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Skew Polynomial Rings: the Schreier Technique

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To Prof. Dr. Nguyễn Tự Cường my dear childhood friend in celebration of his 70th Birthday – Happy Birthday, and many more!

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Abstract
Schreier bases are introduced and used to show that skew polynomial rings are free ideal rings, i.e., rings whose one-sided ideals are free of unique rank, as well as to compute a rank of one-sided ideals together with a description of corresponding bases. The latter fact, a so-called Schreier-Lewin formula (Lewin *Trans. Am. Math. Soc.* **145**, 455–465 1969), is a basic tool determining a module type of perfect localizations which reveal a close connection between classical Leavitt algebras, skew polynomial rings, and free associative algebras.

Keywords Skew polynomial rings · Firs · Semifirs · Leavitt algebras

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1 Introduction

In contrast to what one might expect, skew polynomial rings form a quite large, extremely diverse ring class. Elements of the (left) skew polynomial ring $D[X; \alpha, \delta]$ over a division ring D with a nonzero endomorphism $\alpha : D \to D$ and an α-*derivation* $\delta : D \to D$, i.e., an additive homomorphism satisfying $\delta(cd) = \delta(c)d + \alpha(c)\delta(d)$ for any two $c, d \in D$, are formal polynomials $a_0 + a_1X + \cdots + a_nX^n$ with the usual addition, but with multiplication twisted by putting $Xa = \alpha(a)X + \delta(a)$, respectively. $D[X; \alpha, 0]$ is denoted by $D[X; \alpha]$. Typical examples for endomorphisms α are provided by Lüroth's theorem about endomorphisms of rational function fields. $D[X; \alpha, \delta]$ is a left but not right principal ideal domain unless α is an automorphism. Almost nothing is known about right ideals of skew polynomial rings except an obvious consequence of Cohn's dependence relations [3, Theorem 1.1.1] that finitely generated right ideals are free of unique rank. Because of its importance we present a simple direct and elementary proof that one-sided principal ideal domains are

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semifirs. A ring is a *semifir* if every finitely generated one-sided ideal is free of unique rank. A ring is a *fir*, i.e., a *free ideal ring* if every one-sided ideal is a free module of unique rank. Consequently, a fir is *projective-free*, i.e., all projective modules are free. Following Rosenmann and Rosset [13], Schreier bases are introduced and used to show that skew polynomial rings are firs with a Schreier-Lewin formula [11] determining both rank and corresponding bases of one-sided ideals. The latter formula is a basic tool in the study of perfect localizations of skew polynomial rings which is inspired by the advanced theory of Leavitt algebras [6, 10] and [1]. In particular, we compute precisely both the module type and the Grothendick group of perfect localizations of skew polynomial rings. Furthermore, we describe also their maximal flat epimorphic ring of quotients as the ring of quotients with respect to the Gabriel topology defined by finite codimensional right ideals.

The title is suggested by Lewin's nice article [11]. All rings have the identity $1 \neq 0$, and modules are unitary. A division ring is a ring whose nonzero elements are units. For undefined notions we refer to Stenström's classic [14]. For a systematic investigation of skew polynomial rings, we refer to the classics [3, 5], or [7, 8] of either Cohn or Jacobson, respectively.

2 Skew Polynomial Rings are Free Ideal Rings

As a first step and for the sake of completeness, an obvious consequence of Cohn's theory on dependence relations that one-sided principal ideal domains are semifirs is presented with a short elementary proof. However, it seems to be open whether one-sided principal ideal domains are firs.

Proposition 2.1 *Every one-sided principal ideal domain is a semifir.*

Proof Let A be a left principal ideal domain. Then, A is a ring having IBN, whence every free A-module has a unique rank. Therefore, it suffices to verify by induction on the number of generators that every finitely generated right ideal is free. Since A is a domain, every cyclic right ideal is free. Assume that all right ideals generated by at most $n-1$ $(n > 1)$ generators are free and consider a nonzero right ideal $R = \sum_{i=1}^{n} a_i A$. It is enough to assume that $a_1, \dots, a_n$ are not right linearly independent over A, that is, there are $b_i \in A$ not all equal to 0 with $\sum_{i=1}^{n} a_i b_i = 0$. Then, there is $0 \neq b \in A$ with $b = \sum_{i=1}^{n} c_{ni} b_i$, $b_i = d_i b$ for appropriate $c_{ni}, d_i \in A$. Hence, $\sum_{i=1}^{n} c_{ni} d_i = 1$ and $\sum_{i=1}^{n} a_i d_i = 0$ hold. Therefore, $f_n = c_{n1}e_1 + \cdots + c_{nn}e_n \in {}_AA^n$ maps to $1 \in A$ under the left module homomorphism $\phi : A^n \to A : e_i \mapsto d_i = \phi(e_i) \in A$, where $\{e_i \mid i = 1, \dots, n\}$ is a basis of ${}_AA^n$. Consequently, ${}_AA^n = Af_n \oplus \ker\phi$ holds. This shows that $\ker\phi$ is a projective left A-module. Consequently, $\ker\phi$ is free of rank $n-1$ because A is a left principal ideal domain. If $\{f_i = \sum_{j=1}^{n} c_{ij} e_j \mid i = 1, \dots, n-1\}$ is a basis of $\ker\phi$, then the square matrix (c_{ij}) is invertible with the inverse matrix (v_{ij}). For an arbitrary column vector $(r_1, \dots, r_n)^t \in A_A^n$ the equality

$$\sum_{i=1}^{n} a_i r_i = (a_1, \cdots, a_n)(r_1, \cdots, r_n)^t = \{(a_1, \cdots, a_n)(v_{ij})\}\{(c_{ij})(r_1, \cdots, r_n)^t\}$$

$$= (a_1', \cdots, a_n')(r_1', \cdots, r_n')^t = \sum_{i=1}^{n} a_i' r_i'$$

with $a'_j = \sum_{i=1}^n a_i v_{ij}$ and $r'_j = \sum_{i=1}^n c_{ji} r_i$ for all $j = 1, \ldots, n$ ensures that R can be generated by $a'_1, \ldots, a'_n$, i.e., $R = \sum_{i=1}^n a_i A = \sum_{i=1}^n a'_i A$. In particular, one has

$$0 = \sum_{i=1}^n a_i d_i = \sum_{i=1}^n a'_i d'_i$$

with $d'_j = \sum_{i=1}^n c_{ji} d_i$. The equality $d'_n = \sum_{i=1}^n c_{ni} d_i = 1$ implies $a'_n \in \sum_{i=1}^{n-1} a'_i A$ and so R can be generated by $n - 1$ elements, whence R is free by the induction hypothesis. □

Remark 2.2 The above argument can be used to simplify the demonstration of (b) ⇒ (c) ⇒ (a) in [5, Theorem 1.6.1] characterizing n-firs as algorithmically computable. Moreover, the proof of Proposition 2.1 shows also that one-sided Bezout domains are semifirs. Even more, it is suitable to verify with an obvious modification that a domain whose finitely generated left ideals are free of unique rank, is a semifir. In particular, the proof of Proposition 2.1 can be used to show in a direct and elementary way the symmetry of n-fir, i.e., if every one-sided ideal, say, every left ideal generated by at most n-elements is free of unique rank, then every right ideal generated by at most n elements is also a free right module of unique rank. Note the fact that a unique rank of right free modules generated by at most n elements becomes immediate by applying the duality between finitely generated free left and right modules induced by functors $\mathrm{Hom}_{R(-;R)}$

From now on, D always denotes a division ring together with a non-zero, proper endomorphism $\alpha : D \to D$ and an α-derivation δ. Fix a basis $B_1 = \{b_i \mid i \in I\} \ni 1$ of the right vector space D over $\alpha(D)$ where I is not necessarily a finite set. Then $\alpha^n(B_1) = \{\alpha^n(b_i) \mid i \in I\} = B_{n+1}$ is a basis of the right vector space $\alpha^n(D)_{\alpha^{n+1}(D)}$. Since a skew polynomial ring $A = D[X; \alpha, \delta]$ is obviously a right D-module, we will first construct a basis of A_D in terms of $B_1 = \{b_i \mid i \in I\}$ of $D_{\alpha(D)}$. Put $y_i = b_i X$ for all $i \in I$. For each $d = \sum_i b_i \alpha(d_i) \in D$ the multiplication rule

$$Xd = \alpha(d)X + \delta(d) \tag{1}$$

implies

$$dX = \sum_i b_i \alpha(d_i) X \equiv \sum_i y_i d_i \qquad \bmod D \tag{2}$$

whence $\{y_i \mid i \in I\}$ is a right linearly independent set over D and $\{1,\ y_i \mid i \in I\}$ is a basis of the right D-module $D + DX$. For the sake of simplicity, 1 is considered as the monomial of *length* 0, and a product $p = y_{i_1} \cdots y_{i_n}$ $(i_1, \ldots, i_n \in I)$ has a *length* $|p| = n$. A product $y_{i_1} \cdots y_{i_m}$ of length $m \le n$ is called by definition an *initial segment* of length m of p, and is denoted by $h_p(m) (h_p(0) = 1)$ while a remainder $y_{i_{m+1}} \cdots y_{i_n}$ is a *tail* $t_p(m)$ $(t_p(n) = t_p(|p|) = 1)$ *of colength* m of p. The obvious iteration using (2) shows that for each $l > 0$, monomials $y_{i_j} \cdots y_{i_k}$ of length at most l form a basis of D-D-bimodule $D + DX + \cdots + DX^l$ as a right D-module and

$$DX^l \equiv \sum y_{i_1} \cdots y_{i_l} D = \sum_{|p|=l} pD \qquad \bmod D + DX + \cdots + DX^{l-1}. \tag{3}$$

Note an important fact that the 1-dimensional left D-module DX is not a right D-module unless $\delta = 0$. In this case of $\delta = 0$, all DX^n $(n \in \mathbb{N})$ are also right D-modules. In any case, monomials in y_i form a basis $\mathbf{B}$ of A_D, called a *standard Schreier basis with respect to a basis* B_1 of $D_{\alpha(D)}$, shortly, a *standard Schreier basis* for A_D, and every elements of A can be written uniquely as a right linear combination of monomials in the y_i with coefficients from D on the right.

Schreier bases introduced by Rosenmann and Rosset [13] can be extended to skew polynomial rings as follows.

Definition 2.3 A *Schreier basis* for a right ideal R of a skew polynomial ring $A = D[X; \alpha, \delta]$ with respect to a standard Schreier basis $\mathbf{B}$ of all monomials in $y_i = b_i X$ $(i \in I)$, where $B_1 = \{b_i \mid i \in I\} \ni 1$ is a basis of $D_{\alpha(D)}$, is a subset $B = B_R \subseteq \mathbf{B}$ that spans a right vector space $V = V_R$ over D that is complementary to R (that is, $A = V + R$, $V \cap R = 0$), and is closed to taking initial segments. For each $n \in \mathbb{N}$, let V_n be a right vector space generated by elements of B having length at most n. A Schreier basis B is called a *strong Schreier basis* for R if every monomial in y_i of length n lies in $V_n + R$.

The argument of Rosenmann and Rosset [13, (3.2) Lemma] is used to show the existence of Schreier bases.

Proposition 2.4 *There exists a Schreier basis B_R for any right ideal R of $A = D[X; \alpha, \delta]$.*

Proof The case $R = A$ is trivial because B_A is just the empty set. Therefore, one can assume without loss of generality that R is a proper right ideal, i.e., $1 \notin R$. A Schreier basis $B = B_R$ is constructed inductively by putting firstly ${}_0B = \{1\}$. If $D + R = A$, let ${}_1B = {}_0B$. If $D + R \neq A$, then DX is not a subset of $D + R$. Namely, $DX \subseteq D + R$ would imply $DX^2 \subseteq DX + RX \subseteq D + R$ and hence all $DX^n \subseteq D + R$ hold whence $A = D + R$, a contradiction. Consequently, if $D + R \neq A$, then let ${}_1B'$ be a maximal subset of $\{y_i = 1y_i \mid i \in I\}$ such that ${}_1B'$ is right linearly independent modulo $D + R$ over D. Put ${}_1B = {}_0B \cup {}_1B'$ and V_1 a right D-module spanned by D and ${}_1B$. Then $D + DX \subseteq V_1 + R$ holds. Therefore, one has the equality $D + DX + R = V_1 + R$. Assuming now that ${}_nB'$, ${}_nB$ and V_n $(n > 0)$ have been already constructed such that $D + DX + \cdots + DX^n + R = V_n + R$. If $V_n + R = A$, let ${}_{n+1}B = {}_nB$. If $V_n + R \neq A$, then DX^{n+1} is not a subset of $V_n + R$ because as above one can verify easily that $DX^{n+1} \subseteq V_n + R$ would imply $V_n + R = A$, a contradiction. Consequently, in case $V_n + R \neq A$, let ${}_{n+1}B'$ be a maximal subset

$$\{by_i \mid b \in {}_nB'\}$$

which is a right linearly independent set modulo $V_n + R$ over D. Then, define ${}_{n+1}B = {}_nB \cup {}_{n+1}B'$ and let V_{n+1} be a right D-module spanned by ${}_{n+1}B$. Hence, $DX^{n+1} \subseteq V_{n+1} + R$ holds. If ${}_{n+1}B' = \emptyset$, the process stops at this step and define $B_R = {}_nB$. If the process does not stop after finitely many steps define

$$B = B_R = \bigcup_{n=0}^{\infty} {}_nB\,.$$

B is clearly a strong Schreier basis of R, completing the proof. □

Schreier's technique [11] is now suitable for showing that skew polynomial rings are firs.

Let $\pi : A = V \oplus R \to V$ be the canonical projection of A along R onto the right D-module V spanned by the Schreier basis B of R constructed above. Then, for every element $a \in A$, one has $a - \pi(a) \in R$ whence $as - \pi(a)s \in R$ holds for every $s \in A$. In particular, the equality

$$\pi(as) = \pi(\pi(a)s), \quad \forall a, s \in A \tag{4}$$

holds. Consequently, for every element $b \in B$ and y_i $(i \in I)$, the element by_i is either contained in B whence $\pi(by_i) = by_i$ and so $by_i - \pi(by_i) = 0$, or not contained in B. In that case, by the construction of B, or equivalently, by the definition of a strong Schreier basis, $0 \neq by_i - \pi(by_i) \in R$ holds. Therefore, for every monomial $y = y_{i_1} \cdots y_{i_l}$ of length $l \geq 0$ in B, the associated element

$$u_{y,i} = yy_i - \pi(yy_i), \quad \forall y \in B \,\&\, i \in I \tag{5}$$

is either 0 or a nonzero element of R. We are now ready to reach the main goal of this section.

Theorem 2.5 *If δ is an α-derivation of a division ring with respect to a nonzero, proper endomorphism $\alpha : D \to D$, then $A = D[X; \alpha, \delta]$ is a free ideal ring.*

Proof We have to show that every right ideal R of A is a free A-module. Since the case $R = A$ or $R = 0$ is obvious, one can assume without loss of generality that R is a proper right ideal. For an arbitrary but fixed basis $B_1 = \{b_i \mid i \in I\} \ni 1$ of $D_{\alpha(D)}$ define elements ("variables") $y_i = b_i X$ $(i \in I)$. Moreover, let $B = B_R$ be a strong Schreier basis of R with respect to the standard Schreier basis $\mathbf{B}$ of A_D consisting of all monomials in y_i $(i \in I)$. We show first that the nonzero elements $u_{y,i} = yy_i - \pi(yy_i)$ $(y \in B,\ i \in I)$ generate R freely. For an arbitrary monomial y and $y_i = b_i X$ $(i \in I)$, we have by (4)

$$\pi(y)y_i - \pi(yy_i) = \pi(y)y_i - \pi(\pi(y)y_i).$$

Let $\pi(y) = \sum_{j=1}^{l} z_j d_j$ for some monomials $z_j \in B_R$ and $d_j \in D$. Write $d_j y_i = (\sum_k y_k {}^{ij}c_k) + d_{ij}$ for appropriate ${}^{ij}c_k, d_{ij} \in D$. Then we have, by putting $v = \sum_{j=1}^{l} z_j d_{ij} \in V$ where V is a right D-module spanned by B_R, the following equality

$$\pi(y)y_i = \sum_{j=1}^{l} z_j(d_j y_i) = \sum_{j=1}^{l} z_j \left(d_{ij} + \sum_k y_k {}^{ij}c_k \right) = \sum_{j,k} z_j y_k {}^{ij}c_k + v$$

whence

$$\pi(y)y_i - \pi(yy_i) = \sum_{j,k} (z_j y_k - \pi(z_j y_k)) {}^{ij}c_k = \sum_{j,k} u_{z_j,k} {}^{ij}c_k \in \sum_{y \in B \& i \in I} u_{y,i} A. \tag{6}$$

The canonical projection $1 - \pi$ of A on R via the decomposition $A = V \oplus R$ sends every monomial $y = y_{i_1} \cdots y_{i_n}$ $(n > 0)$ to

$$(1 - \pi)(y) = y - \pi(y) = \sum_{j=0}^{n-1} \left\{ \pi(h_y(j)) y_{i_{j+1}} - \pi(h_y(j+1)) \right\} t_y(j+1) \tag{7}$$

and so by formulas (6), (7) together with $y - \pi(y)$, the image $a - \pi(a)$ of an arbitrary element $a \in A$ is contained in $\sum_{y \in B \& i \in I} u_{y,i} A$. Consequently, $u_{y,i}$ $(y \in B;\ i \in I)$ generate R.

Therefore, it remains to show that the set $\{u_{y,i} \neq 0 \mid y \in B;\ i \in I\}$ is linearly independent over A. Instead of using the lengthy opaque argument of Lewin [11] we use the nice argument of [13, p. 363]. Consider finitely many nonzero elements $u_{z_j,i}$ defined by $z_j \in B_R$ and $y_i = b_i X, i \in I$. Then, $z_j y_i \notin B$ and $u_{z_j,i} = z_j y_i - \pi(z_j y_i) \neq 0$ hold. By definition

$$\pi(z_j y_i) = \sum_k {}^{ij}m_k {}^{ij}d_k \quad \left(0 \neq {}^{ij}d_k \in D \,\&\, {}^{ij}m_k \in B_R,\ |{}^{ij}m_k| \leq |z_j| + 1 \right). \tag{8}$$

Assume indirectly that these $u_{z_j,i}$'s are right linearly dependent over A, i.e., there are elements $a_{ij} \in A$ not all 0 such that $\sum_j u_{z_j,i} a_{ij} = 0$. If yd $(d \in D)$ is a monomial of longest length among monomials appearing in the a_{ij}-s, say, it is a monomial of $a_{\underline{i}\,\underline{j}}$, then in view of the equality (8) the term $z_{\underline{j}} y_{\underline{i}} yd$, cannot cancel. This is an immediate result of the following observations. Because of its maximal length $z_{\underline{j}} y_{\underline{i}}$ is not an initial segment of any elements $z_j y_i$ satisfying $(j, i) \neq (\underline{j}, \underline{i})$, nor of any of ${}^{ij}m_k$, i.e., of monomials appearing in $\pi(z_j y_i)$ (without any exception, even including the case $(j, i) = (\underline{j}, \underline{i})$) in view of the fact that $z_{\underline{j}} y_{\underline{i}}$ is not contained in B_R. This contradiction finishes the proof. □

Theorem 2.5 implies, in view of the structure of projective modules over hereditary rings, that projective right modules over $D[X; \alpha, \delta]$ are free. However, one can extend in an obvious manner the notion of (strong) Schreier basis to free right modules, and by the same proof one gets directly that projective right modules over $D[X; \alpha, \delta]$ are free as

Corollary 2.6 *Let M be a submodule of a free right module F on a set $\{t_j \mid j \in J\}$ over $D[X; \alpha, \delta]$ where $\mathbf{B} = \{y_{i_1}, \ldots, y_{i_n} \mid i_j \in I, n \geq 0\}$ is a standard Schreier basis of F given by a basis $B_1 = \{b_i \mid i \in I\}$ of $D_{\alpha(D)}$ and $y_i = b_i X$ $(i \in I)$. Then $\mathbf{B}_F = \bigcup_{j \in J} t_j \mathbf{B}$ is called a standard Schreier basis of F with respect to $\mathbf{B}$. Let B_M be a strong Schreier basis of M constructed in the same manner as in Proposition 2.4, then F admits a decomposition $F = V_M \oplus M$, where V_M is the right D-module spanned by B_M. Let π_M be the canonical projection of F onto V_M along M and put $u_j = t_j - \pi_M(t_j)$, ${}_j u_{b,i} = t_j b y_i - \pi_M(t_j b y_i)$ for $j \in J$, $i \in I$; $b \in \mathbf{B}$. Then, the nonzero elements u_j, ${}_j u_{b,y_i}$ form a basis of M over $D[X; \alpha, \delta]$.*

We can now easily deduce the Schreier-Lewin formula [11] for the rank of submodules with a finite strong Schreier basis of finitely generated free modules over $D[X; \alpha, \delta]$.

Corollary 2.7 *Let F be a free right module of a finite rank l over $D[x; \alpha, \delta]$ such that D has a finite dimension n over $\alpha(D)$, and M a submodule admitting a finite strong Schreier basis of $m \in \mathbb{N}$ elements, i.e., the factor A-module F/M has dimension m as a right D-module. Then $nm - m + l$ is the rank of M.*

Proof For simplicity, we present a proof for the case $l = 1$ which works word-for-word for the arbitrary case. Consider a right ideal R of $D[X; \alpha, \delta]$. In the notation of preceding results there are exactly mn symbols $u_{b,i}$ $(b \in B_R, i \in I; |I| = n)$. Therefore, by Theorem 2.5 R is a free right A-module of rank $mn - k$ where k is the number of $u_{b,i}$ equal to 0. But this number is exactly $m - 1$ by the construction of B_R and the definition of π and $u_{b,i}$, completing the proof. □

Remark 2.8 In view of Lewin's footnotes [11] that Cohn was able to obtain results in [11] from a general theory of the weak algorithm and the Hilbert series of filtered rings (cf. [3] and [4]), it seems that results of this section can be obtained by Cohn's method, too. Our treatment is, however, elementary. On one hand, it provides evidence for the broad applicability of Cohn's beautifully deep theory; and on the other hand, it supplies another particular, preliminary, introductory example different than the ordinary, classical example of free associative algebras for Cohn's theory. Furthermore, it is worth keeping in mind that by its flexibility, Schreier-Lewin techniques can be applicable to not necessarily unital rings with zero divisors, for example, see [1].

3 Localizations of Skew Polynomial Rings

Inspired by the important work of Rosenmann and Rosset [13], we now turn to the study of perfect bimorphic localizations of skew polynomial rings because of their striking similarity to free associative algebras. In particular, Rosenmann and Rosset [13] proposed a nice method to compute module types of certain rings which do not have IBN. Roughly speaking, their idea is to find a subring A such that A has IBN, and then to reduce the problem to one on (unique) rank of free modules over A which is usually a *projective-free ring*, i.e., a ring that admits only free modules as projective modules. Recall that a *module type* of a ring A without IBN is a pair $(m, n) \in \mathbb{N}^2$ $(m < n)$ such that m, n are the first two smallest integers satisfying $A_A^m \cong A_A^n$. In particular, it is easy to find rings of module type $(1, 2)$, i.e., rings over which all finitely generated free modules are isomorphic. But it is generally quite hard to show that certain free modules are not isomorphic. As an illustration, we include the following obvious folkloric result, with proof to aid the reader.

Proposition 3.1 *If $\alpha : D \to D$ is a non-zero, non-surjective endomorphism, then the right maximal quotient ring of $D[X; \alpha, \delta]$ has the module type* $(1, 2)$. *More generally, if a ring A has a dense right ideal R which is an infinite direct sum of isomorphic modules, then the module type of the maximal right quotient ring $Q_{\max}(A)$ is* $(1, 2)$.

Proof The assumption on α implies that $D[X; \alpha, \delta]$ has a dense right ideal which is free of infinite rank. In particular, if a right ideal R of a ring A is an infinite direct sum of isomorphic modules, then we have an isomorphism $R_R \cong R^n$ for every $n \in \mathbb{N}$. Consequently, this direct decomposition ensures elements $a_i, a_i^* \in Q_{\max}(A)$, $i = 1, \dots, n$ satisfying $\sum_{i=1}^n a_i a_i^* = 1$ and $a_i^* a_j = \delta_j^i$, where δ_j^i is the usual Kronecker symbol. Hence, all finitely generated free modules over $Q_{\max}(A)$ are isomorphic, completing the proof. □

For the goals of this section recall that a ring homomorphism $\phi : A \to B$ is called an *epimorphism* if for any ring C and ring homomorphisms $\alpha, \beta : B \to C$, $\alpha\phi = \beta\phi$ implies $\alpha = \beta$. Dually one gets a notion of a *monomorphism*. Epimorphisms are not necessarily surjective, for example $\mathbb{Q}$ is an epimorphic overring of $\mathbb{Z}$. However, monomorphisms are always injective. Namely, if there is $0 \neq a \in A$ with $\phi(a) = 0$, then $\alpha, \beta : \mathbb{Z}[X] \longrightarrow A$ defined by putting $\alpha(X) = 0$ and $\beta(X) = a$, respectively, are two ring homomorphisms from $Z[X]$ to A satisfying $\phi\alpha = \phi\beta$ but $\alpha \neq \beta$. Two rings are *bimorphic* if there is a ring homomorphism between them which is both a monomorphism and an epimorphism. For example, the ring $\mathbb{Z}$ of integers and the field $\mathbb{Q}$ of rationals are bimorphic. An epimorphism $\phi : A \to B$ is *flat* if ${}_AB$ is a left flat A-module. In this case B is called a *perfect right localization* or a *flat epimorphic right ring of quotients* of A. Flat epimorphisms form a non-trivial intersection of Cohn's localization via epimorphisms and Gabriel's theory of localization via both Gabriel topology and hereditary torsion theory. For a basic theory of perfect localizations we refer to classic books [14] and [12] by Stenström and Popescu, respectively.

If $\phi : A \longrightarrow Q$ is a flat bimorphism, then Q becomes a subring of the maximal ring $Q_{\max}(A)$ of right quotients of A by identifying each $a \in A$ with $\phi(a) \in Q$. It is well-known that $Q_{\max}(A)$ has the largest subring $Q_{\text{tot}}(A)$, called the *maximal flat epimorphic ring of right quotients* of A which is a flat bimorphism of A and contains all flat bimorphisms of A. It is also well-known, and in fact, not hard to see that every element q of $Q_{\max}(A)$ uniquely determines a largest right ideal $\operatorname{dom}(q) = \{a \in A \mid qa \in A\}$ of A, called a *maximal right ideal of definition of* q and q can be identified with an A-homomorphism

$q : \mathrm{dom}(q) \longrightarrow A : a \mapsto qa = q(a)$. This observation greatly simplifies notations when working inside maximal rings of quotients.

From now on, we assume that α is an endomorphism of a division ring D such that $D_{\alpha(D)}$ is of finite dimension $n > 1$. We shall use notations fixed in Section 2. Therefore, $B_1 = \{b_1, \dots, b_n\}$ is a basis of $D_{\alpha(D)}$ and $y_i = b_i X$. Since D is a subring of $A = D[X; \alpha, \delta]$ every right A-module is also a right vector space over D. We'll start with some preparatory results. The first assertion is an adaption of [13, (3.3) Theorem].

Proposition 3.2 *A right ideal R of $A = D[X; \alpha, \delta]$ is of finite codimension over D, i.e., a factor module A/R is a finite dimensional D-module, if and only if R is essential and finitely generated. In particular, a nonzero two-sided ideal of A has a finite codimension over D if and only if it is finitely generated.*

Proof Since every cyclic right ideal is an infinite dimensional D-module, the necessity is obvious in view of Theorem 2.5. Conversely, assume indirectly that there is a finitely generated essential right ideal R which is of infinite (right) codimension over D. If B is a strong Schreier basis of R (constructed as in Proposition 2.4), then $\{u_{b,i} = by_i - \pi(by_i) \neq 0 \mid b \in B, i = 1, \dots, n\}$ is a basis for R as a free module over A. Since A has IBN, and by the assumption that R is finitely generated, then the above set must be finite. In particular, there exists an integer $m > 0$ such that for every $b \in B = B_R$ satisfying $|b| \geq m$ all monomials by_i $(i = 1, \dots, n)$ are also in B. Consequently, all monomials of length $> r$ whose initial segments of length r belong to B, are also contained in B. Hence, for a monomial $b \in B$ of length r the right ideal bA meets R trivially, contradicting the essentiality of R. This shows that $B = B_R$ is finite. The last claim follows trivially from the observation that a nonzero two-sided ideal of a domain is always essential. □

We reach now the first goal of this section as

Theorem 3.3 *Let I be an ideal of right codimension 1 of $A = D[X; \alpha, \delta]$ over D, where α is an endomorphism of a division ring D such that $D_{\alpha(D)}$ has a dimension $n > 1$. Then ideal powers I^l $(l \in \mathbb{N})$ form a perfect Gabriel topology $\mathfrak{I}$ and $(1, n)$ is the module type of the associated ring $A_{\mathfrak{I}}$ of quotients. Moreover, projective modules over $A_{\mathfrak{I}}$ are free and the Grothendieck group $K_0(A_{\mathfrak{I}})$ is cyclic of order $n - 1$.*

Proof By Theorem 2.5 I is a free right A-module of rank n. Consequently, I^l is a free right A-module of rank n^l. This implies by [14, Propositions XI.3.3 and XI.3.4 (d)] that the ideal topology defined by powers I^l $(l \in \mathbb{N})$ is a perfect topology. Therefore, the canonical embedding $D[X; \alpha, \delta] = A \longrightarrow A_{\mathfrak{I}}$ is a flat bimorphism whence $A_{\mathfrak{I}}$ is a subring of both the maximal flat epimorphic ring and the maximal ring of right quotients $Q_{\mathrm{tot}}(A) \subseteq Q_{\max}(A)$. Consequently, by [14, Corollary XI.3.8] an equality $(R \cap A)A_{\mathfrak{I}} = R$ holds for every right ideal R of $A_{\mathfrak{I}}$. Therefore, R is a free $A_{\mathfrak{I}}$-module because $R \cap A$ is a free A-module by Theorem 2.5 and ${}_A A_{\mathfrak{I}}$ is a flat left A-module. Hence, projective right modules over $A_{\mathfrak{I}}$ are free.

To compute the module type of $A_{\mathfrak{I}}$, write $I = \sum_{i=1}^n z_i A = \oplus_{i=1}^n z_i A$ in view of Theorem 2.5. Let z_j^* be the canonical jth projection $z_j^* : I \to A : \sum_{i=1}^n z_i a_i \mapsto a_j$ for each $j = 1, \dots, n$. The obvious equalities

$$z_j^* z_i = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i \end{cases} \quad \& \quad \sum_{i=1}^n z_i z_i^* = 1 \tag{9}$$

in $A_{\mathfrak{J}}$ imply that $A_{\mathfrak{J}}^n \cong A_{\mathfrak{J}}$ holds. Therefore, to verify that $(1, n)$ is the module type of $A_{\mathfrak{J}}$, it is enough to see $m \equiv 1 \mod n-1$ provided $A_{\mathfrak{J}} \cong A_{\mathfrak{J}}^m$. In fact, assume $A_{\mathfrak{J}} = \sum_{i=1}^m c_i A_{\mathfrak{J}} = \oplus_{i=1}^{i=m} c_i A_{\mathfrak{J}}$ such that the annihilator of each $c_i \in A_{\mathfrak{J}}$ is trivial. Consequently, multiplication by c_i on the left is injective on $A_{\mathfrak{J}}$. Therefore, the right ideals $\mathrm{dom}(c_i) \cong c_i\mathrm{dom}(c_i)$ $(i = 1, \ldots, m)$ of A are free A-modules of rank $\equiv 1 \mod (n-1)$ by Schreier-Lewin formula established in Corollary 2.7 and the finite codimensionality of $\mathrm{dom}(c_i)$. This shows that $R = \bigoplus_i c_i \mathrm{dom}\,(c_i)$ is a free A-module of rank $\equiv m \mod (n-1)$. On the other hand, R is an essential right ideal of A. Namely, for any $0 \neq a = \sum_{i=1}^m c_i q_i \in A$ $(q_i \in A_{\mathfrak{J}})$ with at least one of the $q_i \neq 0$, say q_1, there are nonzero elements $r_1 \in q_1\mathrm{dom}(q_1) \cap (\mathrm{dom}(c_1) \cap (\cap_{i=1}^m \mathrm{dom}(q_i)))$; $r_2, \ldots, r_m \in A$ such that $q_i r_1 r_2 \cdots r_i \in \mathrm{dom}(c_i)$ for all $i = 2, \ldots, m$ in view of the fact that all $\mathrm{dom}(c_i)$, $\mathrm{dom}(q_i)$ are essential right ideals of A and $q_1\mathrm{dom}(q_1) \neq 0$ by $q_1 \neq 0$. By the choice of r_1, one has $0 \neq q_1(r_1) = q_1 r_1 \in \mathrm{dom}(c_1)$ whence $0 \neq (c_1 q_1(r_1)) r_2 \cdots r_m = c_1 q_1 (r_1 r_2 \cdots r_m) = c_1 q_1 r_1 \cdots r_m \in c_1\mathrm{dom}(c_1)$ because A is a domain. Similarly the product $a r_1 \cdots r_m \neq 0$ holds, and all $c_i q_i r_1 \cdots r_m$ are not necessarily nonzero elements of $c_i\mathrm{dom}(c_i)$, respectively, for all $i > 1$. Consequently, $0 \neq a r r_1 \cdots r_m = \sum_{i=1}^m c_i q_i (r_1 \cdots r_m) \in R$ holds. Hence, R is a right ideal of A of finite codimension in view of Proposition 3.2. Hence, again by the Schreier-Lewin formula in Corollary 2.7, R is a free right A-module of free rank $\equiv 1 \mod (n-1)$. Therefore, $m - 1 \equiv 0 \mod (n-1)$ holds, i.e., n is the smallest integer satisfying $A_{\mathfrak{J}} \cong A_{\mathfrak{J}}^m$, whence the module type of $A_{\mathfrak{J}}$ is $(1, n)$. Consequently, the Grothendieck group $K_0(A_{\mathfrak{J}})$ is cyclic of order $n-1$ because projective modules over $A_{\mathfrak{J}}$ are free. □

Remark 3.4 There are essential differences between free associative algebras and skew polynomial rings. For example, the annihilator ideals of finite codimensional right ideals are not necessarily finite codimensional. Consequently, the topology defined by finite codimensional right ideals are, in general, not an ideal topology. Furthermore, it is unclear whether there always exist two-sided ideals of codimension 1, even of finite codimension. However, there are simple skew polynomial rings when α is an automorphism and even the identity automorphism. Namely, let $F = K(y)$ be a field of rational functions over a field K of characteristic 0 with the usual derivation $'$, then the skew polynomial ring $F[X; 1, ']$ is a simple ring, whence it does not admit maximal two-sided ideal of codimension 1 over F. The simplicity is obvious in view of the facts that $F[X; 1, ']$ is a localization of the first Weyl algebra $A_1(K)$ generated by x, y over K subject to $yx - xy = 1$ at the set of all monic polynomials on y over K and Weyl algebras over fields of characteristic 0 are simple. It is, in fact, more interesting to study the ring $A_{\mathfrak{J}}$ of quotients with respect to an ideal topology defined by an ideal I of $D[X; \alpha, \delta]$ under the extra assumption that I has codimension 1. It is then quite important to look for (quasi-)normal forms for elements of $A_{\mathfrak{J}}$ and good D-bases for A. If μ^* denotes a product $z_{i_l}^* \cdots z_{i_1}^* = \mu^*$ for every monomial $\mu = z_{i_1} \cdots z_{i_l}$ of z_i $(i = 1, \ldots, n)$ where z_i, z_i^* are defined as in the proof of Theorem 3.3, then it is unclear whether elements of $A_{\mathfrak{J}}$ can be written in the form $\sum_{i=1}^m \mu_i \nu_i^* d_i$ for monomials μ_i, ν_i of "variables" z_j and coefficients $d_i \in D$. Same questions arise when I_D is assumed more generally to be of finite codimension over D.

It seems to be an important and difficult problem whether the associated ring $A_{\mathfrak{J}}$ of quotients considered in Theorem 3.3 is simple. In the particular case of trivial derivation and $I = Ax$ we have a positive answer.

Proposition 3.5 *If α is an endomorphism of a division ring D such that $D_{\alpha(D)}$ has a finite dimension $n > 1$, then the ring $A_{\mathfrak{J}}$ of quotients of $A = D[X; \alpha]$ with respect to the Gabriel topology induced by ideals AX^l $(l \in \mathbb{N})$ is simple.*

Proof We use Cohn's trick presented in the proof of [2, Proposition 8.1]. AX is obviously an ideal of codimension 1. Let $B_1 = \{b_1, \ldots, b_n\}$ be a basis of the right vector space D over $\alpha(D)$. Then Ax is a free right A-module with the free generators $y_i = b_i X$, and monomials in y_i form a basis of A_D. Let y_i^* $(i = 1, \ldots, n)$ be the elements of $A_{\mathfrak{J}}$ sending y_i to 1 and y_j to 0 for every index $j \neq i$. If J is a nonzero ideal of $A_{\mathfrak{J}}$, then $J \cap A$ is a non-zero ideal of A. Let a be a non-zero element of $J \cap A$ such that it can be written as a linear combination $a = \sum_{i=1}^m \mu_i d_i$ of monomials μ_i in the y_j with coefficients d_i with possibly smallest m and among them with possibly smallest degree. By considering $\mu_l^* a$ where μ_l has a minimal length, i.e., of minimal degree; if it is necessary, one sees clearly that some μ_i is a constant, i.e., of length 0. If $m > 1$, by considering $\mu_k^* a \mu_k$, where μ_k has a positive length, one obtain a contradiction. Thus, $m = 1$ and a is a non-zero constant whence the statement follows. □

It is worth noting that even in the situation of Proposition 3.5, it is not clear whether elements of $A_{\mathfrak{J}}$ can be written in the form $\sum_{i=1}^m \mu_i \nu_i^* d_i$ for monomials μ_i, ν_i of "variables" y_j and coefficients $d_i \in D$.

We will now construct the maximal flat epimorphic ring of skew polynomial rings. To reach this aim we follow the idea of Rosenmann and Rosset [13]. First it is routine to verify that finite codimensional right ideals of $A = D[X; \alpha, \delta]$ define a linear right topology on A, named the *fc-topology* by [13]. It is a little bit harder to show that the fc-topology is a Gabriel topology. It is instructive to modify the argument of Rosenmann and Rosset [13] (that right A-modules which can be generated by A-submodules of finite dimension over D, form a hereditary torsion theory in the category of right A-modules because of the fact that D is neither commutative nor commuting with the y_i's) to obtain this claim. For undefined necessary notions and results on torsion theory, Gabriel topologies and localizations, we refer to Stenström classic [14]. However, to help the reader and to get a better grasp of the notorious special axiom T4 (see [14, Chapter VI.5]) in the definition of Gabriel topology, we present here a direct proof. To show that the fc-topology is a Gabriel topology, one has to show that it satisfies axiom T4, that is, a right ideal R is open, i.e., of finite codimension if there is a finite codimensional right ideal J such that $R : b = \{a \in A \,|\, ba \in R\}$ has a finite codimension for all $b \in J$. By Proposition 3.2 and Theorem 2.5 there is a finite basis $\{b_1, \ldots, b_l\}$ for $J = \sum b_i A = \oplus b_i A$. Moreover, there are l right ideals $R_1, \ldots, R_l$ of finite codimension such that $b_i R_i \subseteq R$. Again by Proposition 3.2 and Theorem 2.5 each right ideal R_i $(0 < i < l + 1)$ is an essential right ideal of A which is also a free right A-module of finite rank. Consequently, $\bar{R} = \sum_{i=1}^l b_i R_i \subseteq R$ is clearly a finitely generated right ideal of A. On the other hand, it is obvious that $b_i R_i$ is an essential submodule of $b_i A$ for every index i. Therefore, by [9, Folgerung 5.1.8], $\bar{R} = \sum b_i R_i = \oplus b_i R_i$ is an essential submodule of $\oplus b_i A = J$. Since J is an essential right ideal of A, $\bar{R}$ is an essential right ideal of A. Consequently, being both finitely generated and an essential right ideal of A, $\bar{R}$ is finite codimensional by Proposition 3.2. This implies, in view of the inclusion $\bar{R} \subseteq R$, that R is a finite codimensional right ideal of A, that is, it is open with respect to the fc-topology. Thus, we have shown that the fc-topology is a Gabriel topology and a right ideal of A is open in the fc-topology if and only if it is essential and free of finite rank in view of Proposition 3.2. By [14, Proposition XI.3.4 (d)] every open right ideal defined by a flat epimorphism $\phi : A = D[X; \alpha, \delta] \to Q$ contains a finitely generated essential right ideal and so has a finite codimension by Proposition 3.2. This shows that the fc-topology defines the maximal flatepimorphic localization $Q^{\mathrm{fc}}(D[X;\alpha,\delta]) = Q_{\mathrm{tot}}(D[X;\alpha,\delta])$ of $D[X;\alpha,\delta]$ which is a subring of the maximal ring $Q_{\max}(D[X;\alpha,\delta])$. By the same proof as for Theorem 3.3 we obtain another goal of this section as

Theorem 3.6 *Right finite codimensional ideals of $A = D[X;\alpha,\delta]$ where the dimension of $D_{\alpha(D)}$ is finite, form a perfect Gabriel topology, called the* fc*-topology. The ring $Q^{\mathrm{fc}}(A)$ of quotients with respect to this* fc*-topology is the maximal flat epimorphic localization of A. Its Grothendieck group $K_0(Q^{\mathrm{fc}}(A))$ is cyclic of order $n-1$ and their projective modules are free.*

It seems that the maximal flat epimorphic localization of $D[X;\alpha,\delta]$ is not necessarily simple but I am not able to construct a counterexample. However, if $\delta = 0$, then we have a positive result.

Corollary 3.7 *If α is an endomorphism of a division ring D such that D is a right vector space of dimension $n > 1$ over $\alpha(D)$, then the maximal flat epimorphic localization $Q^{\mathrm{fc}}(D[X;\alpha])$ is simple.*

Proof By assumption, $D[X;\alpha]X$ is a two-sided ideal of codimension 1 whence every element of $D[X;\alpha]$ can be written uniquely as a polynomial in y_i with coefficients from D on the right where $y_i = b_iX$ with respect to a right basis $B_1 = \{b_1,\ldots,b_n\}$ of D over $\alpha(D)$. The proof of Proposition 3.5 can be used to complete the verification. □

In contrast to the case of free unital associative algebras we do not have the description of all flat bimorphisms of $A = D[X;\alpha,\delta]$ because largest ideals contained in finite-codimensional right ideals of A are not necessarily finite-codimensional right ideals over A! However, thanks to [14, Proposition I.6.10], products of ideals from any (fixed) set Λ of maximal two-sided ideals of A which are finite-codimensional right ideals of A, still form a perfect Gabriel topology on A. The proof for [1, Theorem 3.10] yields the following result.

Theorem 3.8 *Let $\alpha : D \longrightarrow D$ be an endomorphism of a divison ring D such that the right dimension of D over $\alpha(D)$ is $n \geq 2$, and $A = D[X;\alpha,\delta]$ be a skew polynomial ring with respect to some α-derivation δ of D. Furthermore, let Λ be an arbitrary set of maximal two-sided ideals which are also finite codimensional right ideals of A, then finite products of elements from Λ form a perfect Gabriel topology $\mathfrak{T}_\Lambda$ of A with the associated ring Q of right quotients. For each ideal $I_\lambda \in \Lambda$ the factor ring A/I_λ is a matrix ring ${}_\lambda D_{m_\lambda}$ over a division ring ${}_\lambda D$ of right dimension l_λ over D. Put $d_\lambda = l\lambda m_\lambda$ and let d be the greatest common divisor of d_λ's. Then, the module type of Q is $(1, d(n-1)+1)$. In addition, if the ideals of Λ are commutable, then the completion of A with respect to $\mathfrak{T}_\lambda$ is the direct products $\prod_I \varprojlim A/I^l$ of inverse limits $\varprojlim A/I^l$ of the canonical inverse systems $\{A/I^l \mid l \in \mathbb{N}\}$, where I runs over all elements of Λ.*

Since the work [1] is under submission, I sketch here the main idea of the proof of Theorem 3.8. First we show that $A_{\mathfrak{I}}$ has module type $(1, d(n-1))$ if I_D is a two-sided ideal of finite codimension over D where d is a dimension of a minimal right ideal of A/I. Here $\mathfrak{I}$ denotes the perfect Gabriel topology defined by powers of I. The proof can be carried out in the same manner as that for Theorem 3.3 after the observation that for an open right ideal R of A, there is a power I^l contained in R, so that the factor module A/R is an A/I^l-module whence the dimension of A/R over D is md if m is a length of a right A module A/R. Consequently, R is a free module of rank $dm(n-1)+1$ by Corollary 2.7. The assertion for a set Λ of maximal ideals of finite right codimension over D is then a trivial consequence of elementary number theory! We now complete the proof by using the following equalities on commutable two-sided maximal ideals $I,\ J$ of A

$$A = (I+J)(I+J) = I + J^2 = I^2 + J = I^2 + J^2 = \cdots = I^l + J^m, \quad \forall l, m \in \mathbb{N},$$

and

$$I \cap J = (I \cap J)(I+J) = IJ + JI = IJ = JI.$$

By iterating these equalities one obtains the following more general equalities for finitely many pairwise coprime, commutable ideals $I_1, \ldots, I_m$

$$I_i^{n_i} + \cap_{j \neq i} I_j^{n_j} = A \qquad \& \qquad \cap I_i^{n_i} = \prod_i I_i^{n_i}, \qquad \forall n_i \in \mathbb{N}.$$

Now the claim on completion becomes obvious.

As a corollary of Theorem 3.3 we reobtain one more direct but "unusual" proof of the classical result [10] of Leavitt in the following

Corollary 3.9 *The classical Leavitt algebra* $L_K(1, n)$ $(n > 1)$ *over a field* K *has module type* $(1, n)$.

Proof Consider the rational function field $D = K(t)$ in one variable t over a field K and let α be the endomorphism of D sending t to t^n. By Lüroth's theorem D is a finite field extension of $D_1 = \alpha(D)$ with basis $\{b_i = t^{i-1} \mid i = 1, \ldots, n\}$. Moreover, K is exactly the set of all elements in D fixed by α. By Theorem 3.3, the subring of $A_{\mathfrak{I}} = D[X, \alpha]_{\mathfrak{I}}$, where I is the two-sided ideal AX of $A = D[X, \alpha]$, generated by $y_i = t_i X$, y_i^* where y_i^* is an A-homomorphism from AX to A sending y_i to 1, y_j to 0 for all $j \neq i$, is isomorphic to the classical Leavitt algebra $L_K(1, n)$ and its module type is clearly $(1, n)$. □

We end with some remarks pointing out the structural diversity of a particular skew polynomial ring $A = D[X; \alpha]$. Since A is a left principal ideal domain, $\{X^n \mid n \in \mathbb{N}\}$ is a left Ore set. Hence, in the localized ring A_X, the equality $\alpha(d) = X^{-1}X\alpha(d) = X^{-1}dX$ implies that $D \cong D_n = X^n D X^{-n}$ $(n \in \mathbb{Z},\ D_0 = D)$ form an ascending chain of rings and α can be extended naturally to an automorphism, denoted again by α of $E = \cup_{n=-\infty}^{\infty} D_n$ by putting $\alpha(X^n d X^{-n}) = X^n \alpha(d) X^{-n} = X^n X^{-1} X \alpha(d) X^{-n} = X^{n-1} d X^{-(n-1)}$. This shows that A_X is isomorphic to a localization of a two-sided skew polynomial ring $E[X, \alpha]$ at the two-sided Ore set $\{X^n \mid n \in \mathbb{N}\}$ and hence A_X is a simple two-sided principal ideal domain! On the other hand, substituting $Y = X + c$ $(0 \neq c \in D)$ for X does not, in general, induce an automorphism of A. However, A can be considered as a skew polynomial ring with respect to Y. The multiplication is modified, in view of the equality $Yd = (X+c)d = \alpha(d)X + cd = \alpha(d)(X+c) + (cd - \alpha(d)c)$, by an inner α-derivation $\delta_c(d) = cd - \alpha(d)c$. Hence, A is now the skew polynomial ring $D[Y, \alpha, \delta_c]$. Conversely, if $A = D[X, \alpha, \delta_c]$ is a skew polynomial ring defined by an endomorphism a of a division ring D together with an inner derivation δ_c given by an element $c \in D$, then A becomes a skew polynomial ring $D[Z; \alpha]$ by putting $Z = X - c$. The situation changes completely when we consider the localized ring $A_{\mathfrak{I}} = D[X, \alpha]_{\mathfrak{I}}$ $(I = AX)$ even in the case when D is commutative and α is an endomorphism of D such that D is finite dimensional over $\alpha(D)$ with a basis $B_1 = \{b_1, \ldots, b_n\}$. For $y_i = b_i X$, let $y_i^* : I \longrightarrow A$ be an A-module homomorphism induced by the rule $y_i \mapsto 1;\ y_j \mapsto 0;\ j \neq i$. It is unclear whether elements of $A_{\mathfrak{I}}$

can be written in a normal form $\sum_{i=1}^{m} \mu_i \nu_i^* d_i$ for monomials μ_i, ν_i of "variables" y_j and coefficients $d_i \in D$.

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