal{A}),$$

where $\mathcal{E} = \mathcal{B} - |\mathcal{A}|$ and $\tau(\mathcal{B})$ is given in (8).

Proof The proof of this theorem runs parallel to proof for the corresponding part of Theorem 7. We may assume that $\mathcal{B} > 0$. By Perron–Frobenius theorem (Chang et al. 2008), there exists a positive vector $x = (x_1, x_2, \dots, x_n)^T$, such that:

$$\mathcal{B}x^{m-1} = \rho(\mathcal{B})x^{[m-1]}.$$

Let $X := \text{diag}(x_1, x_2, \dots, x_n)$, $\mathcal{A}_X := \mathcal{A}X^{-(m-1)} \overbrace{X \dots X}^{m-1}$, and $\mathcal{B}_X := \mathcal{B}X^{-(m-1)} \overbrace{X \dots X}^{m-1}$. Then, by Remark 2 for all $i \in N$, we have:

$$\begin{aligned} \sigma(\mathcal{A}) &= \sigma(\mathcal{A}_X), \quad (\mathcal{A}_X)_{ii\dots i} = a_{ii\dots i}, \\ \sigma(\mathcal{B}) &= \sigma(\mathcal{B}_X), \quad (\mathcal{B}_X)_{ii\dots i} = b_{ii\dots i}. \end{aligned}$$

Let $\mathcal{E} = \mathcal{B} - |\mathcal{A}|$ and $\mathcal{E}_X = \mathcal{B}_X - |\mathcal{A}|_X$. By Theorem 5 for any eigenvalue λ of $\mathcal{A}_X$, we have:

$$\begin{aligned} \prod_{j=1}^m |\lambda - a_{i_j i_j \dots i_j}| &\leq \prod_{j=1}^m r_{i_j}(\mathcal{A}_X) = \prod_{j=1}^m r_{i_j}(\mathcal{B}_X - \mathcal{E}_X) \\ &= \prod_{j=1}^m (r_{i_j}(\mathcal{B}_X) - r_{i_j}(\mathcal{E}_X)) \end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^m \left(\rho(\mathcal{B}) - b_{i_j i_j \dots i_j} - \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{i_j i_2 \dots i_m}=0}}^n \frac{\mathcal{E}_{i_j i_2 \dots i_m} x_{i_2} \dots x_{i_m}}{x_{i_j}^{m-1}} \right) \\
&\leq \prod_{j=1}^m \left(\rho(\mathcal{B}) - b_{i_j i_j \dots i_j} - \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{i_j i_2 \dots i_m}=0}}^n \mathcal{E}_{i_j i_2 \dots i_m} \left(\frac{x_{\min}}{x_{\max}} \right)^{m-1} \right) \\
&\leq \prod_{j=1}^m \left(\rho(\mathcal{B}) - b_{i_j i_j \dots i_j} - (\tau(\mathcal{B}))^{m-1} \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{i_j i_2 \dots i_m}=0}}^n \mathcal{E}_{i_j i_2 \dots i_m} \right) \\
&= \prod_{j=1}^m \left(\rho(\mathcal{B}) - b_{i_j i_j \dots i_j} - (\tau(\mathcal{B}))^{m-1} r_{i_j}(\mathcal{E}) \right),
\end{aligned}$$

where the last inequality follows from (8). $\square$

Similarly, using Theorem 4 and the method of proof employed in Theorem 10, we can obtain improvement of (2) as follows:

Corollary 2 Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \dots i_m})$ be complex tensors of order m dimension n with $|\mathcal{A}| \leq \mathcal{B}$. Then, every eigenvalue of $\mathcal{A}$ lies in the region:

$$\bigcup_{\substack{i, j=1 \\ i \neq j}}^n \{z : |z - a_{i \dots i}|^{m-1} |z - a_{j \dots j}| \leq (\rho(\mathcal{B}) - b_{i \dots i} - (\tau(\mathcal{B}))^{m-1} r_i(\mathcal{E}))^{m-1} (\rho(\mathcal{B}) - b_{j \dots j} - (\tau(\mathcal{B}))^{m-1} r_j(\mathcal{E}))\} := \mathfrak{B}_2(\mathcal{A}).$$

In the following, we consider the relationships between Ky Fan theorem and its improvements.

Theorem 11 Let $\mathcal{A}$ be tensor of order m dimension n , and then:

$$\mathcal{Z}_1(\mathcal{A}) \subseteq \mathfrak{B}_1(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}). \quad (12)$$

Proof First, consider the right-hand inclusion of (12). Fix any i, j ($1 \leq i, j \leq n$; $i \neq j$) and let z be any point of $\mathfrak{B}_1^{(i,j)}(\mathcal{A})$, where:

$$\mathfrak{B}_1^{(i,j)}(\mathcal{A}) := |z - a_{i \dots i}|^{m-1} |z - a_{j \dots j}| \leq (\rho(\mathcal{B}) - b_{i \dots i})^{m-1} (\rho(\mathcal{B}) - b_{j \dots j}).$$

By Corollary 1, we have:

$$|z - a_{i \dots i}|^{m-1} |z - a_{j \dots j}| \leq (\rho(\mathcal{B}) - b_{i \dots i})^{m-1} (\rho(\mathcal{B}) - b_{j \dots j}), \quad i, j \in \langle n \rangle, \quad i \neq j. \quad (13)$$

If $(\rho(\mathcal{B}) - b_{i \dots i})^{m-1} (\rho(\mathcal{B}) - b_{j \dots j}) = 0$, then:

$$z = a_{i \dots i} \quad \text{or} \quad z = a_{j \dots j}.$$

Since $a_{ii\dots i} \in \mathcal{K}^{(i)}(\mathcal{A})$ and $a_{jj\dots j} \in \mathcal{K}^{(j)}(\mathcal{A})$, then:

$$z \in \mathcal{K}^{(i)}(\mathcal{A}) \cup \mathcal{K}^{(j)}(\mathcal{A}).$$

If $(\rho(\mathcal{B}) - b_{ii\dots i})^{m-1}(\rho(\mathcal{B}) - b_{jj\dots j}) > 0$, then by (13), we have:

$$\left(\frac{|z - a_{ii\dots i}|}{\rho(\mathcal{B}) - b_{ii\dots i}} \right)^{m-1} \left(\frac{|z - a_{jj\dots j}|}{\rho(\mathcal{B}) - b_{jj\dots j}} \right) \leq 1. \quad (14)$$

As the factors on the left-hand of (14) cannot all exceed unity, then at least one of these factors is unity, at most, i.e., $z \in \mathcal{K}^{(i)}(\mathcal{A}) \cup \mathcal{K}^{(j)}(\mathcal{A})$. Hence, in either case, it follows that $z \in \mathcal{K}^{(i)}(\mathcal{A}) \cup \mathcal{K}^{(j)}(\mathcal{A})$. Therefore:

$$\mathfrak{B}_1(\mathcal{A}) \subseteq \bigcup_{i=1}^n \mathcal{K}^{(i)}(\mathcal{A}) = \mathcal{K}(\mathcal{A}).$$

Next, we prove that the left-hand inclusion of (12) is valid. Using the first part of Theorem 7, we have:

$$(\rho(\mathcal{B}) - b_{ii\dots i}) > 0, \quad \forall i \in \langle n \rangle.$$

Let z be any point of $\mathcal{Z}_1(\mathcal{A})$, that is:

$$\prod_{j=1}^m |z - a_{ij\dots i_j}| \leq \prod_{j=1}^m (\rho(\mathcal{B}) - b_{ij\dots i_j}). \quad (15)$$

On taking the power of m for the inequality (15), we have:

$$\prod_{j=1}^m |z - a_{ij\dots i_j}|^m \leq \prod_{j=1}^m (\rho(\mathcal{B}) - b_{ij\dots i_j})^m. \quad (16)$$

Since $(\rho(\mathcal{B}) - b_{ij\dots i_j})$ are all positive for $j = 1, 2, \dots, m$, we may equivalently express the inequality (16) in the following form:

$$\begin{aligned} & \left(\frac{|z - a_{i_1\dots i_1}|^{m-1} |z - a_{i_2\dots i_2}|}{(\rho(\mathcal{B}) - b_{i_1\dots i_1})^{m-1} (\rho(\mathcal{B}) - b_{i_2\dots i_2})} \right) \cdot \left(\frac{|z - a_{i_2\dots i_2}|^{m-1} |z - a_{i_3\dots i_3}|}{(\rho(\mathcal{B}) - b_{i_2\dots i_2})^{m-1} (\rho(\mathcal{B}) - b_{i_3\dots i_3})} \right) \\ & \dots \left(\frac{|z - a_{i_{m-1}\dots i_{m-1}}|^{m-1} |z - a_{i_m\dots i_m}|}{(\rho(\mathcal{B}) - b_{i_{m-1}\dots i_{m-1}})^{m-1} (\rho(\mathcal{B}) - b_{i_m\dots i_m})} \right) \leq 1. \end{aligned} \quad (17)$$

As the factors on the left of (17) cannot all exceed unity, then at least one of these factors is unity, at most. That is:

$$\left(\frac{|z - a_{i_1\dots i_1}|^{m-1} |z - a_{i_2\dots i_2}|}{(\rho(\mathcal{B}) - b_{i_1\dots i_1})^{m-1} (\rho(\mathcal{B}) - b_{i_2\dots i_2})} \right) \leq 1$$

or

$$\left(\frac{|z - a_{i_2\dots i_2}|^{m-1} |z - a_{i_3\dots i_3}|}{(\rho(\mathcal{B}) - b_{i_2\dots i_2})^{m-1} (\rho(\mathcal{B}) - b_{i_3\dots i_3})} \right) \leq 1$$

or

...

or

$$\left(\frac{|z - a_{i_m \dots i_m}|^{m-1} |z - a_{i_1 \dots i_1}|}{(\rho(\mathcal{B}) - b_{i_m \dots i_m})^{m-1} (\rho(\mathcal{B}) - b_{i_1 \dots i_1})} \right) \leq 1.$$

Hence, there exists an α ($1 < \alpha < m$), such that:

$$|z - a_{i_\alpha \dots i_\alpha}|^{m-1} |z - a_{i_{\alpha+1} \dots i_{\alpha+1}}| \leq (\rho(\mathcal{B}) - b_{i_\alpha \dots i_\alpha})(\rho(\mathcal{B}) - b_{i_{\alpha+1} \dots i_{\alpha+1}});$$

that is, $z \in \mathfrak{B}_1^{(\alpha, \alpha+1)}(\mathcal{A})$. Therefore, by definition of $\mathfrak{B}_1(\mathcal{A})$, we have $z \in \mathfrak{B}_1(\mathcal{A})$. Thus, $\mathcal{Z}_1(\mathcal{A}) \subseteq \mathfrak{B}_1(\mathcal{A})$. Therefore, the proof is complete. $\square$

Corollary 3 Let $\mathcal{A}$ be tensor of order m dimension n , and then:

$$\mathcal{Z}_2(\mathcal{A}) \subseteq \mathcal{Z}_1(\mathcal{A}) \subseteq \mathfrak{B}_1(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}). \quad (18)$$

Remark 4 Using the method of proof employed in Theorem 11, we conclude:

$$\mathcal{Z}_2(\mathcal{A}) \subseteq \mathfrak{B}_2(\mathcal{A}).$$

The following examples show the improvements of the bounds obtained in this section.

Example 1 Let $\mathcal{A} = (a_{ijk})$ be a tensor of order 3 dimension 2 with entries defined as follows:

$$a_{111} = a_{122} = a_{211} = 1, \quad a_{222} = -1 \quad \text{and other } a_{ijk} = 0.$$

Let $\mathcal{B} = (b_{ijk})$ be a tensor of order 3 dimension 2, where $\mathcal{B} = |\mathcal{A}|$. By computation (Chang et al. 2008), we have $\sigma(\mathcal{A}) = \{\sqrt{2}, -\sqrt{2}\}$, and $\rho(\mathcal{B}) = 2$.

By Ky Fan theorem (Theorem 6), we obtain:

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) := \{z : |z - 1| \leq (2 - 1)\} \cup \{z : |z + 1| \leq (2 - 1)\}.$$

From Corollary 1, we obtain:

$$\sigma(\mathcal{A}) \subseteq \mathfrak{B}_1(\mathcal{A}) := \{z : |z - 1|^2 |z + 1| \leq (2 - 1)^3\} \cup \{z : |z + 1|^2 |z - 1| \leq (2 - 1)^3\}.$$

Also from Theorem 7, we obtain:

$$\sigma(\mathcal{A}) \subseteq \mathcal{Z}_1(\mathcal{A}) := \{z : |z - 1| |z + 1| \leq (2 - 1)^2\}.$$

The blue line in Fig. 1 shows the boundary of $\mathcal{K}(\mathcal{A})$ (Ky Fan theorem), the yellow line shows the boundary of $\mathfrak{B}_1(\mathcal{A})$ (Corollary 1), and the red line shows the boundary of $\mathcal{Z}_1(\mathcal{A})$ (Theorem 7). This example shows that $\mathcal{Z}_1(\mathcal{A}) \subseteq \mathfrak{B}_1(\mathcal{A})$.

Remark 5 In Example 1, $\mathcal{E} = 0$ ($\mathcal{B} = |\mathcal{A}|$), and therefore, the eigenvalue inclusion sets $\mathcal{Z}_1(\mathcal{A})$ and $\mathcal{Z}_2(\mathcal{A})$ are equal. In the following example, we consider $\mathcal{B} \gneq |\mathcal{A}|$.

Example 2 Let $\mathcal{A} = (a_{ijk})$ be an order 3 dimension 2 tensor with entries defined as follows:

$$a_{111} = a_{122} = a_{211} = 1, \quad a_{222} = -1 \quad \text{and other } a_{ijk} = 0.$$

Let $\mathcal{B} = (b_{ijk})$ be an order 3 dimension 2 tensor, where $b_{ijk} = 2$ for any $1 \leq i, j, k \leq 2$. By Yang and Yang (2010, Lemma 5.1), we have $\rho(\mathcal{B}) = 8$.

From Corollary 2, we obtain:

$$\mathfrak{B}_2(\mathcal{A}) := \{z : |z - 1|^2 |z + 1| \leq (8 - 2 - 5 \times 1)^3 (8 - 2 - 5 \times 1)\}$$

Fig. 1 Comparisons of Ky Fan theorem, Corollary 2, and Theorem 7. The “*” symbols show the location of the eigenvalues of $\mathcal{A}$

$$\cup \{z : |z + 1|^2 |z - 1| \leq (8 - 2 - 5 \times 1)^3 (8 - 2 - 5 \times 1)\}.$$

Also from Theorem 10, we obtain:

$$\sigma(\mathcal{A}) \subseteq \mathcal{Z}_2(\mathcal{A}) := \{z : |z - 1| |z + 1| \leq (8 - 2 - 5 \times 1)^2\}.$$

However, from Theorem 4.6 of Wang et al. (2020) (Theorem 9), we obtain:

$$\sigma(\mathcal{A}) \subseteq \{z : |z - 1| |z + 1| \leq (8 - 2)^2 - (5 \times 1)^2\}. \quad (19)$$

It is clear that $\mathcal{Z}_2(\mathcal{A})$ is strictly better than the bounds $\mathfrak{B}_2(\mathcal{A})$ and (19).

3.3 New improvements of Ky Fan theorem

In this section, we obtain another improvement of Ky Fan theorem. First, we prove the following lemma that will be needed.

Lemma 2 Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be a tensor of order m dimension n . For any eigenvalue λ of $\mathcal{A}$, there exists a nonzero $x \in \mathbb{C}^n$, such that:

$$(\lambda - a_{s \dots s}) \left(\lambda - \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{s i_2 \dots i_m}=0}}^n a_{t i_2 \dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}} \right) = a_{t s \dots s} \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{s i_2 \dots i_m}=0}}^n a_{s i_2 \dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}}, \quad (20)$$

for all $s \in \langle n \rangle$ and $t \in T$, where $T = \{t \in \langle n \rangle : t \neq s, x_t \neq 0\}$.

Proof Let λ be eigenvalue of $\mathcal{A}$ with the eigenvector x , that is:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

For each $s \in \langle n \rangle$ and $t \in T$, we have:

$$(\lambda - a_{ss\dots s})x_s^{m-1} = \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n a_{si_2\dots i_m}x_{i_2}\dots x_{i_m}, \quad (21)$$

$$(\lambda - a_{tt\dots t})x_t^{m-1} = \sum_{\substack{\delta_{ti_2\dots i_m}=0 \\ \delta_{si_2\dots i_m}=0}} a_{ti_2\dots i_m}x_{i_2}\dots x_{i_m} + a_{ts\dots s}x_s^{m-1}. \quad (22)$$

If $(\lambda - a_{ss\dots s}) \neq 0$ by (22), one has:

$$\begin{aligned} & (\lambda - a_{ss\dots s})(\lambda - a_{tt\dots t})x_t^{m-1} - \sum_{\substack{\delta_{ti_2\dots i_m}=0 \\ \delta_{si_2\dots i_m}=0}} (\lambda - a_{ss\dots s})a_{ti_2\dots i_m}x_{i_2}\dots x_{i_m} \\ &= (\lambda - a_{ss\dots s})a_{ts\dots s}x_s^{m-1}. \end{aligned} \quad (23)$$

Also if $a_{ts\dots s} \neq 0$ by (21), one has:

$$(\lambda - a_{ss\dots s})a_{ts\dots s}x_s^{m-1} = a_{ts\dots s} \left(\sum_{\substack{\delta_{si_2\dots i_m}=0 \\ \delta_{ti_2\dots i_m}=0}} a_{si_2\dots i_m}x_{i_2}\dots x_{i_m} + a_{st\dots t}x_t^{m-1} \right). \quad (24)$$

By (23) and (24), we have:

$$\begin{aligned} & (\lambda - a_{ss\dots s}) \left((\lambda - a_{tt\dots t}) - \sum_{\substack{\delta_{si_2\dots i_m}=0 \\ \delta_{ti_2\dots i_m}=0}} a_{ti_2\dots i_m} \frac{x_{i_2}\dots x_{i_m}}{x_t^{m-1}} \right) \\ &= a_{ts\dots s} \left(\sum_{\substack{\delta_{si_2\dots i_m}=0 \\ \delta_{ti_2\dots i_m}=0}} a_{si_2\dots i_m} \frac{x_{i_2}\dots x_{i_m}}{x_t^{m-1}} + a_{st\dots t} \right), \end{aligned}$$

where is equal to:

$$(\lambda - a_{ss\dots s}) \left(\lambda - \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n a_{ti_2\dots i_m} \frac{x_{i_2}\dots x_{i_m}}{x_t^{m-1}} \right) = a_{ts\dots s} \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n a_{si_2\dots i_m} \frac{x_{i_2}\dots x_{i_m}}{x_t^{m-1}}.$$

If $(\lambda - a_{ss\dots s}), a_{ts\dots s} = 0$, then the relation (20) is also valid for this case in fact. Thus, the proof is complete. $\square$

Theorem 12 Let $\mathcal{A}$ and $\mathcal{B}$ be tensor of order m dimension n with $|\mathcal{A}| \leq \mathcal{B}$, and then:

$$\sigma(\mathcal{A}) \subseteq \bigcap_{s \in N} \left(\bigcup_{\substack{t \in N \\ s \neq t}} \mathcal{O}_{s,t} \right) := \mathcal{K}_{s,t}(\mathcal{A}),$$

where

$$\begin{aligned} \mathcal{O}_{s,t} &= \left\{ z \in \mathbb{C} : \left| z - \frac{a_{s\dots s} + a_{t\dots t}}{2} \right| \right. \\ &\quad \left. \leq \frac{1}{2} \left| \rho_t(\mathcal{B}) + \sqrt{(|a_{s\dots s} - a_{t\dots t}| + \rho_t(\mathcal{B}))^2 + 4b_{ts\dots s}\rho_s(\mathcal{B})} \right| \right\} \end{aligned}$$

$$\rho_t(\mathcal{B}) = \rho(\mathcal{B}) - b_{tt\dots t} - b_{ts\dots s} \left(\frac{u_s}{u_t} \right)^{m-1} - r_t^s(\mathcal{E}) \left(\frac{u_{\min}}{u_t} \right)^{m-1},$$

$$\rho_s(\mathcal{B}) = (\rho(\mathcal{B}) - b_{ss\dots s}) \left(\frac{u_s}{u_t} \right)^{m-1} - r_s(\mathcal{E}) \left(\frac{u_{\min}}{u_t} \right)^{m-1}, \text{ and } u \text{ is a Perron vector of } \mathcal{B}.$$

Proof Similarly by proof of Lemma 2, we define:

$$z_i = \frac{|x_i|}{u_i}, \quad i \in \langle n \rangle, \quad i \neq s.$$

Let

$$T' := \max_{\substack{j \in \langle n \rangle \\ j \neq s}} \left\{ j : z_j = \frac{|x_j|}{u_j} \right\};$$

it is evident that $T' \subseteq T$. For all $s \in \langle n \rangle$ and $t \in T'$, we set:

$$\alpha_s = \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n a_{si_2\dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}} \quad (25)$$

and

$$\alpha_t = \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n a_{ti_2\dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}} = \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0 \\ \delta_{ti_2\dots i_m}=0}}^n a_{ti_2\dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}} + a_{tt\dots t}. \quad (26)$$

Since u is a Perron vector of the positive tensor $\mathcal{B}$, so:

$$\mathcal{B}u^{m-1} = \rho(\mathcal{B})u^{[m-1]}.$$

Let $\mathcal{E} = \mathcal{B} - |\mathcal{A}|$, and then, the triangle inequality permits us to conclude that:

$$\begin{aligned} |\alpha_s| &\leq \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n |a_{si_2\dots i_m}| \left| \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}} \right| \\ &= \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n (b_{si_2\dots i_m} - e_{si_2\dots i_m}) \left| \frac{(z_{i_2}u_{i_2}) \dots (z_{i_m}u_{i_m})}{(z_t u_t)^{m-1}} \right| \\ &\leq \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n b_{si_2\dots i_m} \frac{u_{i_2} \dots u_{i_m}}{u_t^{m-1}} - \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n e_{si_2\dots i_m} \frac{u_{i_2} \dots u_{i_m}}{u_t^{m-1}} \\ &\leq \frac{1}{u_t^{m-1}} \sum_{i_2, \dots, i_m=1}^n b_{si_2\dots i_m} u_{i_2} \dots u_{i_m} - b_{ss\dots s} u_s^{m-1} - \left(\frac{u_{\min}}{u_t} \right)^{m-1} r_s(\mathcal{E}) \end{aligned}$$

$$= \left(\frac{u_s}{u_t} \right)^{m-1} (\rho(\mathcal{B}) - b_{ss\dots s}) - \left(\frac{u_{\min}}{u_t} \right)^{m-1} r_s(\mathcal{E}) \equiv \rho_s(\mathcal{B}), \quad (27)$$

and

$$\begin{aligned} |\alpha_t - a_{tt\dots t}| &\leq \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0 \\ \delta_{ti_2\dots i_m}=0}}^n |a_{ti_2\dots i_m}| \left| \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}} \right| \\ &= \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0 \\ \delta_{ti_2\dots i_m}=0}}^n (b_{ti_2\dots i_m} - e_{ti_2\dots i_m}) \left| \frac{(z_{i_2} u_{i_2}) \dots (z_{i_m} u_{i_m})}{(z_t u_t)^{m-1}} \right| \\ &\leq \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0 \\ \delta_{ti_2\dots i_m}=0}}^n b_{ti_2\dots i_m} \frac{u_{i_2} \dots u_{i_m}}{u_t^{m-1}} - \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0 \\ \delta_{ti_2\dots i_m}=0}}^n e_{ti_2\dots i_m} \frac{u_{i_2} \dots u_{i_m}}{u_t^{m-1}} \\ &\leq \sum_{\substack{i_2, \dots, i_m=1 \\ \delta_{si_2\dots i_m}=0}}^n b_{ti_2\dots i_m} \frac{u_{i_2} \dots u_{i_m}}{u_t^{m-1}} - b_{tt\dots t} - \left(\frac{u_{\min}}{u_t} \right)^{m-1} r_t^s(\mathcal{E}) \\ &= \rho(\mathcal{B}) - b_{tt\dots t} - b_{ts\dots s} \left(\frac{u_s}{u_t} \right)^{m-1} - \left(\frac{u_{\min}}{u_t} \right)^{m-1} r_t^s(\mathcal{E}) \equiv \rho_t(\mathcal{B}), \quad (28) \end{aligned}$$

where $r_t^s(\mathcal{E})$ is defined in (1). By Lemma 2, we have:

$$(\lambda - a_{ss\dots s})(\lambda - \alpha_t) = a_{ts\dots s}\alpha_s.$$

Therefore:

$$\lambda = \frac{(\alpha_t + a_{ss\dots s}) \pm \sqrt{(\alpha_t + a_{ss\dots s})^2 - 4(a_{ss\dots s}\alpha_t - a_{ts\dots s}\alpha_s)}}{2}.$$

Using the inequalities (27) and (28), we obtain:

$$\begin{aligned} \left| \lambda - \frac{(a_{ss\dots s} + a_{tt\dots t})}{2} \right| &= \left| \frac{(\alpha_t - a_{tt\dots t}) \pm \sqrt{(a_{ss\dots s} - \alpha_t)^2 - 4a_{ts\dots s}\alpha_s}}{2} \right| \\ &\leq \frac{1}{2} \left| \rho_t(\mathcal{B}) + \sqrt{(|a_{ss\dots s} - a_{tt\dots t}| + \rho_t(\mathcal{B}))^2 + 4b_{ts\dots s}\rho_s(\mathcal{B})} \right|. \end{aligned}$$

Since s is arbitrary, we have:

$$\sigma(\mathcal{A}) \subseteq \bigcap_{s \in \langle n \rangle} \left(\bigcup_{\substack{t \in \langle n \rangle \\ t \neq s}} \mathcal{O}_{s,t} \right).$$

The proof of the theorem is complete. $\square$

Theorem 13 Let $\mathcal{A}, \mathcal{B}$ be tensor of order m dimension n with $|\mathcal{A}| \leq \mathcal{B}$, and then:

$$\sigma(\mathcal{A}) \subseteq \bigcap_{s \in N} \left(\bigcup_{\substack{t \in N \\ s \neq t}} \mathcal{M}_{s,t} \right) := \mathcal{K}_2(\mathcal{A}),$$

where

$$\mathcal{M}_{s,t} = \{z \in \mathbb{C} : |z - a_{ss\dots s}| |z - a_{tt\dots t}| \leq |z - a_{ss\dots s}| \rho_t(\mathcal{B}) + b_{ts\dots s} \rho_s(\mathcal{B})\},$$

and $\rho_t(\mathcal{B}), \rho_s(\mathcal{B})$ are the same as those in Theorem 12.

Proof First, for all $s \in \langle n \rangle$ and $t \in T'$, according to the definition of T' , we have $T' \subseteq T$. We write Eq. (20) in the form:

$$\begin{aligned} & \lambda^2 - \lambda \left(\sum_{\substack{i_2 \dots i_m \in \langle n \rangle \\ \delta_{si_2 \dots i_m} = 0 \\ \delta_{ti_2 \dots i_m} = 0}} a_{ti_2 \dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}} + a_{tt\dots t} \right) - a_{ss\dots s} \lambda \\ & + a_{ss\dots s} \left(\sum_{\substack{i_2 \dots i_m \in \langle n \rangle \\ \delta_{si_2 \dots i_m} = 0 \\ \delta_{ti_2 \dots i_m} = 0}} a_{ti_2 \dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}} + a_{tt\dots t} \right) \\ & = a_{ts\dots s} \sum_{\substack{i_2 \dots i_m \in \langle n \rangle \\ \delta_{si_2 \dots i_m} = 0}} a_{si_2 \dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}}. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} & (\lambda - a_{ss\dots s}) (\lambda - a_{tt\dots t}) \\ & = (\lambda - a_{ss\dots s}) \sum_{\substack{i_2 \dots i_m \in \langle n \rangle \\ \delta_{si_2 \dots i_m} = 0 \\ \delta_{ti_2 \dots i_m} = 0}} a_{ti_2 \dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}} + a_{ts\dots s} \sum_{\substack{i_2 \dots i_m \in \langle n \rangle \\ \delta_{si_2 \dots i_m} = 0}} a_{si_2 \dots i_m} \frac{x_{i_2} \dots x_{i_m}}{x_t^{m-1}}. \end{aligned}$$

Using the inequalities (27) and (28), we have:

$$|\lambda - a_{ss\dots s}| |\lambda - a_{tt\dots t}| \leq |\lambda - a_{ss\dots s}| \rho_t(\mathcal{B}) + \rho_s(\mathcal{B}) b_{ts\dots s},$$

i.e., $\lambda \in \mathcal{M}_{s,t}$. Since s is arbitrary, we have:

$$\sigma(\mathcal{A}) \subseteq \bigcap_{s \in N} \left(\bigcup_{\substack{t \in N \\ s \neq t}} \mathcal{M}_{s,t} \right) := \mathcal{K}_2(\mathcal{A}).$$

□

By estimating the ratio of the smallest component and the largest component of the Perron vector of $\mathcal{B}$ (Lemma 1), the following result is obtained.

Theorem 14 Let $\mathcal{A}$, $\mathcal{B}$ be tensor of order m dimension n with $|\mathcal{A}| \leq \mathcal{B}$. Then, every eigenvalue of $\mathcal{A}$ lies in the region:

$$\bigcap_{i \in (n)} \left(\bigcup_{\substack{j \in (n) \\ i \neq j}} \tilde{\mathcal{M}}_{i,j} \right) := \tilde{\mathcal{K}}_2(\mathcal{A}),$$

where

$$\tilde{\mathcal{M}}_{i,j} := \{z \in \mathbb{C} : |z - a_{ii\dots i}| |z - a_{jj\dots j}| \leq |z - a_{ii\dots i}| \tilde{\rho}_j(\mathcal{B}) + b_{ji\dots i} \tilde{\rho}_i(\mathcal{B})\},$$

$$\tilde{\rho}_j(\mathcal{B}) = \rho(\mathcal{B}) - b_{jj\dots j} - (\tau(\mathcal{B}))^{m-1} (b_{ji\dots i} + r_j^i(\mathcal{E})),$$

$$\tilde{\rho}_i(\mathcal{B}) = (\rho(\mathcal{B}) - b_{ii\dots i}) \left(\frac{1}{\tau(\mathcal{B})} \right)^{m-1} - (\tau(\mathcal{B}))^{m-1} r_i(\mathcal{E}), \text{ and } \tau(\mathcal{B}) \text{ is given in (8).}$$

Wang et al. (2020) in Example 4.11 show that their bounds are better than the bound (10). By the same example, we show that our bounds are better than others given in the literature.

Example 3 (Wang et al. 2020, Example 4.11) Consider 3 order 2 dimensional tensors $\mathcal{A} = (a_{ijk})$ and $\mathcal{B} = (b_{ijk})$ defined by:

$$a_{ijk} = \begin{cases} a_{111} = 0; & a_{112} = 0; & a_{121} = 0; & a_{122} = -1; \\ a_{211} = -4; & a_{212} = 0; & a_{221} = 0; & a_{222} = 0; \end{cases}$$

$$b_{ijk} = \begin{cases} b_{111} = 1; & b_{112} = \frac{1}{4}; & b_{121} = \frac{1}{4}; & b_{122} = 2; \\ b_{211} = 4; & b_{212} = \frac{1}{2}; & b_{221} = \frac{1}{2}; & b_{222} = \frac{3}{2}. \end{cases}$$

Obviously, $\mathcal{B} \geq |\mathcal{A}|$ and $\mathcal{A}$ is weakly irreducible. By computations (Liu and Chen 2019), we can calculate:

$$\{\rho(\mathcal{B}), x\} = \{4.8095, (0.6930, 0.8738)\}, \quad r_1(\mathcal{E}) = \frac{3}{2}, \quad r_2(\mathcal{E}) = 1,$$

where $\mathcal{E} = \mathcal{B} - |\mathcal{A}|$. From Theorem 3.3 of Li and Ng (2015), we may compute $\kappa(\mathcal{B}) = \frac{5}{18}$ and:

$$|\lambda| \leq 4.8095 - 1 - \left(\frac{5}{18} \right)^2 \times \frac{3}{2} = 3.6938.$$

From Theorem 4.6 of Wang et al. (2020), we may compute $P = \max \left\{ \left(\frac{5}{18} \right)^2, \frac{1}{4}, 0.1603 \right\} = \frac{1}{4}$ and:

$$|\lambda| \leq \sqrt{(4.8095 - 1) \times \left(4.8095 - \frac{3}{2} \right) - \left(\frac{1}{4} \right)^2 \times \frac{3}{2} \times 1} = 3.5375.$$

By Theorem 4.7 of Wang et al. (2020) (Theorem 7), we get:

$$|\lambda| \leq \sqrt{(4.8095 - 1) \times \left(4.8095 - \frac{3}{2} \right)} = 3.5507.$$

From Theorem 4.9 of Wang et al. (2020), we obtain:

$$|\lambda| \leq 4.8095 - 1 - \frac{1}{4} \times \frac{3}{2} = 3.4345.$$

From Lemma 1, we may compute $\tau(\mathcal{B}) = \max \left\{ \left(\frac{5}{18} \right)^2, \left(\frac{1}{2} \right)^2 \right\} = \frac{1}{4}$, and therefore, by Theorem 10, we have:

$$|\lambda| \leq \sqrt{\left[4.8095 - 1 - \left(\frac{1}{4} \times \frac{3}{2} \right) \right] \times \left[4.8095 - \frac{3}{2} - \left(\frac{1}{4} \times 1 \right) \right]} = 3.2416.$$

From Corollary 2, we obtain:

$$|\lambda| \leq \left(\left[4.8095 - 1 - \left(\frac{1}{4} \times \frac{3}{2} \right) \right]^2 \times \left[4.8095 - \frac{3}{2} - \left(\frac{1}{4} \times 1 \right) \right] \right)^{\frac{1}{3}} = 3.3047.$$

From Theorem 12 by $s = 2$, $t = 1$, we may compute:

$$\begin{aligned} \rho_1(\mathcal{B}) &= (4.8095 - 1) - 2 \times \left(\frac{0.8738}{0.6930} \right)^2 - \frac{1}{2} \times \left(\frac{0.6930}{0.6930} \right)^2 = 0.1298, \\ \rho_2(\mathcal{B}) &= \left(4.8095 - \frac{3}{2} \right) \times \left(\frac{0.8738}{0.6930} \right)^2 - 1 \times \left(\frac{0.6930}{0.6930} \right)^2 = 4.2616, \text{ and} \\ |\lambda| &\leq \frac{1}{2} \left[0.1298 + \sqrt{(0.1298)^2 + 4 \times 2 \times 4.2616} \right] = 2.9851. \end{aligned}$$

Also from Theorem 13 by $s = 2$, $t = 1$, we obtain:

$$|\lambda|^2 \leq |\lambda| \times 0.1298 + 2 \times 4.2616,$$

which this eigenvalue inclusion set is equal to $|\lambda| \leq 2.9851$. This example shows that if we use the ratio of the smallest and largest entries of the Perron vector ($\tau(\mathcal{B})$), then the bound of Theorem 10 is tighter than other bounds. However, if there exist the perron vector of $\mathcal{B}$, then the bounds of Theorems 12 and 13 are tighter.

4 Conclusion

In this paper, we proposed some new improvements of Ky Fan theorem for tensors. We obtain new lower bound for the ratio of the smallest and largest entries of the Perron vector using a technique due to Wang et al. (2020) and Li and Ng (2015). Based on these new lower bounds, we obtain Theorems 10 and 12, which improve the bounds in Li and Ng (2015) and Wang et al. (2020). Each of these new bounds is always better than the existing one. Several numerical examples are given to show that our bounds are better than others given in the literature.

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