

Fractional heat conduction with heat absorption in a sphere under Dirichlet boundary condition

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Abstract The time-fractional heat conduction equation with the Caputo derivative and with heat absorption term proportional to temperature is considered in a sphere in the case of central symmetry. The fundamental solution to the Dirichlet boundary value problem is found, and the solution to the problem under constant boundary value of temperature is studied. The integral transform technique is used. The solutions are obtained in terms of series containing the Mittag-Leffler functions being the generalization of the exponential function. The numerical results are illustrated graphically.

Keywords Non-Fourier heat conduction · Heat absorption · Caputo fractional derivative · Dirichlet boundary condition · Mittag-Leffler function · Laplace transform · Finite Fourier transform

Mathematics Subject Classification 26A33 · 35K05 · 45K05

1 Introduction

The classical parabolic heat conduction equation with the source term proportional to temperature

$$\frac{\partial T}{\partial t} = a \Delta T - bT \quad (1)$$

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was considered, e.g., in Carslaw and Jaeger (1959), Crank (1975), Nyborg (1988), Polyanin (2002). Here, T is temperature, t is time, a stands for the thermal diffusivity coefficient, Δ denotes the Laplace operator, the coefficient b describes the heat absorption (heat release).

In the last few decades, differential equations with derivatives of non-integer order attract the attention of the researchers as such equations provide a very suitable tool for description of many important phenomena in physics, geophysics, chemistry, biology, engineering and solid mechanics (see, for example, Gafiychuk et al. 2008; Herrmann 2011; Magin 2006; Mainardi 2010; Povstenko 2015a; Sabatier et al. 2007; Tarasov 2010; Tenreiro Machado 2011; Uchaikin 2013).

In this paper, we consider the time-fractional equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T - bT, \quad 0 < \alpha \leq 1, \tag{2}$$

where

$$\frac{d^\alpha f(t)}{dt^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, & m-1 < \alpha < m, \\ \frac{d^m f(t)}{dt^m}, & \alpha = m, \end{cases} \tag{3}$$

is the Caputo fractional derivative (Kilbas et al. 2006; Podlubny 1999).

Equation (2) results from the time-nonlocal generalization of the Fourier law with the “long-tail” power kernel. Such a generalization can be interpreted in terms of derivatives and integrals of non-integer order. Equation (2) takes into consideration the memory effects with respect to time. The interested reader is also referred to Povstenko (2011, 2015a).

In the literature, there are papers devoted to investigation of Eq. (2) in the case of one Cartesian spatial coordinate (Damor et al. 2016; Ferrás et al. 2015; Qin and Wu 2016; Vitali et al. 2017). Here, we study Eq. (2) in a spherical domain $0 \leq r < R$ in the case of central symmetry under Dirichlet boundary condition.

The paper is organized as follows. In Sect. 2, we find the fundamental solution to the Dirichlet boundary value problem using the Laplace transform with respect to time t and the finite sin-Fourier transform with respect to the spatial coordinate r . The corresponding problem under constant boundary value of temperature at the surface $r = R$ is investigated in Sect. 3. In both cases, the numerical results are illustrated graphically. Conclusions are presented in Sect. 4.

2 The fundamental solution to the Dirichlet problem

We consider the time-fractional heat conduction equation with heat absorption term in spherical coordinates in the case of central symmetry

$$\frac{\partial^\alpha T(r, t)}{\partial t^\alpha} = a \left[\frac{\partial^2 T(r, t)}{\partial r^2} + \frac{2}{r} \frac{\partial T(r, t)}{\partial r} \right] - bT(r, t), \tag{4}$$

in the domain $0 \leq r < R, 0 < t < \infty$ with $a > 0$ and the order of the Caputo derivative $0 < \alpha \leq 1$.

Equation (4) is studied under zero initial condition

$$t = 0 : T(r, t) = 0 \tag{5}$$

and the Dirichlet boundary condition

$$r = R : T(r, t) = p_0\delta(t) \tag{6}$$

with $\delta(t)$ being the Dirac delta function. The constant multiplier p_0 is introduced in (6) to obtain the non-dimensional quantities used in numerical calculations [see Eq. (17)].

In the general case, the Caputo fractional derivative has the following Laplace transform rule

$$\mathcal{L} \left\{ \frac{d^\alpha f(t)}{dt^\alpha} \right\} = s^\alpha f^*(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad m-1 < \alpha < m, \tag{7}$$

where the asterisk denotes the Laplace transform, s is the transform variable.

In what follows, we will use the finite sin-Fourier transform with respect to the spatial coordinate r in the domain $0 \leq r \leq R$ (see, for example, Povstenko 2015b):

$$\mathcal{F}\{f(r)\} = \tilde{f}(\xi_k) = \int_0^R r f(r) \frac{\sin(r\xi_k)}{\xi_k} dr \tag{8}$$

with the inverse transform

$$\mathcal{F}^{-1}\{\tilde{f}(\xi_k)\} = f(r) = \frac{2}{R} \sum_{k=1}^{\infty} \xi_k \tilde{f}(\xi_k) \frac{\sin(r\xi_k)}{r}, \tag{9}$$

where

$$\xi_k = \frac{k\pi}{R}. \tag{10}$$

This transform is the convenient reformulation of the sin-Fourier series and is used for Dirichlet boundary condition with the prescribed boundary value of a function, since for the Laplace operator in the case of central symmetry we have

$$\mathcal{F} \left\{ \frac{d^2 f(r)}{dr^2} + \frac{2}{r} \frac{df(r)}{dr} \right\} = -\xi_k^2 \tilde{f}(\xi_k) + (-1)^{k+1} R f(R). \tag{11}$$

Applying the integral transforms to (4) under the initial condition (5) and boundary condition (6) gives in the transform domain:

$$\tilde{T}^*(\xi_k, s) = (-1)^{k+1} ap_0 R \frac{1}{s^\alpha + a\xi_k^2 + b}. \tag{12}$$

Inversion of the Laplace and finite sin-Fourier transforms results in the sought-for fundamental solution:

$$T(r, t) = \frac{2ap_0 t^{\alpha-1}}{r} \sum_{k=1}^{\infty} (-1)^{k+1} \xi_k E_{\alpha,\alpha} [-(a\xi_k^2 + b)t^\alpha] \sin(r\xi_k). \tag{13}$$

To obtain (13) the following formula

$$\mathcal{L}^{-1} \left\{ \frac{s^{\alpha-\beta}}{s^\alpha + b} \right\} = t^{\beta-1} E_{\alpha,\beta}(-bt^\alpha) \tag{14}$$

has been used, where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function in two parameters α and β described by the series representation (Gorenflo et al. 2014; Kilbas et al. 2006; Podlubny 1999)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0, z \in C. \tag{15}$$

For $b = 0$, the fundamental solution Eq. (13) coincides with the solution obtained in Povstenko (2008, 2015b).

In the particular case $\alpha = 1$, the Mittag-Leffler function $E_{1,1}(z) = e^z$; hence, the fundamental solution to the Dirichlet problem for the classical heat conduction equation with heat absorption has the form

$$T(r, t) = \frac{2ap_0}{r} \sum_{k=1}^{\infty} (-1)^{k+1} \xi_k \exp[-(a\xi_k^2 + b)t] \sin(r\xi_k). \tag{16}$$

Introducing the non-dimensional quantities

$$\bar{T} = \frac{t}{p_0} T, \quad \bar{r} = \frac{r}{R}, \quad \kappa = \frac{\sqrt{at}^{\alpha/2}}{R}, \quad \bar{b} = bt^{\alpha}, \quad \bar{\xi}_k = R\xi_k = k\pi, \tag{17}$$

we get the solution in the general case

$$\bar{T}(\bar{r}, \kappa) = \frac{2\kappa^2}{\bar{r}} \sum_{k=1}^{\infty} (-1)^{k+1} \bar{\xi}_k E_{\alpha,\alpha}[-(\kappa^2 \bar{\xi}_k^2 + \bar{b})] \sin(\bar{r}\bar{\xi}_k) \tag{18}$$

and for the classical heat conduction

$$\bar{T}(\bar{r}, \kappa) = \frac{2\kappa^2}{\bar{r}} \sum_{k=1}^{\infty} (-1)^{k+1} \bar{\xi}_k \exp[-(\kappa^2 \bar{\xi}_k^2 + \bar{b})] \sin(\bar{r}\bar{\xi}_k), \tag{19}$$

respectively.

The results of numerical calculations are shown in Figs. 1, 2, 3, and 4.

Numerical results demonstrate the significant influence of the order of fractional derivative α and the absorption parameter b on the heat conduction process. When the fractional derivative of order $0 < \alpha < 1$ replaces the standard first-order time derivative in the diffusion (heat conduction) equation, this leads to slow diffusion (see, for example, Chen 2017; Kimmich 2002; Metzler and Klafter 2004). Figures show evidently the slower diffusion with decreasing α . Heat absorption also results in slower heat diffusion.

3 The constant boundary value of temperature

Now we solve the time-fractional heat conduction equation with the heat absorption term under zero initial condition and the Dirichlet boundary condition with the constant boundary value of temperature:

Fig. 1 The fundamental solution to the Dirichlet problem; $\bar{b} = 0.5$, $\kappa = 0.25$

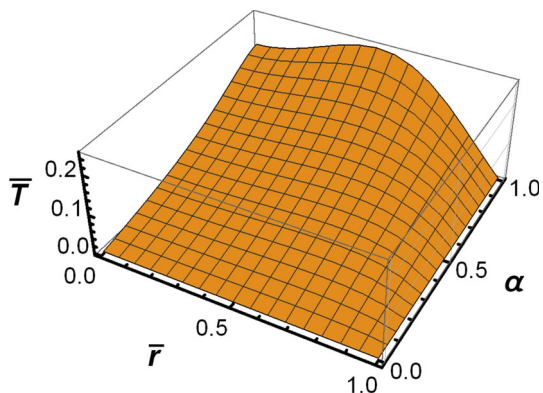


Fig. 2 The fundamental solution to the Dirichlet problem; $\bar{r} = 0.5$, $\kappa = 0.25$

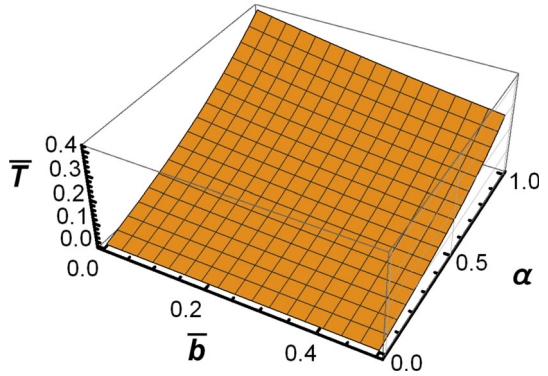


Fig. 3 The fundamental solution to the Dirichlet problem; $\bar{r} = 0.5$, $\bar{b} = 0$

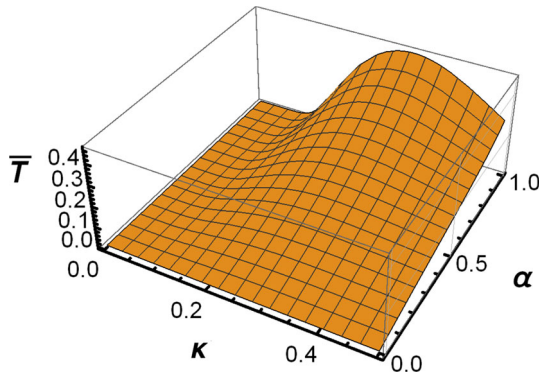
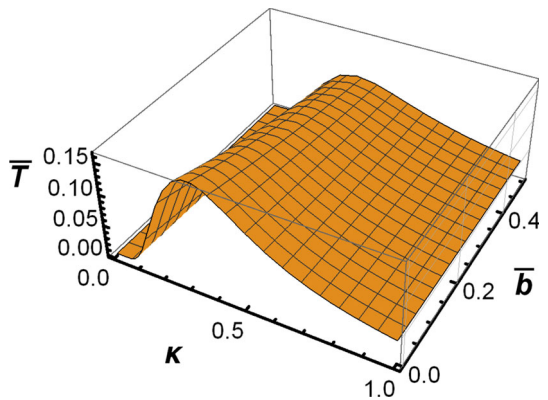


Fig. 4 The fundamental solution to the Dirichlet problem; $\bar{r} = 0.5$, $\alpha = 0.5$



$$\frac{\partial^\alpha T(r, t)}{\partial t^\alpha} = a \left[\frac{\partial^2 T(r, t)}{\partial r^2} + \frac{2}{r} \frac{\partial T(r, t)}{\partial r} \right] - bT(r, t), \tag{20}$$

$$t = 0 : T(r, t) = 0, \tag{21}$$

$$r = R : T(r, t) = T_0. \tag{22}$$

As above, the Laplace transform with respect to time t and the finite sin-Fourier transform with respect to the spatial coordinate r give the solution in the transform domain:

$$\tilde{T}^*(\xi_k, s) = (-1)^{k+1} aT_0R \frac{1}{s(s^\alpha + a\xi_k^2 + b)}. \tag{23}$$

In connection with

$$\frac{1}{s(s^\alpha + a\xi_k^2 + b)} = \frac{1}{a\xi_k^2 + b} \left(\frac{1}{s} - \frac{s^{\alpha-1}}{s^\alpha + a\xi_k^2 + b} \right), \tag{24}$$

we obtain

$$\tilde{T}^*(\xi_k, s) = (-1)^{k+1} aT_0R \frac{1}{a\xi_k^2 + b} \left(\frac{1}{s} - \frac{s^{\alpha-1}}{s^\alpha + a\xi_k^2 + b} \right) \tag{25}$$

and after inverting the integral transforms we arrive at

$$T(r, t) = \frac{2aT_0}{r} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\xi_k}{a\xi_k^2 + b} \{1 - E_\alpha[-(a\xi_k^2 + b)t^\alpha]\} \sin(r\xi_k). \tag{26}$$

In this case $E_\alpha(z)$ is the Mittag-Leffler function in one parameter α having the series representation

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \quad z \in C. \tag{27}$$

Taking into account the following series (Prudnikov et al. 1986)

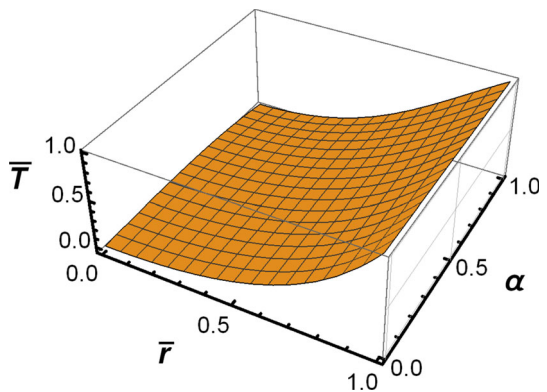
$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 + c^2} \sin(kx) = \frac{\pi}{2} \frac{\sinh(xc)}{\sinh(\pi c)}, \tag{28}$$

we get

$$T(r, t) = \frac{T_0R}{r} \left\{ \frac{\sinh(r\sqrt{b/a})}{\sinh(R\sqrt{b/a})} - \frac{2}{R} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\xi_k}{\xi_k^2 + b/a} E_\alpha[-(a\xi_k^2 + b)t^\alpha] \sin(r\xi_k) \right\}. \tag{29}$$

The advantage of the solution (29) is that the first term satisfies the boundary condition (22), whereas the second term vanishes at $r = R$.

Fig. 5 Solution to the Dirichlet problem with the constant boundary value of temperature; $\bar{b} = 0.1, \kappa = 0.25$



In terms of non-dimensional quantities, we have

$$\bar{T}(\bar{r}, \kappa) = \frac{1}{\bar{r}} \left\{ \frac{\sinh(\bar{r}\sqrt{\bar{b}/\kappa})}{\sinh(\sqrt{\bar{b}/\kappa})} - 2\kappa^2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\bar{\xi}_k}{\bar{\xi}_k^2 + \bar{b}} E_{\alpha} [-(\kappa^2 \bar{\xi}_k^2 + \bar{b})] \sin(\bar{r}\bar{\xi}_k) \right\}, \tag{30}$$

where $\bar{T} = T/T_0$, other non-dimensional parameters are the same as in (17).

Fig. 6 Solution to the Dirichlet problem with the constant boundary value of temperature; $\bar{r} = 0.5, \kappa = 0.25$

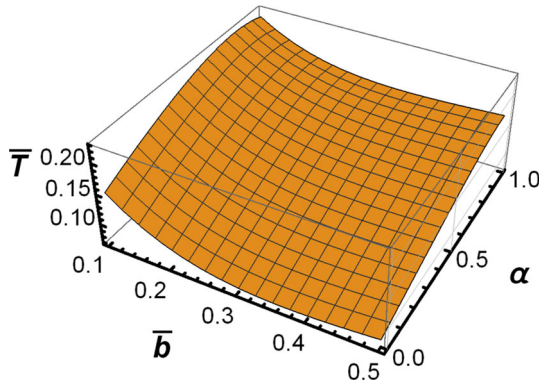


Fig. 7 Solution to the Dirichlet problem with the constant boundary value of temperature; $\bar{r} = 0.5, \bar{b} = 0.1$

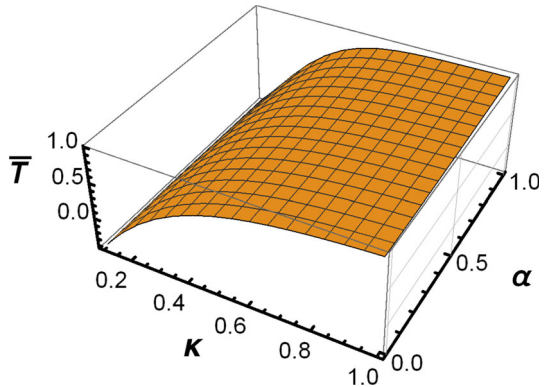
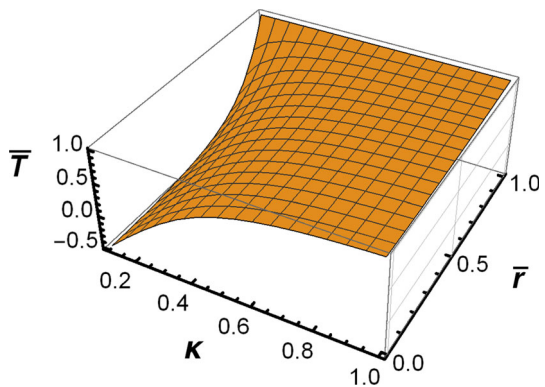


Fig. 8 Solution to the Dirichlet problem with the constant boundary value of temperature; $\alpha = 0.5, \bar{b} = 0.1$



Figures 5, 6, 7, and 8 present the solution (30) for different values on non-dimensional quantities.

4 Conclusions

In this paper, we have investigated the time-fractional heat conduction equation with the Caputo derivative of order $0 < \alpha \leq 1$ and the heat absorption term proportional to temperature. The fundamental solution to the Dirichlet boundary problem and the solution to the problem with constant boundary value of temperature have been found. The solutions have been obtained in terms of series containing the Mittag-Leffler functions being the generalization of the exponential function. To evaluate the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\alpha}(z)$ we have used the algorithms proposed in Gorenflo et al. (2002) (see also the Matlab programs that implement these algorithms Matlab File Exchange 2005). The obtained results can be generalized in the future works for isotropic fractal media within the framework of the non-integer dimensional space approach.

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