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An Inertial Iterative Algorithm for Approximating Common Solutions to Split Equalities of Some Nonlinear Optimization Problems

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Abstract
In this paper, we introduce a new inertial Tseng's extragradient method with self-adaptive step sizes for approximating a common solution of split equalities of equilibrium problem (EP), non-Lipschitz pseudomonotone variational inequality problem (VIP) and fixed point problem (FPP) of nonexpansive semigroups in real Hilbert spaces. We prove that the sequence generated by our proposed method converges strongly to a common solution of the EP, pseudomonotone VIP and FPP of nonexpansive semigroups without any linesearch procedure nor the sequential weak continuity condition often assumed by authors when solving non-Lipschitz VIPs. Finally, we provide some numerical experiments for the proposed method in comparison with related methods in the literature. Our result improves, extends and generalizes several of the existing results in this direction.

Keywords Split equality problems · Equilibrium problem · Variational inequalities · Nonexpansive semigroup · Inertial technique · Self-adaptive step size

Mathematics Subject Classification (2010) 47H09 · 47H10 · 49J20 · 49J40

1 Introduction

Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. The *variational inequality problem* (VIP) is defined as follows: Find $x \in \mathcal{C}$ such that

$$\langle Ax, y - x\rangle \geq 0, \ \forall y \in \mathcal{C}, \tag{1.1}$$

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where $A : \mathcal{H} \to \mathcal{H}$ is an operator. We denote by $VI(\mathcal{C}, A)$ the solution set of the problem (1.1).

Definition 1.1 Let $A : \mathcal{H} \to \mathcal{H}$ be a mapping. Then, A is said to be

(i) *L-co-coercive (or L-inverse strongly monotone)*, if there exists a constant $L > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq L\|Ax - Ay\|^2, \quad \forall x, y \in \mathcal{H},$$

(ii) *Monotone*, if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H}.$$

(iii) *Pseudomonotone*, if

$$\langle Ay, x - y\rangle \geq 0 \implies \langle Ax, x - y\rangle \geq 0, \ \forall x, y \in \mathcal{H},$$

Note that (i) $\implies$ (ii) $\implies$ (iii) but the converses are not always true.

A central problem in nonlinear analysis is the VIP, which was first introduced independently by Fichera [18] and Stampacchia [51]. It plays an important role in the study of several important concepts in pure and applied sciences such as mechanics, neccessary optimality conditions, operations research, systems of nonlinear equations, among others (see [19, 25, 62]). Many authors have analyzed and studied iterative algorithms for approximating the solution of the VIP (1.1) and other related optimization problems, (see [2, 10, 20, 27, 36, 41, 52, 56], and the references therein).

Under certain conditions, there are two common methods used in approximating the solution of the VIP (1.1). These methods are the projection method and the regularized method. To use these methods, a certain level of monotonicity is required for the cost operator. In this work, our main focus is on the projection method. Several authors have proposed and studied projection type algorithms for approximating the solutions of VIP (1.1) (see [1, 13, 14, 22, 30, 43, 60] and other references therein).

Tseng [57] introduced and studied Tseng's extragradient method for approximating the solution of the VIP (1.1). The proposed method is defined as follows:

$$\begin{cases} y_n = P_{\mathcal{C}}(x_n - \lambda Ax_n) \\ x_{n+1} = y_n - \lambda(Ay_n - Ax_n), \quad \forall n \geq 0, \end{cases}$$

where A is monotone, L-Lipschitz continuous and $\lambda \in \left(0, \frac{2}{L}\right)$. The author obtained a weak convergence result under the assumption that $VI(\mathcal{C}, A) \neq \emptyset$.

The equilibrium problem (EP) was introduced by Blum and Oettli [7] and they defined it as follows: Find $x \in \mathcal{C}$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in \mathcal{C}, \tag{1.2}$$

where $\mathcal{C}$ is a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $\Phi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is a bifunction. A point $x \in \mathcal{C}$ that solves this problem is called the equilibrium point. We denote the solution set of EP (1.2) by $EP(\Phi)$. The EP (1.2) has received a lot of attention from several authors due to its application to problems arising in the field of optimization, economics, physics, variational inequalities, among others (see, for example, [39, 42, 47, 53] and other references therein). Several authors have analyzed and proposed various iterative algorithms for approximating the solution of the EP and other related optimization problems, (see, for example, [24, 40, 46] and other references therein).

Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ be real Hilbert spaces. Let $\mathcal{C}, \mathcal{Q}$ be nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{F}_2 : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. The split equality problem (SEP) is defined as follows:

$$\text{Find } x \in \mathcal{C} \text{ and } y \in \mathcal{Q} \text{ such that } \mathcal{F}_1 x = \mathcal{F}_2 y. \tag{1.3}$$

The SEP which was first proposed by Moudafi [37] allows asymmetric and partial relations between the variables x and y. It is used in numerous practical problems such as game theory, medical image reconstruction, partial differential equation, decomposition method, among others (see [29, 50]). We denote the solution set of (1.3) by

$$\Omega_{SEP} := \{(x, y) \in \mathcal{C} \times \mathcal{Q} \,|\, \mathcal{F}_1 x = \mathcal{F}_2 y\}.$$

Several authors have studied several effective methods for solving the SEP (see [50, 58] and other references therein).

If $\mathcal{H}_2 = \mathcal{H}_3$ and $\mathcal{F}_2 = I$ (I is the identity operator), (1.3) reduces to the split feasibility problem (SFP) proposed by Censor et al. [12] and defined as follows:

$$\text{Find } x \in \mathcal{C} \text{ such that } \mathcal{F}_1 x \in \mathcal{Q},$$

where $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. One of the most common method for solving (1.3) is the CQ projection method proposed and studied by Bryne et al. [9]. They defined it as follows:

$$\begin{cases} x_{n+1} = P_{\mathcal{C}}(x_n - \eta_n \mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)) \\ y_{n+1} = P_{\mathcal{Q}}(y_n + \eta_n \mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)), \end{cases} \tag{1.4}$$

where $\eta_n \in \left(\epsilon, \frac{2}{\lambda_{\mathcal{F}_1} + \lambda_{\mathcal{F}_2}} - \epsilon\right)$, and $\lambda_{\mathcal{F}_1}$ and $\lambda_{\mathcal{F}_2}$ are the matrix operator norms $\|\mathcal{F}_1\|$ and $\|\mathcal{F}_2\|$, respectively. Note that the step size η_n in Algorithm (1.4) is dependent on the operator norms, which are difficult and sometimes impossible to compute. Several authors have studied several effective methods for solving SFP (see [50] and other references therein).

Another problem of interest in this study is the fixed point problem (FPP), which is formulated as follows:

$$\text{Find } x \in \mathcal{H} \text{ such that } Tx = x,$$

where $T : \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping. We denote the set of fixed points of T by $F(T)$. Several problems in sciences and engineering can be formulated as the problem of finding solutions of FPP of nonlinear mappings.

If $\mathcal{C}$ and $\mathcal{Q}$ are the sets of fixed points of some nonlinear operators, the SEP (1.3) becomes the split equality common fixed point problem (SECFPP) which is defined as

$$\text{Find } x \in F(T_1) \text{ and } y \in F(T_2) \text{ such that } \mathcal{F}_1 x = \mathcal{F}_2 y, \tag{1.5}$$

where $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$ are the sets of fixed points of T_1 and T_2, respectively, $T_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 : \mathcal{H}_2 \to \mathcal{H}_2$ are nonlinear mappings and $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_3$, $\mathcal{F}_2 : \mathcal{H}_2 \to \mathcal{H}_3$ are bounded linear operators.

If $\mathcal{H}_2 = \mathcal{H}_3$ and $\mathcal{F}_2 = I$, then the SECFPP (1.5) reduces to the following split common fixed point problem (SCFPP) introduced by Censor et al. [11]

$$\text{Find } x \in F(T_1) \text{ such that } \mathcal{F}_1 x \in F(T_2).$$

Several authors have studied and proposed effective methods for solving SCFPP (see [49] and other references therein).

The SECFPP was first studied by Moudafi et al. [37]. They introduced the following simultaneous iterative method for solving the SECFPP

$$\begin{cases} x_{n+1} = T_1(x_n - \eta_n \mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)) \\ y_{n+1} = T_2(y_n + \eta_n \mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)), \end{cases} \tag{1.6}$$

where $\eta_n \in \left(\epsilon, \frac{2}{\lambda_{\mathcal{F}_1}+\lambda_{\mathcal{F}_2}} - \epsilon\right)$, $\lambda_{\mathcal{F}_1}$ and $\lambda_{\mathcal{F}_2}$ are the spectral radius of $\mathcal{F}_1^*\mathcal{F}_1$ and $\mathcal{F}_2^*\mathcal{F}_2$, respectively, and T_1 and T_2 are firmly quasi-nonexpansive mappings. We also observe that the step size of Algorithm (1.6) depends on the operator norms. Hence to implement Algorithm (1.6), one has to compute the operator norms of $\mathcal{F}_1$ and $\mathcal{F}_2$ which are difficult to compute. Several authors have studied and proposed modifications of Algorithm (1.6) for better implementation (see [36, 37, 64] and other references therein).

Recently, Lopéz et al. [32] studied and proposed a method for estimating the step size which does not require prior knowledge of the operator norms for solving the SFP. Dong et al. [17] and J. Zhao [63] also proposed new choices of step size which do not require prior knowledge of the operator norm for solving SECFPP. Zhao [63] studied the SEP and presented the following step size which guarantees convergence of the iterative method without prior knowledge of the operator norm of $\mathcal{F}_1$ and $\mathcal{F}_2$

$$\eta_n \in \left(0, \frac{2\|\mathcal{F}_1 x_n - \mathcal{F}_2 y_n\|^2}{\|\mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)\|^2 + \|\mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)\|^2}\right).$$

The main purpose of this work is to find a common element of split equalities of the VIP, EP and common fixed point of nonexpansive semigroups. Several algorithms have been proposed for approximating the common solution of VIP, EP and FPP due to the applications it has on mathematical models whose constraints can be expressed as VIP, EP and FPP. Particularly, finding common solution problems has application in signal processing, network resource allocation, image recovery, among others (see [26, 33, 34] and other references therein).

Recently, Latif and Eslamian [31] studied and introduced a new algorithm for finding a common element of split equalities of EP, monotone VIP with Lipschitz operator and fixed point problem of nonexpansive semigroups satisfying the uniformly asymptotically regularity (u.a.r) condition in Hilbert spaces. The authors obtained strong convergence result for the proposed algorithm. However, their proposed algorithm has certain drawbacks. For instance, their method requires computing two projections each per iteration onto $\mathcal{C}$ and $\mathcal{Q}$, which makes it computationally expensive to implement. Moreover, the associated cost operators for the VIP are required to be monotone and Lipschitz continuous and the step size of the algorithm depends on the Lipschitz constants of these operators. In addition, the authors needed to impose the uniformly asymptotically regularity condition on the nonexpansive semigroups to obtain their result. All of these drawbacks limit the scope of application of their proposed method.

The inertial technique has been employed by several authors to increase the convergence rate of iterative methods. Polyak [45] studied the convergence of the following inertial extrapolation algorithm

$$x_{n+1} = x_n + \alpha_1(x_n - x_{n-1}) - \alpha_2 A x_n, \ \forall n \geq 0,$$

where α_1 and α_2 are two real numbers. Recently, there has been an increased interest in studying inertial type algorithm (see [2, 5, 6, 23, 28, 59] and other references therein).

Motivated by the above results in the literature and other related results in this direction, we propose and study an inertial Tseng's extragradient algorithm for the SEP for finding a common element of solution of the EP, VIP and common fixed point of nonexpansive semigroups with the following features:

(i) Different from other existing methods for finding a common element of the solution of the EP, VIP and fixed point problem of nonexpansive semigroups, our method only requires that the underlying operator for the VIP be pseudomonotone, uniformly continuous and without the weak sequential continuity condition often used in the literature. Also, we do not need to assume the u.a.r condition employed by authors in the literature to obtain our strong convergence result.
(ii) Different from other existing methods in the literature for solving non-Lipschitz VIP, our method does not require any linesearch technique but rather uses an easily implementable self-adaptive step size technique that generates non-monotonic sequence of step sizes. Also, our method only requires one projection each per iteration onto the feasible sets $\mathcal{C}$ and $\mathcal{Q}$.
(iii) Our method employs the inertial extrapolation technique to increase the rate of convergence (see [4–6] and other references therein).
(iv) The proof of our strong convergence result does not rely on the usual "two cases approach" widely used in many papers to prove strong convergence results.

Finally, we provide some numerical experiments for our proposed method in comparison with the related method in the literature to show the applicability of our proposed method.

The rest of the paper is organized as follows: In Section 2 we present some definitions and lemmas needed to obtain the strong convergence result. In Section 3, we present our proposed method and discuss some of its important features. In Section 4, the convergence of our method is investigated and in Section 5, we present some numerical experiments of our method in comparison with a related method in the literature. We conclude in Section 6.

2 Preliminaries

In this section, we recall some lemmas, results and definitions which will be required in subsequent sections to obtain our strong convergence result. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot\ \cdot \rangle$, and associated norm $|| \cdot ||$ defined by $||x|| = \sqrt{\langle x, x\rangle}$, $\forall x \in \mathcal{H}$. We denote the strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$", respectively. Also, we denote the set of weak limits of $\{x_n\}$ by $w_\omega(x_n)$, that is

$$w_\omega(x_n) := \left\{x \in \mathcal{H} \ : \ x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\right\}.$$

Definition 2.1 Let $T : \mathcal{H} \to \mathcal{H}$ be a mapping. Then, T is said to be

(i) *L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H};$$

if $L \in [0, 1)$, then T is called a *contraction;*
(ii) *Uniformly continuous*, if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that

$$\|Tx - Ty\| < \epsilon \quad \text{whenever} \quad \|x - y\| < \delta, \ \ \forall x, y \in \mathcal{H};$$

(iii) *Sequentially weakly continuous*, if for each sequence $\{x_n\}$, we have $x_n \rightharpoonup x \in \mathcal{H}$ implies that $Tx_n \rightharpoonup Tx \in \mathcal{H}$;
(iv) *Nonexpansive* if T is 1-Lipschitz continuous;
(v) *Firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \ \ \forall x, y \in \mathcal{H}.$$

More information on firmly nonexpansive mappings can be found, for example, in [21, Section 11]. Observe that uniform continuity is a weaker notion than Lipschitz continuity.

Definition 2.2 A one-parameter family mapping $\mathcal{T} = \{T(s) : 0 \leq s < +\infty\}$ from $\mathcal{H}_1$ into itself is said to be a nonexpansive semigroup if it satisfies the following conditions:

(i) $T(0)x = x, \ \forall x \in \mathcal{H}_1$;
(ii) $T(s+u) = T(s)T(u)$ for all $s, u \geq 0$;
(iii) For each $x \in \mathcal{H}_1$, the mapping $T(s)x$ is continuous;
(iv) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}_1$ and $s \geq 0$.

We denote the common fixed point set of the semigroup $\mathcal{T}$ by $F(\mathcal{T}) = \{x \in \mathcal{C} : T(s)x = x, \ \forall s \geq 0\}$. It is well known that $F(\mathcal{T})$ is closed and convex [8].

Lemma 2.3 [48, 55] *Let $\mathcal{C}$ be a nonempty bounded closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $\mathcal{T} = \{T(s) : s \geq 0\}$ from $\mathcal{C}$ be a nonexpansive semigroup on $\mathcal{C}$. Then for all $h \geq 0$,*

$$\limsup_{t\to\infty,\ x\in\mathcal{C}} \left\| \frac{1}{t}\int_0^t T(s)x - T(h)\Big(\frac{1}{t}\int_0^t T(s)x dx\Big)\right\| = 0.$$

Lemma 2.4 [55] *Let $\mathcal{C}$ be a nonempty bounded closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $\{x_n\}$ be a sequence and let $\mathcal{T} = \{T(s) : s \geq 0\}$ from $\mathcal{C}$ be a nonexpansive semigroup on $\mathcal{C}$, if the following conditions are satisfied*

(i) $x_n \rightharpoonup x$;
(ii) $\limsup_{s\to\infty} \limsup_{n\to\infty} \|T(s)x_n - x_n\| = 0$,

then, $x \in F(\mathcal{T})$.

It is well known that if D is a convex subset of $\mathcal{H}$, then $T : D \to \mathcal{H}$ is uniformly continuous if and only if, for every $\epsilon > 0$, there exists a constant $M < +\infty$ such that

$$\|Tx - Ty\| \leq M\|x - y\| + \epsilon, \ \ \forall x, y \in D. \tag{2.1}$$

For the proof of (2.1), see [61, Theorem 1].

Lemma 2.5 [38] *Let $\mathcal{H}$ be a real Hilbert space, then the following assertions hold:*

(1) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \ \forall x, y \in \mathcal{H}$;
(2) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2, \ \forall x, y \in \mathcal{H}, \ \alpha \in \mathbb{R}$;
(3) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \forall x, y \in \mathcal{H}$.

Lemma 2.6 [16] *Assume that $A : \mathcal{H} \to \mathcal{H}$ is a continuous and pseudomonotone operator. Then, x is a solution of (1.1) if and only if $\langle Ay, y - x\rangle \geq 0, \ \forall y \in \mathcal{C}$.*

Lemma 2.7 [35] *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{C}$ be a nonempty closed and convex subset of $\mathcal{H}$. If the mapping $h : [0, 1] \to \mathcal{H}$ defined as $h(t) := A(tx + (1-t)y)$ is continuous for all $x, y \in \mathcal{C}$ (i.e. h is hemicontinuous), then $M(A, \mathcal{C}) := \{x \in \mathcal{C} : \big\langle Ay, y - x\big\rangle \geq 0, \ \forall y \in \mathcal{C}\} \subset VI(\mathcal{C}, A)$. Moreover, if A is pseudo-monotone, then $VI(\mathcal{C}, A)$ is closed, convex and $M(\mathcal{C}, A) = VI(\mathcal{C}, A)$.*

Recall that for a nonempty, closed and convex subset $\mathcal{C}$ of $\mathcal{H}$, the metric projection denoted by $P_{\mathcal{C}}$, is a map defined on $\mathcal{H}$ onto $\mathcal{C}$ which assigns to each $x \in \mathcal{H}$, the unique point in $\mathcal{C}$, denoted by $P_{\mathcal{C}}x$ such that

$$||x - P_{\mathcal{C}}x|| = \inf\{||x - y|| : y \in \mathcal{C}\}.$$

Lemma 2.8 [21] *Let $\mathcal{C}$ be a closed and convex subset of a real Hilbert space $\mathcal{H}$ and $x, y \in \mathcal{H}$. Then*

(i) $\|P_{\mathcal{C}}x - P_{\mathcal{C}}y\|^2 \leq \langle P_{\mathcal{C}}x - P_{\mathcal{C}}y,\ x - y\rangle$;
(ii) $\|P_{\mathcal{C}}x - y\|^2 \leq \|x - y\|^2 - \|x - P_{\mathcal{C}}x\|^2$.

Assumption 2.9 [7] *Let $\Phi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

1) $\Phi(x, x) = 0,\ \forall x \in \mathcal{C}$;
2) *Φ is monotone, i.e.,* $\Phi(x, y) + \Phi(y, x) \leq 0,\ \forall x \in \mathcal{C}$;
3) *For each* $x, y, z \in \mathcal{C},\ \limsup_{t\to 0} \Phi(tz + (1 - t)x, y) \leq \Phi(x, y)$;
4) *For each* $x \in \mathcal{C},\ y \to \Phi(x, y)$ *is convex and lower semi continuous.*

Lemma 2.10 [15] *Let $\Phi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a bifunction satisfying Assumption 2.9. For any $r > 0$ and $x \in \mathcal{H}$, define a mapping $U_r^{\Phi} : \mathcal{H} \to \mathcal{C}$ as follows*

$$U_r^{\Phi}(x) = \left\{z \in \mathcal{C} : \Phi(z, y) + \frac{1}{r}\left\langle y - z, z - x\right\rangle \geq 0,\ \ \forall y \in \mathcal{C}\right\}.$$

Then, we have the following

(1) *U_r^{Φ} is nonempty and single valued;*
(2) *U_r^{Φ} is firmly nonexpansive;*
(3) *$F(U_r^{\Phi}) = EP(\Phi)$ is closed and convex.*

Definition 2.11 Assume that $T : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator with $F(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in $\mathcal{H}$, the following implication holds:

$$x_n \rightharpoonup x \text{ and } (I - T)x_n \to 0 \implies x \in F(T).$$

Lemma 2.12 [54] *Suppose $\{\lambda_n\}$ and $\{\theta_n\}$ are two nonnegative real sequences such that*

$$\lambda_{n+1} \leq \lambda_n + \phi_n,\ \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \phi_n < \infty$, then $\lim_{n\to\infty} \lambda_n$ exists.

Lemma 2.13 [3] *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\gamma_n\}$ be a sequence of real numbers in $(0, 1)$ with conditions $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n d_n,\ \ n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.14 [44] *Each Hilbert space H satisfies the Opial condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$ holds for every $y \in H$ with $y \neq x$.*

3 Proposed Method

In this section, we present our proposed method and discuss its features. We begin with the following assumptions under which our strong convergence result is obtained.

Assumption 3.1 *Suppose that the following conditions hold:*

(a) *The feasible sets $\mathcal{C}$ and $\mathcal{Q}$ are nonempty, closed and convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.*
(b) *$A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ are pseudomonotone and uniformly continuous.*
(c) *The mapping $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ satisfies the following property: whenever $\{x_n\} \subset \mathcal{C}$, $x_n \rightharpoonup x^*$, one has $\|Ax^*\| \leq \liminf_{n\to\infty} \|Ax_n\|$ and whenever $\{x_n\} \subset \mathcal{Q}$, $x_n \rightharpoonup x^*$, one has $\|Bx^*\| \leq \liminf_{n\to\infty} \|Bx_n\|$, respectively.*
(d) *$\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{F}_2 : \mathcal{H}_2 \to \mathcal{H}_3$ are bounded linear operators.*
(e) *$\Phi_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, $\Phi_2 : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ are bifunctions satisfying Assumption 2.9 and Φ_2 is upper semi continuous in the first argument.*
(f) *$\mathcal{T}_a = \{T_1(s) : 0 \leq s < \infty\}$ and $\mathcal{T}_b = \{T_2(u) : 0 \leq u < \infty\}$ are one-parameter nonexpansive semigroups on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.*
(g) *The solution set $\Gamma = \{x \in EP(\Phi_1) \cap VI(\mathcal{C}, A) \cap F(\mathcal{T}_a),\ y \in EP(\Phi_2) \cap VI(\mathcal{Q}, B) \cap F(\mathcal{T}_b) : \mathcal{F}_1 x = \mathcal{F}_2 y\} \neq \emptyset$.*
(h) *$\{\alpha_n\} \subset (0, 1)$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$, $\lim_{n\to\infty} \alpha_n = 0$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.*
(i) *Let $\{\epsilon_n\}$ and $\{\zeta_n\}$ be positive sequences such that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ and $\lim_{n\to\infty} \frac{\zeta_n}{\alpha_n} = 0$, respectively.*
(j) *Let $\{\sigma_n\}$ and $\{\mu_n\}$ be nonnegative sequences such that $\sum_{n=1}^{\infty} \sigma_n < +\infty$ and $\sum_{n=1}^{\infty} \mu_n < +\infty$, respectively, $\{t_{n,1}\}, \{t_{n,2}\} \subset (0, +\infty)$, $\liminf r_{n,1} > 0$, $\liminf r_{n,2} > 0$.*

Algorithm 3.2 **Step 0:** *Choose sequences $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that the conditions from Assumption 3.1* (h)–(i) *hold. Select an initial point $(x_0, y_0) \in \mathcal{H}_1 \times \mathcal{H}_2$, let $\eta \geq 0$, $a_i \in (0, 1)$, $i = 1, 2$, $\lambda_1 > 0$, $\rho_1 > 0$, $\theta > 0$, $\tau > 0$ and set $n := 1$.*
Step 1: *Given the iterates x_{n-1}, y_{n-1} and x_n, y_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$ and τ_n such that $0 \leq \tau_n \leq \bar{\tau}_n$, where*

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1} \\ \theta & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: *Compute*

$$w_n = (1 - \alpha_n)\Big(x_n + \theta_n(x_n - x_{n-1})\Big)$$

and

$$\varphi_n = (1 - \alpha_n)\Big(y_n + \tau_n(y_n - y_{n-1})\Big).$$

Step 3: *Compute*

$$\begin{aligned} z_n &= w_n - \eta_n \mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n), \\ \phi_n &= U_{r_{n,1}}^{\Phi_1} z_n, \\ u_n &= P_{\mathcal{C}}(\phi_n - \lambda_n A\phi_n), \\ v_n &= u_n - \lambda_n(Au_n - A\phi_n), \\ x_{n+1} &= (1 - \beta_n)v_n + \beta_n \frac{1}{t_{n,1}} \int_0^{t_{n,1}} T_1(s)v_n\, ds \end{aligned}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{a_1\|u_n-\phi_n\|}{\|Au_n-A\phi_n\|},\ \lambda_n+\sigma_n\right\} & \text{if } Au_n \neq A\phi_n \\ \lambda_n+\sigma_n & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 4: *Compute*

$$\bar{\tau}_n := \begin{cases} \min\left\{\tau,\ \frac{\zeta_n}{\|y_n-y_{n-1}\|}\right\} & \text{if } y_n \neq y_{n-1} \\ \tau & \text{otherwise.} \end{cases} \tag{3.3}$$

Step 5 *Compute*

$$k_n = \varphi_n + \eta_n \mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n).$$

Step 6: *Compute*

$$\begin{aligned} \psi_n &= U_{r_{n,2}}^{\Phi_2} k_n, \\ s_n &= P_Q(\psi_n - \rho_n B\psi_n), \\ b_n &= s_n - \rho_n(Bs_n - B\psi_n), \\ y_{n+1} &= (1-\gamma_n)b_n + \gamma_n \frac{1}{t_{n,2}} \int_0^{t_{n,2}} T_2(u)b_n\, du \end{aligned}$$

and

$$\rho_{n+1} = \begin{cases} \min\left\{\frac{a_2\|s_n-\psi_n\|}{\|Bs_n-B\psi_n\|},\ \rho_n+\mu_n\right\} & \text{if } Bs_n \neq B\psi_n \\ \rho_n+\mu_n & \text{otherwise,} \end{cases} \tag{3.4}$$

where the step size η_n is chosen such that for small enough $\epsilon > 0$,

$$\eta_n \in \left[\epsilon,\ \frac{2\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2}{\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2} - \epsilon\right],$$

if $\mathcal{F}_1 w_n \neq \mathcal{F}_2\varphi_n$; otherwise, $\eta_n = \eta$.
Set $n := n+1$ and go back to **Step 1**.

Remark 3.3 The step sizes generated in (3.2) and (3.4) are allowed to increase per iteration. This reduces their dependence on the initial step sizes. When n is large enough the step size may not increase. We assume that Algorithm 3.2 does not terminate in a finite number of iterations.

Remark 3.4 By conditions (h) and (i), from (3.1) we observe that

$$\lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0 \ \text{ and } \ \lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0. \tag{3.5}$$

Similarly, from (3.3) we have

$$\lim_{n\to\infty} \tau_n\|y_n - y_{n-1}\| = 0 \ \text{ and } \ \lim_{n\to\infty} \frac{\tau_n}{\alpha_n}\|y_n - y_{n-1}\| = 0.$$

Remark 3.5 We note that condition (c) of Assumption 3.1 is weaker than the sequentially weakly continuity condition.

We present an example which satisfies condition (c) of Assumption 3.1.

Example 3.6 Let $A : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ be an operator defined by

$$Ax^* = x^*\|x^*||, \quad \forall x^* \in \ell_2.$$

Suppose that $\{x_n\} \subset \ell_2(\mathbb{R})$ such that $x_n \rightharpoonup x^*$. Then, by the weakly lower semi-continuity of the norm we obtain

$$\|x^*\| \leq \liminf_{n\to+\infty} \|x_n\|.$$

Thus,

$$\|Ax^*\| = \|x^*\|^2 \leq \left(\liminf_{n\to+\infty} \|x_n\|\right)^2 \leq \liminf_{n\to+\infty} \|x_n\|^2 = \liminf_{n\to+\infty} \|Ax_n\|.$$

Hence, A satisfies condition (c) of Assumption 3.1.

Remark 3.7 Since the sequences of step sizes generated by the algorithm in (3.2) and (3.4) are well defined and the limits $\lim_{n\to\infty} \lambda_n$ and $\lim_{n\to\infty} \rho_n$ exist (see Lemma 4.1). Then, the limit

$$\lim_{n\to\infty} \left(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\right) = 1 - a_1^2 > 0. \tag{3.6}$$

Thus, there exists $n_{0_1} > 0$ such that for all $n > n_{0_1}$, we have $\left(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\right) > 0.$

Similarly, we have that

$$\lim_{n\to\infty} \left(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\right) = 1 - a_2^2 > 0, \tag{3.7}$$

and there exists $n_{0_2} > 0$ such that for all $n > n_{0_2}$, we have $\left(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\right) > 0$. Now, we set $n_0 = \max\{n_{0_1}, n_{0_2}\}$.

Remark 3.8 From the definition of η_n, that is,

$$\eta_n \in \left[\epsilon, \quad \frac{2\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2}{\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2} - \epsilon\right]$$

we have

$$(\eta_n + \epsilon)\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big] \leq 2\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2.$$

Expanding the last inequality, we have

$$\begin{aligned} &\eta_n \cdot \epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|)^2\Big] \\ \leq &\eta_n\Big(2\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2 - \eta_n\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big]\Big). \end{aligned} \tag{3.8}$$

4 Convergence Analysis

Lemma 4.1 *Let $\{\lambda_n\}$ and $\{\rho_n\}$ be sequences generated by Algorithm 3.2. Then, we have $\lim_{n\to\infty} \lambda_n = \lambda$, where $\lambda \in [\min\{\frac{a_1}{K_1}, \lambda_1\}, \lambda_1 + b_1]$, $b_1 = \sum_{n=1}^{\infty} \sigma_n$ for some $K_1 > 0$ and $\lim_{n\to\infty} \rho_n = \rho$, where $\rho \in [\min\{\frac{a_2}{K_2}, \rho_1\}, \rho_1 + b_2]$, $b_2 = \sum_{n=1}^{\infty} \mu_n$ for some $K_2 > 0$.*

Proof Since A is uniformly continuous, we obtain from (2.1) that for any given $\epsilon > 0$, there exists a constant $M < +\infty$ such that $\|Au_n - A\phi_n\| \leq M\|u_n - \phi_n\| + \epsilon$. Thus, when $Au_n - A\phi_n \neq 0$ for all $n \geq 1$ we have

$$\frac{a_1\|u_n - \phi_n\|}{\|Au_n - A\phi_n\|} \geq \frac{a_1\|u_n - \phi_n\|}{M\|u_n - \phi_n\| + \epsilon} \geq \frac{a_1\|u_n - \phi_n\|}{(M + \epsilon_1)\|u_n - \phi_n\|} = \frac{a_1}{K_1},$$

where $\epsilon = \min\{\epsilon_1\|u_n - \phi_n\| \, : \, n \in \mathbb{N}\}$ for some $\epsilon_1 > 0$ and $K_1 = M + \epsilon_1$. Hence, from the definition of λ_{n+1}, the sequence $\{\lambda_n\}$ is bounded below by $\min\{\frac{a_1}{K_1}, \lambda_1\}$ and above by $\lambda_1 + b_1$. By Lemma 2.12, it follows that $\lim_{n\to\infty} \lambda_n$ denoted by $\lambda = \lim_{n\to\infty} \lambda_n$ exists. Clearly, we have $\lambda \in [\min\{\frac{a_1}{K_1}, \lambda_1\}, \lambda_1 + b_1]$.

Similarly, we have $\lim_{n\to\infty} \rho_n = \rho$, and $\rho \in \min\{\frac{a_2}{K_2}, \rho_1\}, \rho_1 + b_2$. □

Lemma 4.2 *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then*

$$\|z_n - x^*\|^2 + \|k_n - y^*\|^2 \leq \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2.$$

Proof Let $(x^*, y^*) \in \Gamma$. Then, by applying Lemma 2.5, we have

$$\begin{aligned}\|z_n - x^*\|^2 =& \|w_n - \eta_n\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n) - x^*\|^2\\ =& \|w_n - x^*\|^2 + \eta_n^2\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2 - 2\eta_n\left\langle w_n - x^*, \ \mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\right\rangle\\ =& \|w_n - x^*\|^2 + \eta_n^2\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2 - 2\eta_n\left\langle \mathcal{F}_1 w_n - \mathcal{F}_1 x^*, \ \mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\right\rangle\\ =& \|w_n - x^*\|^2 + \eta_n^2\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2 - \eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_1 x^*\|^2\\ &- \eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2 + \eta_n\|\mathcal{F}_2\varphi_n - \mathcal{F}_1 x^*\|^2.\end{aligned} \tag{4.1}$$

Similarly, we have

$$\begin{aligned}\|k_n - y^*\|^2 =& \|\varphi_n + \eta_n\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n) - y^*\|^2\\ =& \|\varphi_n - y^*\|^2 + \eta_n^2\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2 - \eta_n\|\mathcal{F}_2\varphi_n - \mathcal{F}_2 y^*\|^2\\ &- \eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2 + \eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_2 y^*\|^2.\end{aligned} \tag{4.2}$$

Adding (4.1) and (4.2), we have

$$\begin{aligned}&\|z_n - x^*\|^2 + \|k_n - y^*\|^2\\ =& \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 + \eta_n^2\Big[\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2\\ &+ \|\mathcal{F}_2^*(\mathcal{F}_1 w_n - F_2\varphi_n)\|^2\Big] - \eta_n\Big[\|\mathcal{F}_1 w_n - \mathcal{F}_1 x^*\|^2 + \|\mathcal{F}_2\varphi_n - \mathcal{F}_2 y^*\|^2\Big]\\ &- 2\eta_n\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2 + \eta_n\Big[\|\mathcal{F}_1 w_n - \mathcal{F}_2 y^*\|^2 + \|\mathcal{F}_2\varphi_n - \mathcal{F}_1 x^*\|^2\Big].\end{aligned}$$

By (3.8) and the fact that $\mathcal{F}_1 x^* = \mathcal{F}_2 y^*$, we have

$$\begin{aligned}&\|z_n - x^*\|^2 + \|k_n - y^*\|^2\\ =& \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 - \eta_n\Big[2\|\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n\|^2\\ &- \eta_n\Big(\|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big)\Big]\\ =& \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 - \eta_n \cdot \epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big]\\ &\leq \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2,\end{aligned} \tag{4.3}$$

which is the desired result. □

Lemma 4.3 *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then*

$$\|v_n - x^*\|^2 \le \|z_n - x^*\|^2 - \|z_n - \phi_n\|^2 - \left(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\right)\|u_n - \phi_n\|^2$$

and

$$\|b_n - y^*\|^2 \le \|k_n - y^*\|^2 - \|k_n - \psi_n\|^2 - \left(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\right)\|s_n - \psi_n\|^2.$$

Proof Let $(x^*, y^*) \in \Gamma$. Since $U_{r_{n,1}}^{\Phi_1}$ is firmly nonexpansive, it follows from Lemma 2.8 that

$$\|\phi_n - x^*\|^2 = \|U_{r_{n,1}}^{\Phi_1} z_n - x^*\| \le \|z_n - x^*\|^2 - \|z_n - \phi_n\|^2. \tag{4.4}$$

Similarly, we have

$$\|\psi_n - y^*\|^2 = \|U_{r_{n,2}}^{\Phi_2} k_n - y^*\| \le \|k_n - y^*\|^2 - \|k_n - \psi_n\|^2.$$

From (3.2), we obtain

$$\lambda_{n+1} = \min\left\{\frac{a_1\|u_n - \phi_n\|}{\|Au_n - A\phi_n\|}, \lambda_n + \sigma_n\right\} \le \frac{a_1\|u_n - \phi_n\|}{\|Au_n - A\phi_n\|},$$

which implies that

$$\|Au_n - A\phi_n\| \le \frac{a_1}{\lambda_{n+1}}\|u_n - \phi_n\|, \quad \forall n \ge 1. \tag{4.5}$$

Similarly, we have

$$\|Bs_n - B\psi_n\| \le \frac{a_2}{\rho_{n+1}}\|s_n - \psi_n\|, \quad \forall n \ge 1.$$

From the definition of v_n in **Step 3** and Lemma 2.5, we have

$$\begin{aligned}
\|v_n - x^*\|^2 \le& \|u_n - \lambda_n(Au_n - A\phi_n) - x^*\|^2 \\
=& \|u_n - x^*\|^2 + \lambda_n^2\|Au_n - A\phi_n\|^2 - 2\lambda_n\langle Au_n - A\phi_n, u_n - x^*\rangle \\
=& \|\phi_n - x^*\|^2 + \|u_n - \phi_n\|^2 + 2\langle u_n - \phi_n, \phi_n - x^*\rangle \\
&+ \lambda_n^2\|Au_n - A\phi_n\|^2 - 2\lambda_n\langle Au_n - A\phi_n, u_n - x^*\rangle \\
=& \|\phi_n - x^*\|^2 + \|u_n - \phi_n\|^2 - 2\langle u_n - \phi_n, u_n - \phi_n\rangle \\
&+ 2\langle u_n - \phi_n, u_n - x^*\rangle - 2\lambda_n\langle Au_n - A\phi_n, u_n - x^*\rangle + \lambda_n^2\|Au_n - A\phi_n\|^2 \\
=& \|\phi_n - x^*\|^2 - \|u_n - \phi_n\|^2 + 2\langle u_n - \phi_n, u_n - x^*\rangle \\
&- 2\lambda_n\langle Au_n - A\phi_n, u_n - x^*\rangle + \lambda_n^2\|Au_n - A\phi_n\|^2.
\end{aligned} \tag{4.6}$$

Since $u_n = P_{\mathcal{C}}(\phi_n - \lambda_n A\phi_n)$ and $x^* \in \mathcal{C}$, we obtain from the characteristic property of $P_{\mathcal{C}}$ that

$$\langle u_n - \phi_n + \lambda_n A\phi_n, u_n - x^*\rangle \le 0.$$

This implies that

$$\langle u_n - \phi_n, u_n - x^*\rangle \le -\lambda_n\langle A\phi_n, u_n - x^*\rangle. \tag{4.7}$$

Also since $u_n \in \mathcal{C}$ and $x^* \in \Gamma$ we have

$$\langle Au_n, \ \ u_n - x^* \rangle \geq 0, \ \ \forall n \geq 0. \tag{4.8}$$

Applying (4.4), (4.5), (4.7) and (4.8) in (4.6), we obtain

$$\begin{aligned}\|v_n - x^*\|^2 \leq & \|\phi_n - x^*\|^2 - \|u_n - \phi_n\|^2 \\ & - 2\lambda_n \langle A\phi_n, u_n - x^* \rangle - 2\lambda_n \langle Au_n - A\phi_n, u_n - x^* \rangle + \lambda_n^2 \|Au_n - A\phi_n\|^2 \\ = & \|\phi_n - x^*\|^2 - \|u_n - \phi_n\|^2 - 2\lambda_n \langle Au_n, u_n - x^* \rangle + \lambda_n^2 \|Au_n - A\phi_n\|^2 \\ \leq & \|\phi_n - x^*\|^2 - \|u_n - \phi_n\|^2 + \lambda_n^2 \frac{a_1^2}{\lambda_{n+1}^2} \|u_n - \phi_n\|^2 \\ = & \|z_n - x^*\|^2 - \|z_n - \phi_n\|^2 - \left(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\right) \|u_n - \phi_n\|^2. \end{aligned} \tag{4.9}$$

Following the same line of argument, we have

$$\|b_n - y^*\|^2 \leq \|k_n - y^*\|^2 - \|k_n - \psi_n\|^2 - \left(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\right) \|s_n - \psi_n\|^2, \tag{4.10}$$

which completes the proof. □

Lemma 4.4 *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 3.2 satisfying Assumption 3.1. Then $\{(x_n, y_n)\}$ is bounded.*

Proof Let $x^* \in \Gamma$. From the definition of w_n and Lemma 2.5, we have

$$\begin{aligned}\|w_n - x^*\| &= \|(1 - \alpha_n)(x_n + \theta_n(x_n - x_{n-1})) - x^*\| \\ &= \|(1 - \alpha_n)(x_n - x^*) + (1 - \alpha_n)\theta_n(x_n - x_{n-1}) - \alpha_n x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + (1 - \alpha_n)\theta_n\|x_n - x_{n-1}\| + \alpha_n\|x^*\| \\ &= (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\Big[(1 - \alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|x^*\|\Big]. \end{aligned} \tag{4.11}$$

By (3.5), we have

$$\lim_{n\to\infty}\Big[(1 - \alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|x^*\|\Big] = \|x^*\|.$$

Thus, there exists a constant $M_1 > 0$ such that $(1 - \alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|x^*\| \leq M_1$ for all $n \in \mathbb{N}$. Thus, from (4.11) it follows that

$$\|w_n - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n M_1.$$

Consequently, we have

$$\|w_n - x^*\|^2 \leq (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n(1 - \alpha_n)M_1\|x_n - x^*\| + \alpha_n^2 M_1^2. \tag{4.12}$$

Following similar procedure, we have

$$\|\varphi_n - y^*\|^2 \leq (1 - \alpha_n)^2\|y_n - y^*\|^2 + 2\alpha_n(1 - \alpha_n)M_2\|y_n - y^*\| + \alpha_n^2 M_2^2. \tag{4.13}$$

Adding (4.12) and (4.13), we obtain

$$\begin{aligned}
&\|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 \\
\leq&(1-\alpha_n)^2\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] \\
&+ 2\alpha_n(1-\alpha_n)\Big(M_1\|x_n - x^*\| + M_2\|y_n - y^*\|\Big) + \alpha_n^2(M_1^2 + M_2^2) \\
\leq&(1-\alpha_n)\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] \\
&+ 2\alpha_n\Big(M_1\|x_n - x^*\| + M_2\|y_n - y^*\|\Big) + \alpha_n(M_1^2 + M_2^2) \\
=&(1-\alpha_n)\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] + \alpha_n c_n,
\end{aligned} \tag{4.14}$$

where $c_n = 2(M_1\|x_n - x^*\| + M_2\|y_n - y^*\|) + M_1^2 + M_2^2$. From **STEP 3**, and by applying Lemma 2.5, (4.9) together with Remark 3.6, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =&\Big\|(1-\beta_n)v_n + \beta_n\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - x^*\Big\|^2 \\
=&\Big\|(1-\beta_n)(v_n - x^*) + \beta_n\Big(\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - x^*\Big)\Big\|^2 \\
=&(1-\beta_n)\|v_n - x^*\|^2 + \beta_n\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - x^*\Big\|^2 \\
&- \beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\Big\|^2 \\
=&(1-\beta_n)\|v_n - x^*\|^2 + \beta_n\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - \frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)x^* ds\Big\|^2 \\
&- \beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\Big\|^2 \\
\leq&(1-\beta_n)\|v_n - x^*\|^2 + \beta_n\|v_n - x^*\|^2 - \beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\Big\|^2 \\
=&\|v_n - x^*\|^2 - \beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\Big\|^2 \\
\leq&\|z_n - x^*\|^2 - \|z_n - \phi_n\|^2 - \Big(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\Big)\|u_n - \phi_n\|^2 \\
&- \beta_n(1-\beta_n)\Big\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\Big\|^2 \qquad (4.15)\\
\leq&\|z_n - x^*\|^2. \qquad (4.16)
\end{aligned}$$

Similarly, from **STEP 5**, and by applying Lemma 2.5, (4.10) together with Remark 3.6, we have

$$\begin{aligned}
\|y_{n+1} - y^*\|^2 \leq&\|k_n - y^*\|^2 - \|k_n - \psi_n\|^2 - \Big(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\Big)\|s_n - \psi_n\|^2 \\
&- \gamma_n(1-\gamma_n)\Big\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}} T_2(u)b_n du - b_n\Big\|^2 \qquad (4.17)\\
\leq&\|k_n - y^*\|^2. \qquad (4.18)
\end{aligned}$$

From (4.3), (4.14), (4.16) and (4.18), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &\le \|z_n-x^*\|^2+\|k_n-y^*\|^2\\ &\le \|w_n-x^*\|^2+\|\varphi_n-y^*\|^2\\ &\le (1-\alpha_n)\Big[\|x_n-x^*\|^2+\|y_n-y^*\|^2\Big]+\alpha_n c_n\\ &\le \max\ \{\|x_n-x^*\|^2+\|y_n-y^*\|^2,\ \ c_n\}\\ &\ \ \vdots\\ &\le \max\ \{\|x_{n_0}-x^*\|^2+\|y_{n_0}-y^*\|^2,\ \ c_{n_0}\}.\end{aligned}$$

Thus, $\{(x_n, y_n)\}$ is bounded. Consequently, $\{z_n\}$, $\{v_n\}$, $\{k_n\}$ and $\{b_n\}$ are also bounded. □

Lemma 4.5 *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then,*

$$\begin{aligned}&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\ \le&(1-\alpha_n)\Big[\|x_n-x^*\|^2+\|y_n-y^*\|^2\Big]+\alpha_n d_n\\ &-\eta_n\cdot\epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n-\mathcal{F}_2\varphi_n)\|^2+\|\mathcal{F}_1^*(\mathcal{F}_1 w_n-\mathcal{F}_2\varphi_n)\|^2\Big]\\ &-\|z_n-\phi_n\|^2-\|k_n-\psi_n\|^2-\left(1-\frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\right)\|u_n-\phi_n\|^2-\left(1-\frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\right)\|s_n-\psi_n\|^2\\ &-\beta_n(1-\beta_n)\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}}T_1(s)v_n ds-v_n\|^2-\gamma_n(1-\gamma_n)\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}}T_2(u)b_n du-b_n\|^2,\end{aligned}$$

where $d_n = [2(1-\alpha_n)\|x_n-x^*\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\theta_n\|x_n-x_{n-1}\|\cdot\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\|x^*\|\|w_n-x_{n+1}\|+2\langle x^*, x^*-x_{n+1}\rangle]+[2(1-\alpha_n)\|y_n-y^*\|\frac{\tau_n}{\alpha_n}\|y_n-y_{n-1}\|+\tau_n\|y_n-y_{n-1}\|\cdot\frac{\tau_n}{\alpha_n}\|y_n-y_{n-1}\|+2\|y^*\|\|\varphi_n-y_{n+1}\|+2\langle y^*, y^*-y_{n+1}\rangle]$.

Proof Let $(x^*, y^*)\in\Gamma$. From Lemma 2.5 and the definition of w_n, we have

$$\begin{aligned}\|w_n-x^*\|^2=&\|(1-\alpha_n)(x_n-x^*)+(1-\alpha_n)\theta_n(x_n-x_{n-1})-\alpha_n x^*\|^2\\ \le&\|(1-\alpha_n)(x_n-x^*)+(1-\alpha_n)\theta_n(x_n-x_{n-1})\|^2+2\alpha_n\langle -x^*,\ \ w_n-x^*\rangle\\ \le&(1-\alpha_n)^2\|x_n-x^*\|^2+2(1-\alpha_n)\theta_n\|x_n-x^*\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2\\ &+2\alpha_n\langle -x^*,\ \ w_n-x_{n+1}\rangle+2\alpha_n\langle -x^*,\ \ x_{n+1}-x^*\rangle\\ \le&(1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\Big[2(1-\alpha_n)\|x_n-x^*\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\\ &+\theta_n\|x_n-x_{n-1}\|\cdot\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\\ &+2\|x^*\|\|w_n-x_{n+1}\|+2\langle x^*, x^*-x_{n+1}\rangle\Big].\end{aligned}\tag{4.19}$$

Following the same line of argument, we have

$$\begin{aligned}\|\varphi_n-y^*\|^2\le&(1-\alpha_n)\|y_n-y^*\|^2+\alpha_n\Big[2(1-\alpha_n)\|y_n-y^*\|\frac{\tau_n}{\alpha_n}\|y_n-y_{n-1}\|\\ &+\tau_n\|y_n-y_{n-1}\|\cdot\frac{\tau_n}{\alpha_n}\|y_n-y_{n-1}\|+2\|y^*\|\|\varphi_n-y_{n+1}\|+2\langle y^*, y^*-y_{n+1}\rangle\Big].\end{aligned}\tag{4.20}$$

Adding (4.19) and (4.20) we have

$$\|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 \le (1-\alpha_n)\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] + \alpha_n d_n. \tag{4.21}$$

From (4.3), (4.15), (4.17) and (4.21), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\\
\le&\|z_n - x^*\|^2 + \|k_n - y^*\|^2 - \|z_n - \phi_n\|^2 - \|k_n - \psi_n\|^2 - \Big(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\Big)\|u_n - \phi_n\|^2\\
&- \Big(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\Big)\|s_n - \psi_n\|^2 - \beta_n(1-\beta_n)\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\|^2\\
&- \gamma_n(1-\gamma_n)\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}} T_2(u)b_n du - b_n\|^2\\
&\le \|w_n - x^*\|^2 + \|\varphi_n - y^*\|^2 - \eta_n \cdot \epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big]\\
&- \|z_n - \phi_n\|^2 - \|k_n - \psi_n\|^2 - \Big(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\Big)\|u_n - \phi_n\|^2 - \Big(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\Big)\|s_n - \psi_n\|^2\\
&- \beta_n(1-\beta_n)\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\|^2 - \gamma_n(1-\gamma_n)\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}} T_2(u)b_n du - b_n\|^2\\
&\le (1-\alpha_n)\Big[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\Big] + \alpha_n d_n\\
&- \eta_n \cdot \epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 w_n - \mathcal{F}_2\varphi_n)\|^2\Big] - \|z_n - \phi_n\|^2 - \|k_n - \psi_n\|^2\\
&- \Big(1 - \frac{\lambda_n^2 a_1^2}{\lambda_{n+1}^2}\Big)\|u_n - \phi_n\|^2 - \Big(1 - \frac{\rho_n^2 a_2^2}{\rho_{n+1}^2}\Big)\|s_n - \psi_n\|^2\\
&- \beta_n(1-\beta_n)\|\frac{1}{t_{n,1}}\int_0^{t_{n,1}} T_1(s)v_n ds - v_n\|^2 - \gamma_n(1-\gamma_n)\|\frac{1}{t_{n,2}}\int_0^{t_{n,2}} T_2(u)b_n du - b_n\|^2,
\end{aligned}$$

which is the required result. □

Now we are in a position to state the main result of this work.

Theorem 4.6 *Let $\{(x_n, y_n)\}$ be a sequence generated by Algorithm 3.2 such that Assumption 3.1 holds. Then, the sequence $\{(x_n, y_n)\}$ converges strongly to $(\hat{x}, \hat{y}) = P_\Gamma(0_{\mathcal{H}_1}, 0_{\mathcal{H}_2}) \in \Gamma$.*

Proof Let $(\hat{x}, \hat{y}) = P_\Gamma(0_{\mathcal{H}_1}, 0_{\mathcal{H}_2}) \in \Gamma$. Then, it follows from Lemma 4.5 that

$$\|x_{n+1} - \hat{x}\|^2 + \|y_{n+1} - \hat{y}\|^2 \le (1-\alpha_n)\Big[\|x_n - \hat{x}\|^2 + \|y_n - \hat{y}\|^2\Big] + \alpha_n \hat{d}_n, \tag{4.22}$$

where $\hat{d}_n = [2(1-\alpha_n)\|x_n - \hat{x}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| \cdot \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\|\hat{x}\|\|w_n - x_{n+1}\| + 2\langle \hat{x}, \hat{x} - x_{n+1}\rangle] + [2(1-\alpha_n)\|y_n - \hat{y}\|\frac{\tau_n}{\alpha_n}\|y_n - y_{n-1}\| + \tau_n\|y_n - y_{n-1}\| \cdot \frac{\tau_n}{\alpha_n}\|y_n - y_{n-1}\| + 2\|\hat{y}\|\|\varphi_n - y_{n+1}\| + 2\langle \hat{y}, \hat{y} - y_{n+1}\rangle]$. Now, we claim that the sequence $\{\|x_n - \hat{x}\| + \|y_n - \hat{y}\|\}$ converges to zero. To show this, by Lemma 2.13 it suffices to show that $\limsup_{k\to\infty} \hat{d}_{n_k} \le 0$ for every subsequence $\{\|x_{n_k} - \hat{x}\| + \|y_{n_k} - \hat{y}\|\}$ of $\{\|x_n - \hat{x}\| + \|y_n - \hat{y}\|\}$ satisfying

$$\liminf_{k\to\infty}\Big(\big(\|x_{n_{k+1}} - \hat{x}\| + \|y_{n_{k+1}} - \hat{y}\|\big) - \big(\|x_{n_k} - \hat{x}\| + \|y_{n_k} - \hat{y}\|\big)\Big) \ge 0. \tag{4.23}$$

Suppose that $\{\|x_{n_k}-\hat{x}\|+\|y_{n_k}-\hat{y}\|\}$ is a subsequence of $\{\|x_n-\hat{x}\|+\|y_n-\hat{y}\|\}$ such that (4.23) holds. Again, from Lemma 4.5, we obtain

$$\begin{aligned}
&\eta_{n_k}\cdot\epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2+\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2\Big]\\
&+\|z_{n_k}-\phi_{n_k}\|^2+\|k_{n_k}-\psi_{n_k}\|^2\\
&+\Big(1-\frac{\lambda_{n_k}^2 a_1^2}{\lambda_{n_{k+1}}^2}\Big)\|u_{n_k}-\phi_{n_k}\|^2+\Big(1-\frac{\rho_{n_k}^2 a_2^2}{\rho_{n_{k+1}}^2}\Big)\|s_{n_k}-\psi_{n_k}\|^2\\
&+\beta_{n_k}(1-\beta_{n_k})\|\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds-v_{n_k}\|^2\\
&+\gamma_{n_k}(1-\gamma_{n_k})\|\frac{1}{t_{n_{k,2}}}\int_0^{t_{n_{k,2}}}T_2(u)b_{n_k}du-b_{n_k}\|^2\\
\le&(1-\alpha_{n_k})\Big[\|x_{n_k}-\hat{x}\|^2+\|y_{n_k}-\hat{y}\|^2\Big]-\Big[\|x_{n_{k+1}}-\hat{x}\|^2+\|y_{n_{k+1}}-\hat{y}\|^2\Big]+\alpha_{n_k}\hat{d}_{n_k}.
\end{aligned}$$

From (4.23) and the condition on α_{n_k} we have

$$\begin{aligned}
&\lim_{k\to\infty}\Big(\eta_{n_k}\cdot\epsilon\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2+\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2\Big]\\
&+\|z_{n_k}-\phi_{n_k}\|^2+\|k_{n_k}-\psi_{n_k}\|^2\\
&+\Big(1-\frac{\lambda_{n_k}^2 a_1^2}{\lambda_{n_{k+1}}^2}\Big)\|u_{n_k}-\phi_{n_k}\|^2+\Big(1-\frac{\rho_{n_k}^2 a_2^2}{\rho_{n_{k+1}}^2}\Big)\|s_{n_k}-\psi_{n_k}\|^2\\
&+\beta_{n_k}(1-\beta_{n_k})\|\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds-v_{n_k}\|^2\\
&+\gamma_{n_k}(1-\gamma_{n_k})\|\frac{1}{t_{n_{k,2}}}\int_0^{t_{n_{k,2}}}T_2(u)b_{n_k}du-b_{n_k}\|^2\Big)=0.
\end{aligned}$$

From (3.6), (3.7) and the conditions on the control parameters, we have

$$\lim_{k\to\infty}\|z_{n_k}-\phi_{n_k}\|=0,\ \lim_{k\to\infty}\|k_{n_k}-\psi_{n_k}\|=0,\ \lim_{k\to\infty}\|u_{n_k}-\phi_{n_k}\|=0,\ \lim_{k\to\infty}\|s_{n_k}-\psi_{n_k}\|=0, \tag{4.24}$$

$$\lim_{k\to\infty}\Big\|\frac{1}{t_{n_{k,2}}}\int_0^{t_{n_{k,2}}}T_2(u)b_{n_k}du-b_{n_k}\Big\|=0,\ \lim_{k\to\infty}\Big\|\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T_1(s)v_{n_k}ds-v_{n_k}\Big\|=0. \tag{4.25}$$

Also, we have

$$\lim_{k\to\infty}\Big[\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2+\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|^2\Big]=0$$

which implies that

$$\begin{aligned}
&\lim_{k\to\infty}\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|=0,\\
&\lim_{k\to\infty}\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k})\|=0,\\
&\lim_{k\to\infty}\|\mathcal{F}_1 w_{n_k}-\mathcal{F}_2\varphi_{n_k}\|=0.
\end{aligned}$$

From the definition of z_{n_k}, k_{n_k} and the previous inequality we have

$$\begin{aligned} \|z_{n_k} - w_{n_k}\| &= \eta_{n_k}\|\mathcal{F}_1^*(\mathcal{F}_1 w_{n_k} - \mathcal{F}_2\varphi_{n_k})\| \to 0, \text{ as } k \to \infty. \\ \|k_{n_k} - \varphi_{n_k}\| &= \eta_{n_k}\|\mathcal{F}_2^*(\mathcal{F}_1 w_{n_k} - \mathcal{F}_2\varphi_{n_k})\| \to 0, \text{ as } k \to \infty. \end{aligned} \tag{4.26}$$

Also, from the definition of v_{n_k}, b_{n_k} and (4.24), we have

$$\|v_{n_k} - u_{n_k}\| = \lambda_{n_k}\|Au_{n_k} - A\phi_{n_k}\| \le \frac{\lambda_{n_k}a_1}{\lambda_{n_{k+1}}}\|u_{n_k} - \phi_{n_k}\| \to 0, \text{ as } k \to \infty.$$

$$\|b_{n_k} - s_{n_k}\| = \rho_{n_k}\|Bs_{n_k} - B\psi_{n_k}\| \le \frac{\rho_{n_k}a_2}{\rho_{n_{k+1}}}\|s_{n_k} - \psi_{n_k}\| \to 0, \text{ as } k \to \infty.$$

From (4.25) and Lemma 2.3 we have

$$\begin{aligned} \|v_{n_k} - T_1(v)v_{n_k}\| \le & \left\| v_{n_k} - \frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}} T_1(s)v_{n_k}ds \right\| \\ & + \left\| \frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}} T_1(s)v_{n_k}ds - T_1(v)\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}} T_1(s)v_{n_k}ds \right\| \\ & + \left\| T_1(v)\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}} T_1(s)v_{n_k}ds - T_1(v)v_{n_k} \right\| \to 0, \text{ as } k \to \infty. \end{aligned} \tag{4.27}$$

Similarly, we have

$$\lim_{k\to\infty}\|b_{n_k} - T_2(b)b_{n_k}\| = 0.$$

From the definition of $x_{n_{k+1}}$ and (4.25), we have

$$\begin{aligned} \|x_{n_{k+1}} - v_{n_k}\| &= \left\|(1-\beta_{n_k})v_{n_k} + \beta_{n_k}\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}} T_1(s)v_{n_k}ds - v_{n_k}\right\| \\ &= \beta_{n_k}\left\|\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}} T_1(s)v_{n_k}ds - v_{n_k}\right\| \to 0, \text{ as } k \to \infty. \end{aligned}$$

Similarly, we have

$$\lim_{k\to\infty}\|y_{n_{k+1}} - b_{n_k}\| = 0.$$

Now, from **Step 2** and by Remark 3.4, we get

$$\begin{aligned} \|w_{n_k} - x_{n_k}\| &= \|(1-\alpha_{n_k})(x_{n_k} + \theta_{n_k}(x_{n_k} - x_{n_k-1})) - x_{n_k}\| \\ &= \|(1-\alpha_{n_k})(x_{n_k} - x_{n_k}) + (1-\alpha_{n_k})\theta_{n_k}(x_{n_k} - x_{n_k-1}) - \alpha_{n_k}x_{n_k}\| \\ &\le (1-\alpha_{n_k})\|x_{n_k} - x_{n_k}\| + (1-\alpha_{n_k})\theta_{n_k}\|x_{n_k} - x_{n_k-1}\| + \alpha_{n_k}\|x_{n_k}\| \to 0, \quad k \to \infty. \end{aligned} \tag{4.28}$$

Similarly, we have

$$\lim_{k\to\infty}\|\varphi_{n_k} - y_{n_k}\| = 0. \tag{4.29}$$

From (4.24)–(4.29) we have

$$\lim_{k\to\infty}\|x_{n_k} - \phi_{n_k}\| = 0, \lim_{k\to\infty}\|x_{n_{k+1}} - w_{n_k}\| = 0, \lim_{k\to\infty}\|y_{n_{k+1}} - \varphi_{n_k}\| = 0. \tag{4.30}$$

From (4.28) and (4.30) we have

$$\lim_{k\to\infty} \|x_{n_{k+1}} - x_{n_k}\| = 0. \tag{4.31}$$

Similarly, from (4.29) and (4.30)

$$\lim_{k\to\infty} \|y_{n_{k+1}} - y_{n_k}\| = 0. \tag{4.32}$$

To complete the proof, we show that $w_\omega\big(x_n, y_n\big) \subset \Gamma$, where $w_\omega\big(x_n, y_n\big)$ is the set of weak limits of $\big\{(x_n, y_n)\big\}$. Since $\big\{(x_n, y_n)\big\}$ is bounded we have that $w_\omega\big(x_n, y_n\big)$ is nonempty. Let $(x^*, y^*) \in w_\omega\big(x_n, y_n\big)$ be an arbitrary element. From (4.26), (4.28) and (4.29) we have $x^* \in w_\omega(x_n)$ and $y^* \in w_\omega(y_n)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Since $\lim_{k\to\infty} \|x_{n_k} - \phi_{n_k}\| = 0$, we have that $\phi_{n_k} \rightharpoonup x^* \in \mathcal{C}$ as $k \to \infty$. From the characteristic property of $P_{\mathcal{C}}$, we have

$$\langle x - u_{n_k},\ \ \phi_{n_k} - \lambda_{n_k} A\phi_{n_k} - u_{n_k}, \rangle \le 0, \quad x \in \mathcal{C},$$

which implies that

$$\frac{1}{\lambda_{n_k}} \big\langle \phi_{n_k} - u_{n_k},\ \ x - u_{n_k} \big\rangle \le \big\langle A\phi_{n_k},\ x - u_{n_k} \big\rangle, \quad \forall x \in \mathcal{C}.$$

Consequently, we have

$$\frac{1}{\lambda_{n_k}} \big\langle \phi_{n_k} - u_{n_k},\ x - u_{n_k} \big\rangle + \big\langle A\phi_{n_k},\ \ u_{n_k} - \phi_{n_k} \big\rangle \le \big\langle A\phi_{n_k},\ \ x - \phi_{n_k} \big\rangle, \quad \forall x \in \mathcal{C}. \tag{4.33}$$

Applying the fact that $\lim_{k\to\infty} \|\phi_{n_k} - u_{n_k}\| = 0$ and $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$ to (4.33), we have

$$0 \le \liminf_{k\to\infty} \big\langle A\phi_{n_k},\ \ x - \phi_{n_k} \big\rangle, \quad \forall x \in \mathcal{C}. \tag{4.34}$$

Also, we have that

$$\big\langle Au_{n_k},\ x - u_{n_k} \big\rangle = \big\langle Au_{n_k} - A\phi_{n_k},\ x - \phi_{n_k} \big\rangle + \big\langle A\phi_{n_k},\ x - \phi_{n_k} \big\rangle + \big\langle Au_{n_k},\ \phi_{n_k} - u_{n_k} \big\rangle.$$

Since A is uniformly continuous on $\mathcal{H}$ and $\lim_{k\to\infty} \|\phi_{n_k} - u_{n_k}\|$, we have

$$\lim_{k\to\infty} \|A\phi_{n_k} - Au_{n_k}\| = 0. \tag{4.35}$$

From (4.34)–(4.35), we have

$$0 \le \liminf_{k\to\infty} \big\langle Au_{n_k},\ x - u_{n_k} \big\rangle, \quad \forall x \in \mathcal{C}. \tag{4.36}$$

Let $\{\delta_k\}$ be a sequence of positive numbers such that $\delta_{k+1} \le \delta_k$, $\forall k \ge 1$ and $\delta_k \to 0$ as $k \to \infty$. Then, for each $k \ge 1$, we denote by N_k the smallest positive integer such that

$$\langle Au_{n_j}, x - u_{n_j} \rangle + \delta_k \ge 0, \quad \forall j \ge N_k, \tag{4.37}$$

where the existence of N_k follows from (4.36). We have that $\{N_k\}$ is increasing since $\{\delta_k\}$ is decreasing. Furthermore, since $\{u_{n_k}\} \subset \mathcal{C}$ we can suppose $Au_{N_k} \ne 0$ (otherwise, u_{N_k} is a solution) and we set for each $k \ge 1$, $h_{N_k} = \frac{Au_{N_k}}{\|Au_{N_k}\|^2}$. Then we have that $\langle Au_{N_k}, h_{N_k} \rangle = 1$ for each $k \ge 1$. Thus, by (4.37), we have that

$$\langle Au_{N_k}, x + \delta_k h_{N_k} - u_{N_k} \rangle \ge 0,$$

which implies by the pseudo-monotonicity of A that

$$\langle A(x+\delta_k h_{N_k}), x+\delta_k h_{N_k} - u_{N_k}\rangle \geq 0. \tag{4.38}$$

Since $u_{n_k} \subset C$, the sequence $\{u_{n_k}\}$ converges weakly to $x^* \in \mathcal{C}$. If $Ax^* = 0$, then $x^* \in VI(\mathcal{C}, A)$. On the contrary, we suppose $Ax^* \neq 0$. Since A satisfies condition (c), we have

$$0 < \|Ax^*\| \leq \liminf_{k\to\infty} \|Au_{n_k}\|.$$

Since $\{u_{N_k}\} \subset \{u_{n_k}\}$, we obtain that

$$0 \leq \limsup_{k\to\infty} \|\delta_k h_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\delta_k}{\|Au_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \delta_k}{\liminf_{k\to\infty} \|Au_{n_k}\|} = 0.$$

Therefore, $\lim_{k\to\infty} \|\delta_k h_{N_k}\| = 0$. Letting $k \to \infty$ in (4.38) gives

$$\langle Ax, x - x^*\rangle \geq 0, \quad \forall x \in \mathcal{C},$$

which implies by Lemma 2.7 that $x^* \in VI(\mathcal{C}, A)$. By similar argument, we have that $y^* \in VI(\mathcal{Q}, B)$.

Now, to show that $x^* \in F(\mathcal{T}_a)$ and $y^* \in F(\mathcal{T}_b)$. On the contrary, we suppose that $T_1(v)x^* \neq x^*$ and $T_2(b)x^* \neq y^*$ for all $v \geq 0$ and $b \geq 0$. Then, it follows from the Opial condition of Hilbert space and from (4.27) that

$$\begin{aligned}
\liminf_{k\to\infty} \|v_{n_k} - x^*\| &< \liminf_{k\to\infty} \|v_{n_k} - T_1(v)x^*\| \\
&\leq \liminf_{k\to\infty} \Big\{\|v_{n_k} - T_1(v)v_{n_k}\| + \|T_1(v)v_{n_k} - T_1(v)x^*\|\Big\} \\
&\leq \liminf_{k\to\infty} \Big\{\|v_{n_k} - T_1(v)v_{n_k}\| + \|v_{n_k} - x^*\|\Big\} \\
&= \liminf_{k\to\infty} \|v_{n_k} - x^*\|,
\end{aligned}$$

which is a contradiction. Thus, it follows that $T_1(v)x^* = x^*$ for all $v \geq 0$ which implies that $x^* \in F(\mathcal{T}_a)$. Similarly, $y^* \in F(\mathcal{T}_b)$.

Next, from (4.24) we have that $\lim_{k\to\infty} \|\phi_{n_k} - z_{n_k}\| = \lim_{k\to\infty} \|U^{\Phi_1}_{r_{n_k},1} z_{n_k} - z_{n_k}\| = 0$, and since $z_{n_k} \rightharpoonup x^*$ it follows from the demiclosed property of nonexpansive mappings that $x^* \in EP(\Phi_1)$. Similarly, we have that $y^* \in EP(\Phi_2)$. Since $\mathcal{F}_1 x^* - \mathcal{F}_2 y^* \in w_\omega(\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n)$, it follows from the weakly lower semi-continuity of the norm that

$$\|\mathcal{F}_1 x^* - \mathcal{F}_2 y^*\| \leq \liminf_{n\to\infty} \|\mathcal{F}_1 w_n - \mathcal{F}_2 \varphi_n\| = 0.$$

Hence, we have that $(x^*, y^*) \in \Gamma$. Since $(x^*, y^*) \in w_\omega\big(x_n, y_n\big)$ was chosen arbitrarily, it follows that $w_\omega\big(x_n, y_n\big) \subset \Gamma$. To conclude, we show that

$$\limsup_{k\to\infty}\Big(\Big\langle \hat{x},\ \hat{x} - x_{n_{k+1}}\Big\rangle + \Big\langle \hat{y},\ \hat{y} - y_{n_{k+1}}\Big\rangle\Big) \leq 0.$$

By the boundedness of $\{(x_{n_k}, y_{n_k})\}$, it follows that there exists a subsequence $\{(x_{n_{k_j}}, y_{n_k})\}$ of $\{(x_{n_k}, y_{n_k})\}$ which converges weakly to some $(\bar{x}, \bar{x}) \in \mathcal{H}$, and such that

$$\lim_{j\to\infty}\Big(\Big\langle \hat{x},\ \hat{x} - x_{n_{k_j}}\Big\rangle + \Big\langle \hat{y},\ \hat{y} - y_{n_{k_j}}\Big\rangle\Big) = \limsup_{k\to\infty}\Big(\big\langle \hat{x},\ \hat{x} - x_{n_k}\big\rangle + \big\langle \hat{y},\ \hat{y} - y_{n_k}\big\rangle\Big). \tag{4.39}$$

From (4.39) and the fact that $(\hat{x}, \hat{y}) = P_\Gamma(0_{\mathcal{H}_1}, 0_{\mathcal{H}_2}) \in \Gamma$ we have

$$\begin{aligned} &\limsup_{k\to\infty}\Big(\langle \hat{x}, \ \hat{x} - x_{n_k}\rangle + \langle \hat{y}, \ \hat{y} - y_{n_k}\rangle \Big) \\ &= \lim_{j\to\infty}\Big(\Big\langle \hat{x}, \ \hat{x} - x_{n_{k_j}}\Big\rangle + \Big\langle \hat{y}, \ \hat{y} - y_{n_{k_j}}\Big\rangle \Big) \\ &= \langle \hat{x}, \ \hat{x} - \bar{x}\rangle + \langle \hat{y}, \ \hat{y} - \bar{y}\rangle \leq 0. \end{aligned} \tag{4.40}$$

From (4.31), (4.32) and (4.40), it follows that

$$\begin{aligned} &\limsup_{k\to\infty}\Big(\langle \hat{x}, \ \hat{x} - x_{n_k+1}\rangle + \langle \hat{y}, \ \hat{y} - y_{n_k+1}\rangle \Big) \\ &= \limsup_{k\to\infty}\Big(\langle \hat{x}, \ \hat{x} - x_{n_k}\rangle + \langle \hat{y}, \ \hat{y} - y_{n_k}\rangle \Big) \\ &= \langle \hat{x}, \ \hat{x} - \bar{x}\rangle + \langle \hat{y}, \ \hat{y} - \bar{y}\rangle \leq 0. \end{aligned} \tag{4.41}$$

Thus, by (4.30) and (4.41) we have $\limsup_{k\to\infty} \hat{d}_{n_k} \leq 0$. Now, applying Lemma 2.13 to (4.22) we have $\{\|x_n - \hat{x}\| + \|y_n - \hat{y}\|\}$ converges to zero, which implies that $\lim_{n\to\infty} \|x_n - \hat{x}\| = 0$ and $\lim_{n\to\infty} \|y_n - \hat{y}\| = 0$. Therefore, $(\{x_n\}, \{y_n\})$ converges strongly to $(\hat{x}, \hat{y})$. □

5 Numerical Experiment

In this section, we discuss the numerical behavior of our method, (Proposed Alg.) Algorithm 3.2 in comparison with the method in Appendix A proposed by Latif and Eslamian

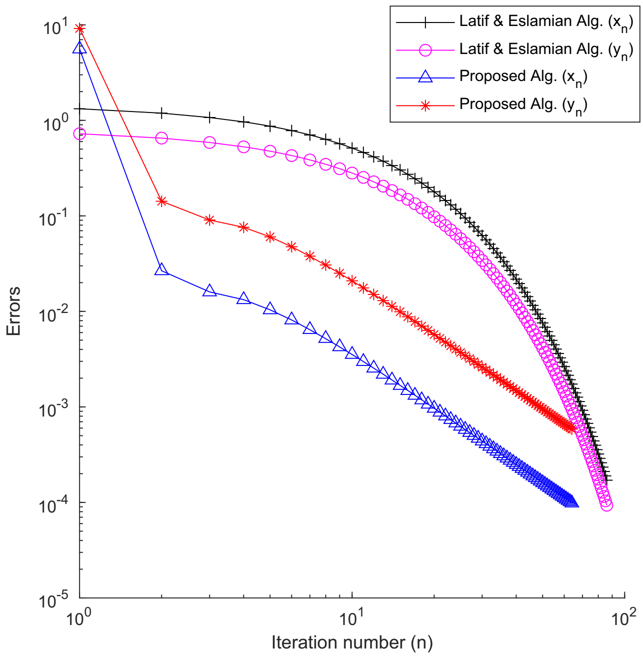

Fig. 1 Example 5.1: Case 1

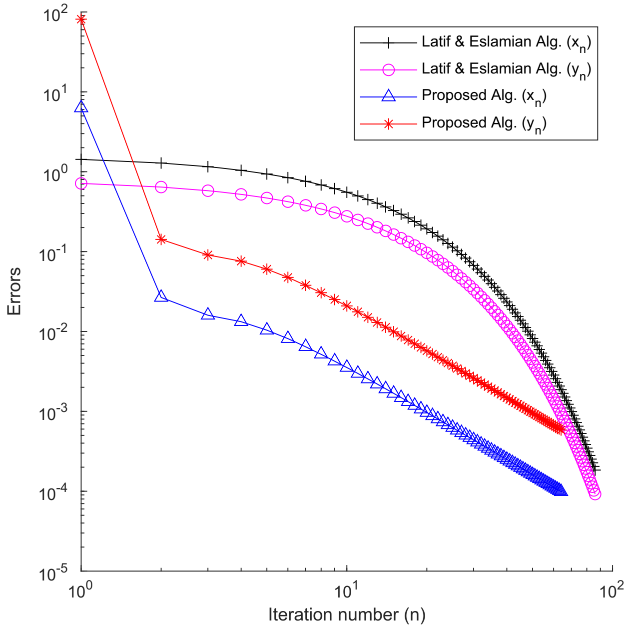

Fig. 2 Example 5.1: Case 2

[31] (Latif and Eslamian Alg.), which is the only related result we could find in the literature. We plot the graph of errors against the number of iterations in each case of both examples

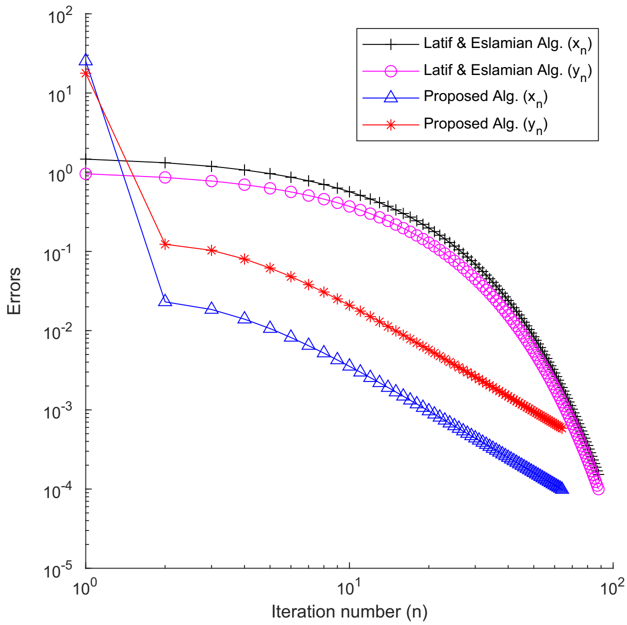

Fig. 3 Example 5.1: Case 3

using $|x_{n+1} - x_n| < 10^{-4}$ and $\|x_{n+1} - x_n\| < 10^{-4}$ in Example 5.1 and Example 5.2 respectively as the stopping criterion. The numerical computations are reported in Figs. 1, 2, 3, 4, 5, 6, 7, and 8 and Tables 1 and 2 with all implementations performed using Matlab 2021 (b).

In our computation, we choose $\theta = 3.5,\ \tau = 2.44,\ \lambda_1 = 1.5,\ \rho_1 = 1.8,\ a_1 = 0.8, a_2 = 0.9,\ \epsilon_n = \zeta_n = \frac{1}{(2n+1)^3},\ \alpha_n = \frac{3}{2n+1},\ \beta_n = \frac{1}{4},\ \gamma_n = \frac{1}{4},\ \rho_n = \sigma_n = \frac{100}{(n+1)^2}, \eta = 0.5, r_{n,1} = 2.8, r_{n,2} = 3.5, t_{n,1} = 4.5, t_{n,2} = 5.5, s = u = 1.5$. For Appendix A, we choose $\alpha = 0.85,\ \varsigma_n = \kappa_n = \frac{1}{6},\ \xi_n = \delta_n = \frac{1-\alpha_n}{2}$.

Example 5.1 Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathbb{R}$ the set of all real numbers with the inner product $\langle x, y\rangle = xy,\ \forall x, y \in \mathbb{R}$ and induced norm $|\cdot|$. For $r_i > 0,\ i = 1, 2$, consider $\mathcal{C} = [-10, 10]$ and $\mathcal{Q} = [0, 20]$. We define the bifunction $\Phi_1 : \mathcal{C}\times\mathcal{C} \to \mathbb{R}$ and $\Phi_2 : \mathcal{Q}\times\mathcal{Q} \to \mathbb{R}$ as follows:

$$U_{r_1}^{\Phi_1}(u) = \frac{u}{3r_1 + 1},\quad \forall x \in \mathcal{C}$$

and

$$U_{r_2}^{\Phi_2}(v) = \frac{v}{r_2 + 1},\quad \forall y \in \mathcal{Q}.$$

Let $\mathcal{F}_1 x = 2x$ and $\mathcal{F}_2 x = 5x$ which implies that $\mathcal{F}_1^* x = 2x$ and $\mathcal{F}_2^* x = 5x$. Next we define $A : \mathcal{H}_1 \to \mathcal{H}_1$ as $Ax = 2x$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ as $Bx = 3x$. We define the mappings $T_1(s) : \mathbb{R} \to \mathbb{R}$ and $T_2(u) : \mathbb{R} \to \mathbb{R}$ as follows; $T_1(s)x = 10^{-s}x$ and $T_2(u)y = 10^{-2u}y$. Clearly, we observe that $T_1(s)$ and $T_2(u)$ are nonexpansive semigroups.

We choose $\mathcal{V}_1 = x_0, \mathcal{V}_2 = y_0$ and consider the following cases for the numerical experiments of this example.

Case 1: Take $(x_0, y_0) = (-13.5, 8.0)$ and $(x_1, y_1) = (5.7, -9.1)$.

Case 2: Take $(x_0, y_0) = (15.1, 7.9)$ and $(x_1, y_1) = (6.4, 81.3)$.

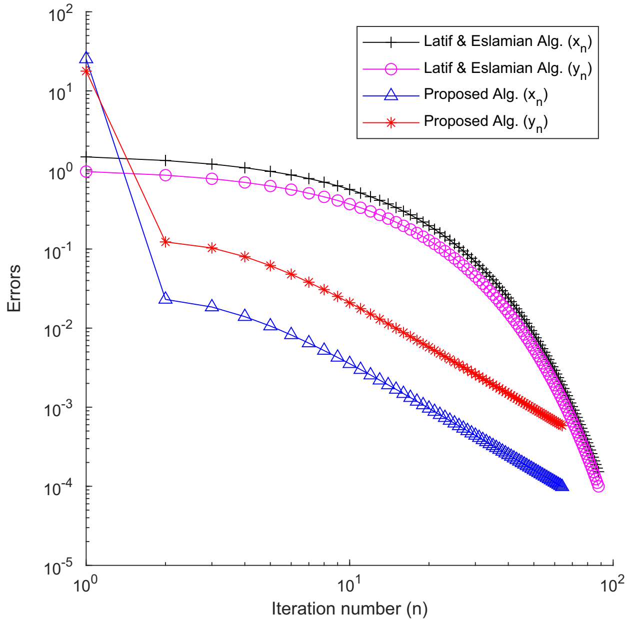

Fig. 4 Example 5.1: Case 4

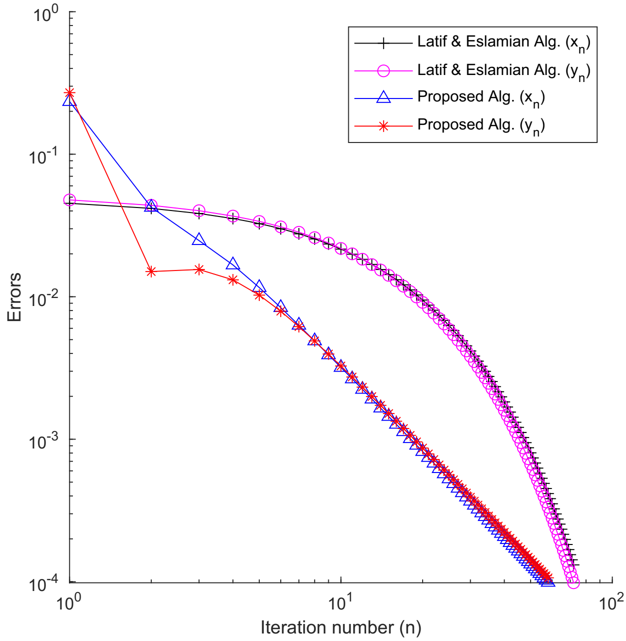

Fig. 5 Example 5.2: Case 1

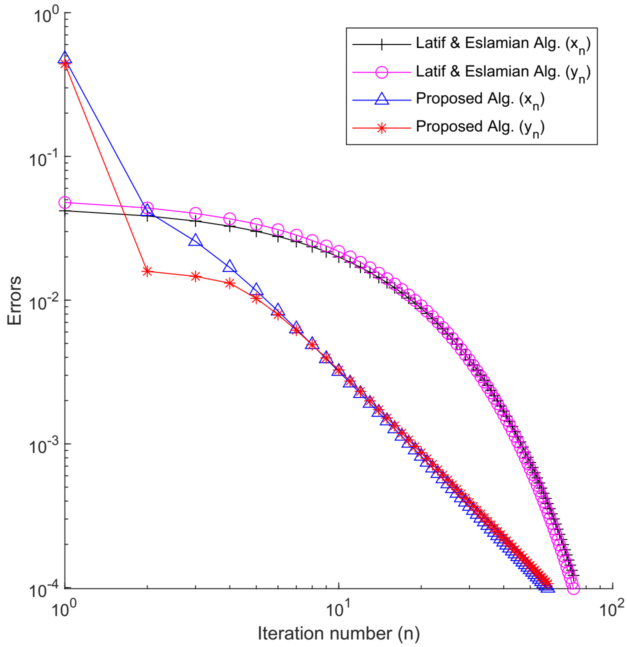

Fig. 6 Example 5.2: Case 2

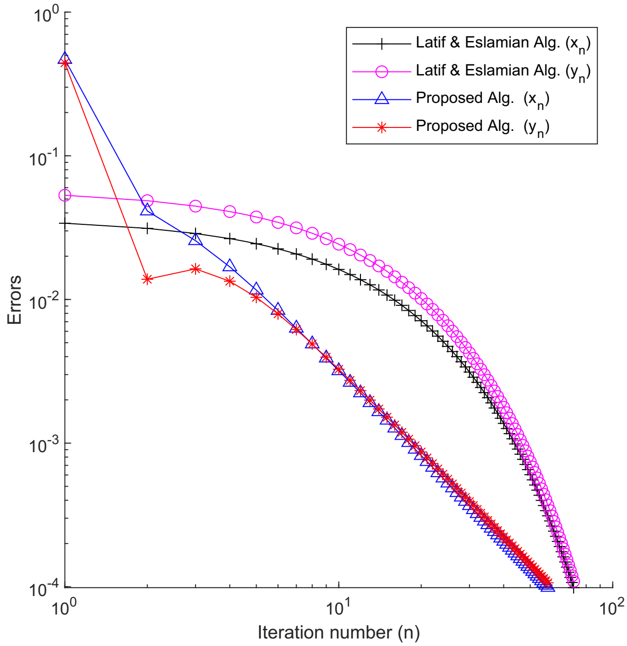

Fig. 7 Example 5.2: Case 3

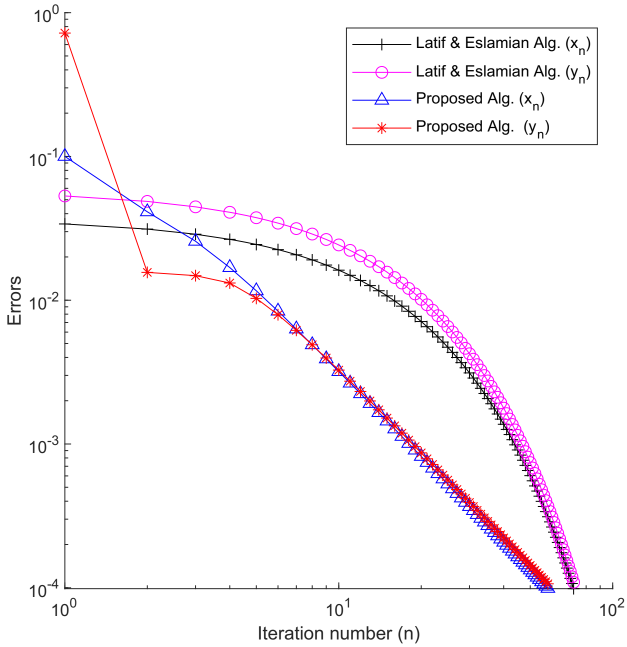

Fig. 8 Example 5.2: Case 4

Table 1 Numerical Results for Example 5.1

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Latif & Eslamian Alg.	86	0.0085	86	0.0062	89	0.0084	88	0.0064
Proposed Alg. 3.2	64	0.0132	64	0.0078	64	0.0093	64	0.0018

Case 3: Take $(x_0, y_0) = (10.9, -11.8)$ and $(x_1, y_1) = (-37.2, 26.8)$.
Case 4: Take $(x_0, y_0) = (-14.9, -9.8)$ and $(x_1, y_1) = (-25.2, -17.7)$.

Example 5.2 Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = (l_2(\mathbb{R}), \|\cdot\|_2)$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}, ||x||_2 = \sqrt{(\sum_{i=1}^{\infty} |x_i|^2)}$ and $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ for all $x \in \ell_2(\mathbb{R})$. For $r_i > 0$, $i = 1, 2$, we define the sets $\mathcal{C} := \{x \in \ell_2 : \|x\| \leq 1\}$ and $\mathcal{Q} := \{y \in \ell_2 : \|y\| \leq 1\}$. Let $\mathcal{F}_1 : \mathcal{H}_1 \to \mathcal{H}_2$, $\mathcal{F}_2 : \mathcal{H}_2 \to \mathcal{H}_3$ be defined by $\mathcal{F}_1 x = \frac{x}{3}$ and $\mathcal{F}_2 x = \frac{2x}{5}$ respectively which implies that $\mathcal{F}_1^* y = \frac{y}{3}$ and $\mathcal{F}_2^* y = \frac{2y}{5}$. Clearly, $\mathcal{F}_1$ and $\mathcal{F}_2$ are bounded linear operators. We define $\Phi_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ and $\Phi_2 : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ by $\Phi_1(x, y) = \langle L_1 x, y - x\rangle$ and $\Phi_2(x, y) = \langle L_2 x, y - x\rangle$, where $L_1 x = \frac{x}{3}$ and $L_2 x = \frac{x}{2}$. Observe that Φ_1 and Φ_2 satisfy Assumption 2.9. After simple calculation and applying Lemma 2.10, we obtain

$$U_{r_1}^{\Phi_1}(u) = \frac{3u}{r_1 + 3}, \quad \forall x \in \mathcal{C},$$

and

$$U_{r_2}^{\Phi_2}(v) = \frac{2v}{r_2 + 2}, \quad \forall y \in \mathcal{Q}.$$

Let $A : \mathcal{H}_1 \to \mathcal{H}_1$ be defined by $A(x_1, x_2, x_3, \ldots) = (x_1 e^{-x_1^2}, 0, 0, \ldots)$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ as $B(x_1, x_2, x_3, \ldots) = (5x_1 e^{-x_1^2}, 0, 0, \ldots)$. Clearly, we see that A and B are pseudomonotone mappings. We define the mappings $T_1(s) : \mathbb{R} \to \mathbb{R}$ and $T_2(u) : \mathbb{R} \to \mathbb{R}$ as follows; $T_1(s)x = 10^{-5s}x$ and $T_2(u)y = 10^{-3u}y$. Clearly, we observe that $T_1(s)$ and $T_2(u)$ are nonexpansive semigroups.

We choose $\mathcal{V}_1 = x_0$, $\mathcal{V}_2 = y_0$ and consider different initial values as follows:
Case 1: $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$, $y_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$; $x_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots)$, $y_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots)$;
Case 2: $x_0 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots)$, $y_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$; $x_1 = (-\frac{1}{3}, \frac{1}{6}, -\frac{1}{18}, \ldots)$, $y_1 = (-\frac{1}{3}, \frac{1}{6}, -\frac{1}{18}, \ldots)$;
Case 3: $x_0 = (\frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \ldots)$, $y_0 = (\frac{5}{9}, \frac{5}{18}, -\frac{5}{36}, \ldots)$; $x_1 = (-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots)$, $y_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \ldots)$;
Case 4: $x_0 = (\frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \ldots)$, $y_0 = (\frac{5}{9}, \frac{5}{18}, \frac{5}{36}, \ldots)$; $x_1 = (\frac{1}{9}, \frac{1}{18}, \frac{1}{36}, \ldots)$, $y_1 = (-\frac{7}{12}, \frac{7}{24}, -\frac{7}{36})$.

6 Conclusion

In this paper, we studied the split equalities of the VIP, EP and FPP of nonexpansive semigroups. We introduced a Tseng's extragradient method with self-adaptive step size for

Table 2 Numerical Results for Example 5.2

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Latif & Eslamian Alg.	72	0.0134	72	0.0192	72	0.0147	72	0.0073
Proposed Alg. 3.2	58	0.0211	58	0.0171	58	0.0263	58	0.0100

approximating a common solution of the split equalities of the VIP, EP and FPP of nonexpansive semigroups in the framework of real Hilbert spaces when the cost operator of the VIP is pseudomonotone and non-Lipschitz. Without the sequential weak continuity condition on the cost operator, we obtained a strong convergence result of our proposed method. While the cost operator is non-Lipschitz, our algorithm does not involve any linesearch procedure and our strong convergence result was obtained without the usual “two cases approach" widely used in many papers. Finally, we presented some numerical experiments of our proposed method in comparison with a related method in the literature to show the applicability of our method. Our result improves, extends and generalizes several other results in the literature.

Appendix A Algorithm 1 of Latif et al. [31]

Choose sequences $\{\beta_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\delta_n\}_{n=1}^{\infty}$ such that $\beta_n + \alpha_n + \delta_n = 1$. Select initial point $x_0 \in \mathcal{H}_1$, $y_0 \in \mathcal{H}_2$, let $\vartheta \geq 0$. Set $n := 1$.

$$\begin{cases} z_n = x_n - \vartheta_n \mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n), \\ \phi_n = U_{r_{n,1}}^{\Phi_1} z_n, \\ u_n = P_{\mathcal{C}}(\phi_n - \varsigma_n A\phi_n), \\ p_n = P_{\mathcal{C}}(\phi_n - \varsigma_n A u_n), \\ x_{n+1} = \alpha_n \mathcal{V}_1 + \xi_n p_n + \delta_n T_1(s) p_n \\ k_n = y_n + \vartheta_n \mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n), \\ \psi_n = U_{r_{n,2}}^{\Phi_2} k_n, \\ s_n = P_{\mathcal{Q}}(\psi_n - \kappa_n B\psi_n), \\ l_n = P_{\mathcal{Q}}(\psi_n - \kappa_n B s_n), \\ y_{n+1} = \alpha_n \mathcal{V}_2 + \xi_n l_n + \delta_n T_2(u) l_n, \end{cases}$$

where the step size ϑ_n is chosen such that for small enough $\epsilon > 0$,

$$\vartheta_n \in \left[\epsilon, \ \frac{2\|\mathcal{F}_1 x_n - \mathcal{F}_2 y_n\|^2}{\|\mathcal{F}_2^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)\|^2 + \|\mathcal{F}_1^*(\mathcal{F}_1 x_n - \mathcal{F}_2 y_n)\|^2} - \epsilon\right],$$

if $\mathcal{F}_1 x_n \neq \mathcal{F}_2 y_n$; otherwise, $\vartheta_n = \eta$.

Set $n := n + 1$ and go back to **Step 1**.

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Declarations

Competing interests The authors declare that they have no competing interests.

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