Generalizing Choi-Like Maps

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Abstract A problem of further generalization of generalized Choi maps $\Phi_{[a,b,c]}$ acting on $\mathbb{M}_3$ introduced by Cho, Kye, and Lee is discussed. Some necessary conditions for positivity of the generalized maps are provided as well as some sufficient conditions. Also, some sufficient conditions for decomposability of these maps are shown.

Keywords Positive map · Indecomposable · Choi matrix · PPT · Separable state

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1 Introduction

The paper concerns linear maps acting between matrix algebras. For $n \in \mathbb{N}$, by $\mathbb{M}_n$, we denote the algebra of square $n \times n$ -matrices $X = (x_{ij})$ with complex coefficients. Let $\Phi : \mathbb{M}_m \rightarrow \mathbb{M}_n$ be a linear map. We say that Φ is a *positive map* if $\Phi(X)$ is a positive definite matrix in $\mathbb{M}_n$ for every positive definite matrix $X \in \mathbb{M}_m$. Be reminded that for $k \in \mathbb{N}$, there are the following identifications: $\mathbb{M}_k(\mathbb{M}_m) = \mathbb{M}_k \otimes \mathbb{M}_m = \mathbb{M}_{km}$, where

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$\mathbb{M}_k(\mathbb{M}_m)$ denotes the C^* -algebra of $k \times k$ -matrices (X_{ij}) with $X_{ij} \in \mathbb{M}_m$. Define a map $\Phi_k : \mathbb{M}_k(\mathbb{M}_m) \rightarrow \mathbb{M}_k(\mathbb{M}_n)$ by $\Phi_k(X_{ij}) = (\Phi(X_{ij}))$. We say that the map Φ is k -positive if Φ_k is a positive map. If Φ is k -positive for every $k \in \mathbb{N}$, then it is called a *completely positive map*.

The transposition map $\theta_n : \mathbb{M}_n \rightarrow \mathbb{M}_n : (x_{ij}) \mapsto (x_{ji})$ is known to be not completely positive map (even not 2-positive) [22]. We say that a map $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_n$ is k -copositive (respectively *completely copositive*) if the composition $\theta_n \circ \Phi$ is k -positive (respectively completely positive). A map Φ is said to be *decomposable* if it can be expressed as a sum $\Phi = \Phi_1 + \Phi_2$, where Φ_1 is a completely positive map while Φ_2 is a completely copositive one. Otherwise, Φ is called *indecomposable*. For further details, we refer the reader to the book of Størmer [22].

The systematic study of positive maps on C^* -algebras was started by a pioneering work of Erling Størmer in the 1960s of the last century [20]. Although the definitions are easy and the above notions seem to be rather elementary, the problem of description of all positive maps is still unsolved even in the case of low dimensional matrix algebras. In particular, still, there is no effective characterization of decomposable maps. In recent years, the importance of the theory of positive maps drastically increased because of its applications in mathematical physics, especially in rapidly emerging theory of quantum information [6, 10, 13].

It was shown by Størmer [20] and Woronowicz [25] that every positive map $\Phi : \mathbb{M}_2 \rightarrow \mathbb{M}_2$ (Størmer) or $\Phi : \mathbb{M}_2 \rightarrow \mathbb{M}_3$ (Woronowicz) is decomposable. The first example of an indecomposable positive map was given by Choi [3, 4]. It is a map $\Phi : \mathbb{M}_3 \rightarrow \mathbb{M}_3$ given by

$$\Phi(X) = \begin{pmatrix} x_{11} + x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} + x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} + x_{22} \end{pmatrix}. \quad (1.1)$$

Although there is no systematic prescription of indecomposable maps, there are numerous examples scattered across literature [5, 7–9, 14–19, 23, 24].

An interesting approach to generalization of (1.1) was presented in [1]. For any triple of nonnegative numbers a, b, c define a map $\Phi_{[a,b,c]} : \mathbb{M}_3 \rightarrow \mathbb{M}_3$ by

$$\Phi_{[a,b,c]}(X) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} + bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{22} + ax_{33} \end{pmatrix},$$

with x_{ij} being the matrix elements of $X \in \mathbb{M}_3(\mathbb{C})$. One proves

Theorem 1.1 ([1]) *The map $\Phi_{[a,b,c]}$ is positive but not completely positive if and only if*

- (1) $0 \leq a < 2$,
- (2) $a + b + c \geq 2$,
- (3) *if $a \leq 1$, then $bc \geq (1 - a)^2$.*

Moreover, being positive it is indecomposable if and only if $4bc < (2 - a)^2$.

Slightly different class was considered by Kye [12]

$$\Phi_{[a;c_1,c_2,c_3]}[X] = \begin{pmatrix} ax_{11} + c_1x_{33} & -x_{12} & -x_{13} \\ -x_{21} & c_2x_{11} + ax_{22} & -x_{23} \\ -x_{31} & -x_{32} & c_3x_{22} + ax_{33} \end{pmatrix},$$

with $a, c_1, c_2, c_3 \geq 0$. He proved the following

Theorem 1.2 A map $\Phi_{[a;c_1,c_2,c_3]}$ is positive but not completely positive if and only if

(1) $1 \leq a < 2$,

(2) $c_1 c_2 c_3 \geq (2 - a)^3$.

Moreover, $\Phi_{[a;c_1,c_2,c_3]}$ is indecomposable and atomic.

Interestingly, Osaka shown

Theorem 1.3 ([17]) A map $\Phi_{[1;c_1,c_2,c_3]}$ is extremal if $c_1 c_2 c_3 = 1$.

Let us be reminded that for a linear map $\Phi : \mathbb{M}_m \rightarrow \mathbb{M}_n$, one can define its Choi matrix [2]. It is an element $C_\Phi \in \mathbb{M}_m(\mathbb{M}_n)$ defined by

$$C_\Phi = (\Phi(E_{ij})), \quad (1.2)$$

where $\{E_{ij} : 1 \leq i, j \leq n\}$ is a system of matrix units in $\mathbb{M}_n$. The mapping $\Phi \mapsto C_\Phi$ is the so called Choi-Jamiołkowski isomorphism between $L(\mathbb{M}_m, \mathbb{M}_n)$ and $\mathbb{M}_m(\mathbb{M}_n)$ [11]. One has

Theorem 1.4 ([2]) A map Φ is completely positive if and only if its Choi matrix C_Φ is positive definite.

It turns out also that it is possible to characterize positivity and decomposability of Φ in terms of the matrix C_Φ . To this end, let us recall that C_Φ can be considered as an element of the tensor product $\mathbb{M}_m \otimes \mathbb{M}_n$. An element $\rho \in M_m \otimes M_n$ is called a PPT matrix if both ρ and $(\text{id}_{\mathbb{M}_m} \otimes \theta_n)(\rho)$ are positive definite.

Theorem 1.5 ([10, 13, 21]) Let Φ be a linear map.

(i) Φ is positive if and only if

$$\langle \xi \otimes \eta, C_\Phi \xi \otimes \eta \rangle \geq 0$$

for every vectors $\xi \in \mathbb{C}^m$, $\eta \in \mathbb{C}^n$.

(ii) Φ is decomposable if and only if

$$\text{Tr}(\rho C_\Phi) \geq 0$$

for every PPT matrix $\rho \in \mathbb{M}_m \otimes \mathbb{M}_n$.

The property (i) in the above theorem is called *block-positivity* [13].

2 Choi-Like Maps

In this paper, we consider the following generalization of the maps described in Introduction. Let

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & b_2 \\ b_3 & c_3 & a_3 \end{pmatrix}, \quad (2.1)$$

with $a_i, b_j, c_k \geq 0$, and define $\Phi_A : \mathbb{M}_3 \rightarrow \mathbb{M}_3$ as follows

$$\Phi_A(X) = \begin{pmatrix} a_1x_{11} + b_1x_{22} + c_1x_{33} & -x_{12} & -x_{13} \\ -x_{21} & c_2x_{11} + a_2x_{22} + b_2x_{33} & -x_{23} \\ -x_{31} & -x_{32} & b_3x_{11} + c_3x_{22} + a_3x_{33} \end{pmatrix}. \quad (2.2)$$

Our aim is to describe some conditions for positivity of Φ_A as well as for decomposability.

Firstly, let us make the following observation.

Proposition 2.1 *A map Φ_A is completely positive if and only if*

$$\begin{pmatrix} a_1 & -1 & -1 \\ -1 & a_2 & -1 \\ -1 & -1 & a_3 \end{pmatrix} \quad (2.3)$$

is positive definite matrix.

Proof Observe that the Choi matrix of the map Φ_A is of the form

$$C_{\Phi_A} = \left(\begin{array}{ccc|ccc|ccc} a_1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a_2 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_2 & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a_3 \end{array} \right),$$

where we used dots instead of zeros. Now, we apply Theorem 1.4. Since b_j, c_k are non-negative, positive definiteness of C_{Φ_A} is equivalent to positive definiteness of the submatrix (2.3). $\square$

Now, let us discuss some necessary conditions for positivity.

Theorem 2.2 *A necessary condition for positivity of Φ_A is positivity of $\Phi_{[\bar{a}, \bar{b}, \bar{c}]}$, where*

$$\bar{a} = \frac{1}{3}(a_1 + a_2 + a_3), \quad \bar{b} = \frac{1}{3}(b_1 + b_2 + b_3), \quad \bar{c} = \frac{1}{3}(c_1 + c_2 + c_3). \quad (2.4)$$

Proof Let Φ_A be a positive map and consider a linear map

$$\bar{\Phi}_A(X) = \frac{1}{3} (\Phi_A(X) + S\Phi_A(S^*XS)S^* + S^*\Phi_A(SXS^*)S),$$

where S denotes a unitary shift $Se_i = e_{i+1}$. It is clear that $\bar{\Phi}_A$ is positive. Moreover, $\bar{\Phi}_A = \Phi_{[\bar{a}, \bar{b}, \bar{c}]}$ with $\bar{a}, \bar{b}, \bar{c}$ defined in (2.4). $\square$

Let us introduce the following notation

$$a_* = (a_1a_2a_3)^{1/3}, \quad b_* = (b_1b_2b_3)^{1/3}, \quad c_* = (c_1c_2c_3)^{1/3}.$$

To find sufficient conditions, consider a positive map $\Phi[a, b, c]$ and define

$$\Phi_{[a,b,c]}^V(X) := V^{-1/2}\Phi_{[a,b,c]}\left(V^{1/2}XV^{1/2}\right)V^{-1/2},$$

where $V = \sum_{i=1}^3 p_i E_{ii}$ with $p_i > 0$. It is clear that $\Phi_{[a,b,c]}^V$ is positive. Moreover, one easily finds that $\Phi_{[a,b,c]}^V = \Phi_A$ with

$$a_i = a_*, \quad b_i = \frac{p_{i+1}}{p_i} b, \quad c_i = \frac{p_{i+2}}{p_i} c, \quad i = 1, 2, 3, \quad (2.5)$$

that is, $A = V^{-1} A_{[a,b,c]} V$. Note that $b_* = b$, $c_* = c$, and $b_i c_{i+1} = bc$. Hence, one arrives at the following

Theorem 2.3 *Assume that there exist a, b, c and V such that $\Phi_{[a,b,c]}$ is a positive map and all coefficients of the matrix*

$$M = A - V^{-1} A_{[a,b,c]} V,$$

are non-negative numbers. Then Φ_A is a positive map.

Corollary 2.4 *Assume that there exist $a, b, c \geq 0$ such that*

- (1) $\Phi_{[a,b,c]}$ defines a positive map,
- (2) $\min\{a_1, a_2, a_3\} \geq a$, $b_* \geq b$, $c_* \geq c$,
- (3) $b_i c_{i+1} \geq bc$ for $i = 1, 2, 3$.

Then Φ_A is a positive map.

Proof We have $b_1 b_2 b_3 \geq b^3$ and $c_1 c_2 c_3 \geq c^3$. We construct a positive map with b'_k and c'_k such that $b_k \geq b'_k$ and $c_k \geq c'_k$. Let us take p_1, p_2, p_3 such that

$$b_1 = b'_1 := \frac{p_2}{p_1} b, \quad b_2 = b'_2 := \frac{p_3}{p_2} b, \quad b_3 \geq b'_3 := \frac{p_1}{p_3} b.$$

Clearly, $b_1 b_2 b_3 \geq b'_1 b'_2 b'_3 = b^3$. Now, $b_1 c_2 \geq bc$ and hence, let us take $c_2 \geq c'_2 := \frac{p_1}{p_2} c$ and similarly for c_3 and c_1 .

Now, the map with b'_k and c'_k is positive due to (2.5). Hence, the map with b_k and c_k is positive as well. $\square$

Remark 2.5 If $b_1 b_2 b_3 = 0$, then a sufficient condition reduces to

- (1) $a_1, a_2, a_3 \geq a$,
- (2) $c_1 c_2 c_3 \geq (2-a)^3$,

with $1 \leq a < 2$. Interestingly, if $a_1 = a_2 = a_3 = a$, then $a \geq 1$ and $c_1 c_2 c_3 \geq (2-a)^3$ are necessary and sufficient [12]. Indeed, if Φ_A is positive and $c_1 c_2 c_3 = c^3$ let us define V by taking

$$c_1 = \frac{p_3}{p_1} c, \quad c_2 = \frac{p_1}{p_2} c, \quad c_3 = \frac{p_2}{p_3} c.$$

Note the map $\Phi_{[a,b,c]}$ has to be positive and $b = 0$, hence $a + c \geq 2$. Therefore, $a + \sqrt[3]{c_1 c_2 c_3} \geq 2$ which implies $c_1 c_2 c_3 \geq (2-a)^3$.

We end this section with the following remark.

Remark 2.6 Let us observe that condition (3) in Theorem 1.1 can equivalently be written as

$$a + \sqrt{bc} \geq 1.$$

On the other hand, the condition (2) of Theorem 1.2 can be rewritten as

$$a + c_* \geq 2.$$

Let us also remind that for a matrix

$$A = \begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix}$$

one can define a map $\Phi_A : \mathbb{M}_2 \rightarrow \mathbb{M}_2$ similarly to (2.2). It was shown by Kossakowski (see [6, Example 7.1]) that Φ_A is positive if and only if $a_i \geq 0$, $b_i \geq 0$ and

$$\sqrt{a_1 a_2} + \sqrt{b_1 b_2} \geq 1.$$

Having all these observations in mind, one can conjecture that positivity of the map (2.2) is related to the following set of conditions:

$$\sqrt{a_i a_{i+1}} + \sqrt{b_i c_{i+1}} \geq 1, \quad (2.6)$$

$$a_* + b_* + c_* \geq 2. \quad (2.7)$$

In next sections, we shall discuss these conditions.

3 Condition (2.6)

Now, we consider further generalization. Let $A = (a_{ij})_{i,j=1}^n$ be a matrix with nonnegative coefficients. Let us define a map $\Phi_A : \mathbb{M}_n \rightarrow \mathbb{M}_n$ by

$$\Phi_A(X) = \Delta_A(X) - X, \quad (3.1)$$

where

$$\Delta_A(X) = \text{diag} \left(\sum_{j=1}^n a'_{1j} x_{jj}, \sum_{j=1}^n a'_{2j} x_{jj}, \dots, \sum_{j=1}^n a'_{nj} x_{jj} \right).$$

In the above formula $a'_{ij} = a_{ij} + \delta_{ij}$, where δ_{ij} stands for the Kronecker delta.

Observe that positivity of Φ_A is equivalent to the inequality $\langle \eta, \Delta_A(\zeta \zeta^*) \eta \rangle \geq \langle \eta, \zeta \zeta^* \eta \rangle$ for every $\zeta, \eta \in \mathbb{C}^n$. One can easily show that the above is equivalent to

$$\sum_{i,j=1}^n a'_{ij} p_i^2 q_j^2 \geq \left(\sum_{i=1}^n p_i q_i \right)^2, \quad p_i, q_i \geq 0.$$

In the following main result of this section, we discuss the role of the condition (2.6).

Theorem 3.1 Let Φ_A be given by (3.1). If the map Φ_A is positive, then

$$\sqrt{a_{ii}a_{jj}} + \sqrt{a_{ij}a_{ji}} \geq 1, \quad i \neq j \quad (3.2)$$

Proof Let $i \neq j$ be fixed. Observe that

$$\sum_{kl} a'_{kl} p_k^2 q_l^2 - \left(\sum_k p_k q_k \right)^2 \quad (3.3)$$

$$= \sum_{kl} a'_{kl} p_k^2 q_l^2 - \sum_{kl} p_k p_l q_k q_l \quad (3.4)$$

$$= \sum_k a_{kk} p_k^2 q_k^2 + \sum_{k < l} (a_{kl} p_k^2 q_l^2 + a_{lk} p_l^2 q_k^2) - 2 \sum_{k < l} p_k p_l q_k q_l \quad (3.5)$$

$$= \sum_k a_{kk} p_k^2 q_k^2 + \sum_{k < l} (\sqrt{a_{kl}} p_k q_l - \sqrt{a_{lk}} p_l q_k)^2 + 2 \sum_{k < l} (\sqrt{a_{kl}a_{lk}} - 1) p_k p_l q_k q_l. \quad (3.6)$$

Consider the case $p_k = 0$ and $q_k = 0$ for every $k \notin \{i, j\}$. Then, the above expression is equal to

$$a_{ii} p_i^2 q_i^2 + a_{jj} p_j^2 q_j^2 + (\sqrt{a_{ij}} p_i q_j - \sqrt{a_{ji}} p_j q_i)^2 + 2(\sqrt{a_{ij}a_{ji}} - 1) p_i p_j q_i q_j$$

$$= (\sqrt{a_{ii}} p_i q_i - \sqrt{a_{jj}} p_j q_j)^2 + (\sqrt{a_{ij}} p_i q_j - \sqrt{a_{ji}} p_j q_i)^2 \quad (3.7)$$

$$+ 2(\sqrt{a_{ii}a_{jj}} + \sqrt{a_{ij}a_{ji}} - 1) p_i p_j q_i q_j. \quad (3.8)$$

Now, take

$$p_i = q_j = 1, \quad p_j = \left(\frac{a_{ij}a_{ii}}{a_{ji}a_{jj}} \right)^{1/4}, \quad q_i = \left(\frac{a_{ij}a_{jj}}{a_{ji}a_{ii}} \right)^{1/4}.$$

Then, the expression from lines (3.7) and (3.8) reduces to

$$2(\sqrt{a_{ii}a_{jj}} + \sqrt{a_{ij}a_{ji}} - 1) \left(\frac{a_{ij}}{a_{ji}} \right)^{1/2}.$$

It follows from the assumption that the expression (3.3) is nonnegative for every choice of p_k, q_k . Thus, we arrive at the inequality

$$\sqrt{a_{ii}a_{jj}} + \sqrt{a_{ij}a_{ji}} - 1 \geq 0$$

which is equivalent to (3.2). $\square$

Having in mind, Theorem 1.1 one cannot expect that a converse theorem is also true. However, it turns out that a slight modification of (3.2) gives a sufficient condition for positivity and even decomposability.

Theorem 3.2 If

$$\frac{1}{n-1} \sqrt{a_{ii}a_{jj}} + \sqrt{a_{ij}a_{ji}} \geq 1, \quad i \neq j \quad (3.9)$$

then the map Φ_A is positive and decomposable.

Proof Let us observe that

$$\begin{aligned} \sum_i a_{ii} p_i^2 q_i^2 &= \frac{1}{n-1} \sum_{i < j} \left(a_{ii} p_i^2 q_i^2 + a_{jj} p_j^2 q_j^2 \right) \\ &= \sum_{i < j} \left(\sqrt{\frac{a_{ii}}{n-1}} p_i q_i - \sqrt{\frac{a_{jj}}{n-1}} p_j q_j \right)^2 + 2 \sum_{i < j} \frac{\sqrt{a_{ii} a_{jj}}}{n-1} p_i p_j q_i q_j. \end{aligned} \quad (3.10)$$

According to the lines (3.3)–(3.6) and (3.10), one has

$$\begin{aligned} &\sum_{ij} a'_{ij} p_i^2 q_j^2 - \left(\sum_i p_i q_i \right)^2 \\ &= \sum_i a_{ii} p_i^2 q_i^2 + \sum_{i < j} (\sqrt{a_{ij}} p_i q_j - \sqrt{a_{ji}} p_j q_i)^2 + 2 \sum_{i < j} (\sqrt{a_{ij} a_{ji}} - 1) p_i p_j q_i q_j \\ &= \sum_{i < j} \left(\sqrt{\frac{a_{ii}}{n-1}} p_i q_i - \sqrt{\frac{a_{jj}}{n-1}} p_j q_j \right)^2 + \sum_{i < j} (\sqrt{a_{ij}} p_i q_j - \sqrt{a_{ji}} p_j q_i)^2 \\ &\quad + 2 \sum_{i < j} \left(\frac{\sqrt{a_{ii} a_{jj}}}{n-1} + \sqrt{a_{ij} a_{ji}} - 1 \right) p_i p_j q_i q_j. \end{aligned}$$

It follows from (3.9) that the above expression is nonnegative for every nonnegative p_i and q_i , so Φ_A is positive.

Now, we will show that Φ_A is decomposable. Let $\rho \in \mathbb{M}_n(\mathbb{M}_n)$ be a PPT matrix. We have $\rho = (\rho_{ij})$, where $\rho_{ij} \in \mathbb{M}_n$. For $i \neq j$, let r_{ij} be the (i, j) -coefficient of the matrix ρ_{ij} . For any i, j , let β_{ij} denote the j -th diagonal term in the matrix ρ_{ii} . Thus, the matrix ρ looks like

$$\rho = \begin{pmatrix} \begin{array}{ccccc|ccccc|c|ccccc} \beta_{11} & * & * & \cdots & * & * & r_{12} & * & \cdots & * & \cdots & * & * & * & \cdots & r_{1n} \\ * & \beta_{12} & * & \cdots & * & * & * & * & \cdots & * & \cdots & * & * & * & \cdots & * \\ * & * & \beta_{13} & \cdots & * & * & * & * & \cdots & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & \beta_{1n} & * & * & * & \cdots & * & \cdots & * & * & * & \cdots & * \end{array} \\ \hline \begin{array}{ccccc|ccccc|c|ccccc} * & * & * & \cdots & * & \beta_{21} & * & * & \cdots & * & \cdots & * & * & * & \cdots & * \\ r_{21} & * & * & \cdots & * & * & \beta_{22} & * & \cdots & * & \cdots & * & * & * & \cdots & r_{2n} \\ * & * & * & \cdots & * & * & * & \beta_{23} & \cdots & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * & * & * & \cdots & \beta_{2n} & \cdots & \cdots & * & * & * & \cdots & * \end{array} \\ \hline \begin{array}{ccccc|ccccc|c|ccccc} \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \end{array} \\ \hline \begin{array}{ccccc|ccccc|c|ccccc} * & * & * & \cdots & * & * & * & * & \cdots & * & \cdots & \beta_{n1} & * & * & \cdots & * \\ * & * & * & \cdots & * & * & * & * & \cdots & * & \cdots & * & \beta_{n2} & * & \cdots & * \\ * & * & * & \cdots & * & * & * & * & \cdots & * & \cdots & * & * & \beta_{n3} & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ r_{n1} & * & * & \cdots & * & * & r_{n2} & * & \cdots & * & \cdots & * & * & * & \cdots & \beta_{nn} \end{array} \end{pmatrix}$$

while

$$\rho^\Gamma = \begin{pmatrix} \beta_{11} & * & * & \cdots & * & * & * & \cdots & * & \cdots & * & * & * & \cdots & * \\ * & \beta_{12} & * & \cdots & * & r_{12} & * & * & \cdots & * & \cdots & * & * & * & \cdots & * \\ * & * & \beta_{13} & \cdots & * & * & * & * & \cdots & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & \beta_{1n} & * & * & * & \cdots & * & \cdots & r_{1n} & * & * & \cdots & * \\ * & r_{21} & * & \cdots & * & \beta_{21} & * & * & \cdots & * & \cdots & * & * & * & \cdots & * \\ * & * & * & \cdots & * & * & \beta_{22} & * & \cdots & * & \cdots & * & * & * & \cdots & * \\ * & * & * & \cdots & * & * & * & \beta_{23} & \cdots & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * & * & * & * & \cdots & \beta_{2n} & \cdots & * & r_{2n} & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & r_{n1} & * & * & * & \cdots & * & \cdots & \beta_{n1} & * & * & \cdots & * \\ * & * & * & \cdots & * & * & * & * & \cdots & r_{n2} & \cdots & * & \beta_{n2} & * & \cdots & * \\ * & * & * & \cdots & * & * & * & * & \cdots & * & \cdots & * & * & \beta_{n3} & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * & * & * & * & \cdots & * & \cdots & * & * & * & \cdots & \beta_{nn} \end{pmatrix},$$

where $\rho^\Gamma = (\theta_n(\rho_{ij}))$ is the partially transposed ρ , and stars stand for any numbers. Since ρ and ρ^Γ are positive definite, $\beta_{ij} \geq 0$, $r_{ji} = \bar{r}_{ij}$, and

$$\beta_{ii}\beta_{jj} \geq |r_{ij}|^2, \quad \beta_{ij}\beta_{ji} \geq |r_{ij}|^2, \quad i \neq j. \quad (3.11)$$

Now, observe that

$$\begin{aligned} \text{Tr}(\rho C_{\Phi_A}) &= \sum_{i,j=1}^n a_{ij}\beta_{ij} - 2 \sum_{1 \leq i < j \leq n} \text{Re}(r_{ij}) \\ &\geq \sum_{i=1}^n a_{ii}\beta_{ii} + \sum_{1 \leq i < j \leq n} (a_{ij}\beta_{ij} + a_{ji}\beta_{ji}) - 2 \sum_{1 \leq i < j \leq n} |r_{ij}| \\ &= \frac{1}{n-1} \sum_{1 \leq i < j \leq n} (a_{ii}\beta_{ii} + a_{jj}\beta_{jj}) + \sum_{1 \leq i < j \leq n} (a_{ij}\beta_{ij} + a_{ji}\beta_{ji}) - 2 \sum_{1 \leq i < j \leq n} |r_{ij}| \\ &\geq 2 \sum_{i < j} \left(\frac{1}{n-1} \sqrt{a_{ii}a_{jj}\beta_{ii}\beta_{jj}} + \sqrt{a_{ij}a_{ji}\beta_{ij}\beta_{ji}} - |r_{ij}| \right) \end{aligned} \quad (3.12)$$

$$\geq 2 \sum_{i < j} |r_{ij}| \left(\frac{1}{n-1} \sqrt{a_{ii}a_{jj}} + \sqrt{a_{ij}a_{ji}} - 1 \right) \geq 0. \quad (3.13)$$

The expression (3.12) was obtained by arithmetic-geometric mean inequality, while (3.13) is due to inequalities (3.11). Since ρ is an arbitrary PPT matrix, we conclude that Φ_A is decomposable (cf. Theorem 1.5 (ii)). $\square$

For $n = 2$, we immediately get the following corollary (cf. Remark 2.6).

Corollary 3.3 *For $n = 2$, a map ϕ_A is positive if and only if*

$$\sqrt{a_{11}a_{22}} + \sqrt{a_{12}a_{21}} \geq 1.$$

Remark 3.4 For $n = 3$, the above theorem assures that the condition (2.6) is necessary for positivity of Φ_A .

4 Condition (2.7)

Now, we are going to discuss necessity of condition (2.7). Let us start with the following

Theorem 4.1 Assume that the matrix (2.1) is of the form

$$A = \begin{pmatrix} a_1 & b & c \\ c & a_2 & b \\ b & c & a_3 \end{pmatrix} \quad (4.1)$$

for some arbitrary, $a_1, a_2, a_3 \geq 1$ and $b, c > 0$. If the map Φ_A given by (2.2) is positive, then

$$(a_1 a_2 a_3)^{1/3} + b + c \geq 2. \quad (4.2)$$

Proof Observe that the Choi matrix of the map Φ_A is of the form

$$C_A = \left(\begin{array}{ccc|ccc} a_1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a_2 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a_3 \end{array} \right).$$

It follows from positivity of Φ_A that C_A is block positive, i.e.,

$$\langle \xi \otimes \eta, C_A \xi \otimes \eta \rangle \geq 0 \quad (4.3)$$

for every $\xi, \eta \in \mathbb{C}^3$.

Let

$$\xi = \eta = \left(\left(a_1^{-1} a_2 a_3 \right)^{\frac{1}{12}}, \left(a_1 a_2^{-1} a_3 \right)^{\frac{1}{12}}, \left(a_1 a_2 a_3^{-1} \right)^{\frac{1}{12}} \right)^T.$$

Then,

$$\xi \otimes \eta = \left(\left(a_1^{-1} a_2 a_3 \right)^{\frac{1}{6}}, a_3^{\frac{1}{6}}, a_2^{\frac{1}{6}} \middle| a_3^{\frac{1}{6}}, \left(a_1 a_2^{-1} a_3 \right)^{\frac{1}{6}}, a_1^{\frac{1}{6}} \middle| a_2^{\frac{1}{6}}, a_1^{\frac{1}{6}}, \left(a_1 a_2 a_3^{-1} \right)^{\frac{1}{6}} \right)^T.$$

Hence,

$$\begin{aligned} \langle \xi \otimes \eta, C_A \xi \otimes \eta \rangle &= a_1^{\frac{2}{3}} a_2^{\frac{1}{3}} a_3^{\frac{1}{3}} + a_1^{\frac{1}{3}} a_2^{\frac{2}{3}} a_3^{\frac{1}{3}} + a_1^{\frac{1}{3}} a_2^{\frac{1}{3}} a_3^{\frac{2}{3}} + b a_3^{\frac{1}{3}} + b a_1^{\frac{1}{3}} + b a_2^{\frac{1}{3}} \\ &\quad + c a_2^{\frac{1}{3}} + c a_3^{\frac{1}{3}} + c a_1^{\frac{1}{3}} - 2 a_3^{\frac{1}{3}} - 2 a_1^{\frac{1}{3}} - 2 a_2^{\frac{1}{3}} \\ &= \left(a_1^{\frac{1}{3}} + a_2^{\frac{1}{3}} + a_3^{\frac{1}{3}} \right) \left((a_1 a_2 a_3)^{\frac{1}{3}} + b + c - 2 \right). \end{aligned}$$

Thus, inequality (4.3) leads to (4.2). $\square$

Remark 4.2 The vector ξ in the above proof can be taken even simpler:

$$\xi = \eta = \left(a_1^{-\frac{1}{6}}, a_2^{-\frac{1}{6}}, a_3^{-\frac{1}{6}} \right)^T. \quad (4.4)$$

Remark 4.3 For the case $n = 2$, one can prove a similar result for

$$A = \begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix},$$

where $a_i \geq 1, b_i > 0$ are arbitrary. In this case, one takes

$$\xi = \left(a_1^{-\frac{1}{4}} b_2^{\frac{1}{4}}, a_2^{-\frac{1}{4}} b_1^{\frac{1}{4}} \right)^T, \quad \eta = \left(a_1^{-\frac{1}{4}} b_2^{-\frac{1}{4}}, a_2^{-\frac{1}{4}} b_1^{-\frac{1}{4}} \right)^T.$$

Theorem 4.4 Assume that the matrix (2.1) is of the form

$$A = \begin{pmatrix} a_1 & b_1 & 0 \\ 0 & a_2 & b_2 \\ b_3 & 0 & a_3 \end{pmatrix}$$

for some arbitrary, $a_1, a_2, a_3 \geq 1$ and $b_1, b_2, b_3 > 0$. If the map Φ_A given by (2.2) is positive, then

$$(a_1 a_2 a_3)^{1/3} + (b_1 b_2 b_3)^{1/3} \geq 2.$$

Proof The same idea as above. Observe that the Choi matrix of Φ_A has the form

$$C_A = \left(\begin{array}{ccc|ccc|ccc} a_1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a_2 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_2 & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a_3 \end{array} \right).$$

Consider the following vectors ξ and η (see (4.4)):

$$\xi = \begin{pmatrix} a_1^{-\frac{1}{6}} b_1^{\frac{1}{6}} b_3^{-\frac{1}{6}} \\ a_2^{-\frac{1}{6}} b_2^{\frac{1}{6}} b_1^{-\frac{1}{6}} \\ a_3^{-\frac{1}{6}} b_3^{\frac{1}{6}} b_2^{-\frac{1}{6}} \end{pmatrix}, \quad \eta = \begin{pmatrix} a_1^{-\frac{1}{6}} b_1^{-\frac{1}{6}} b_3^{\frac{1}{6}} \\ a_2^{-\frac{1}{6}} b_2^{-\frac{1}{6}} b_1^{\frac{1}{6}} \\ a_3^{-\frac{1}{6}} b_3^{-\frac{1}{6}} b_2^{\frac{1}{6}} \end{pmatrix}.$$

Then, the vector $\xi \otimes \eta$ is of the form

$$\xi \otimes \eta = \begin{pmatrix} a_1^{-\frac{1}{3}} \\ a_1^{-\frac{1}{6}} a_2^{-\frac{1}{6}} b_1^{\frac{1}{3}} b_2^{-\frac{1}{6}} b_3^{-\frac{1}{6}} \\ a_1^{-\frac{1}{6}} a_3^{-\frac{1}{6}} b_1^{\frac{1}{6}} b_2^{\frac{1}{6}} b_3^{-\frac{1}{3}} \\ \hline a_1^{-\frac{1}{6}} a_2^{-\frac{1}{6}} b_1^{-\frac{1}{3}} b_2^{\frac{1}{6}} b_3^{\frac{1}{6}} \\ a_2^{-\frac{1}{3}} \\ a_2^{-\frac{1}{6}} a_3^{-\frac{1}{6}} b_1^{-\frac{1}{6}} b_2^{\frac{1}{3}} b_3^{-\frac{1}{6}} \\ \hline a_1^{-\frac{1}{6}} a_3^{-\frac{1}{6}} b_1^{-\frac{1}{6}} b_2^{-\frac{1}{6}} b_3^{\frac{1}{3}} \\ a_2^{-\frac{1}{6}} a_3^{-\frac{1}{6}} b_1^{\frac{1}{6}} b_2^{-\frac{1}{3}} b_3^{\frac{1}{6}} \\ a_3^{-\frac{1}{3}} \end{pmatrix}.$$

Like in the proof of Theorem 4.1, we calculate

$$\begin{aligned} & \langle \xi \otimes \eta, C_A \xi \otimes \eta \rangle \\ &= a_1^{\frac{1}{3}} + a_2^{\frac{1}{3}} + a_3^{\frac{1}{3}} + a_1^{-\frac{1}{3}} a_3^{-\frac{1}{3}} b_1^{\frac{1}{3}} b_2^{\frac{1}{3}} b_3^{\frac{1}{3}} + a_1^{-\frac{1}{3}} a_2^{-\frac{1}{3}} b_1^{\frac{1}{3}} b_2^{\frac{1}{3}} b_3^{\frac{1}{3}} + a_2^{-\frac{1}{3}} a_3^{-\frac{1}{3}} b_1^{\frac{1}{3}} b_2^{\frac{1}{3}} b_3^{\frac{1}{3}} \\ & \quad - 2a_1^{-\frac{1}{3}} a_2^{-\frac{1}{3}} - 2a_1^{-\frac{1}{3}} a_3^{-\frac{1}{3}} - 2a_2^{-\frac{1}{3}} a_3^{-\frac{1}{3}} \\ &= a_1^{\frac{1}{3}} + a_2^{\frac{1}{3}} + a_3^{\frac{1}{3}} + a_2^{\frac{1}{3}} a_*^{-1} b_* + a_3^{\frac{1}{3}} a_*^{-1} b_* + a_1^{\frac{1}{3}} a_*^{-1} b_* - 2a_3^{\frac{1}{3}} a_*^{-1} - 2a_2^{\frac{1}{3}} a_*^{-1} - 2a_1^{\frac{1}{3}} a_*^{-1} \\ &= a_*^{-1} \left(a_1^{\frac{1}{3}} + a_2^{\frac{1}{3}} + a_3^{\frac{1}{3}} \right) (a_* + b_* - 2). \quad \square \end{aligned}$$

5 Problem of Sufficiency

Let us consider the matrix A given by (4.1). We have shown that the following conditions (cf. (3.2), (4.2))

$$\begin{aligned} (a_i a_{i+1})^{1/2} + (bc)^{1/2} &\geq 1, \quad i = 1, 2, 3, \\ (a_1 a_2 a_3)^{1/3} + b + c &\geq 2 \end{aligned}$$

are necessary for positivity of Φ_A . For $a_1 = a_2 = a_3$, these conditions are also sufficient [1]. Note that they are special cases of conditions (2.6) and (2.7).

However, these conditions are not sufficient for general case. Namely, we have the following no-go result.

Proposition 5.1 *Let $a_1, a_2, a_3 > 0$ and $b \geq 0$ be such that*

$$(a_1 a_2 a_3)^{1/3} + b = 2.$$

Let A be the matrix in given by (4.1) with $c=0$. Then, Φ_A is positive if and only if $a_1 = a_2 = a_3$.

Proof Sufficiency part follows from Theorem 1.1. We will show necessity. Assume that Φ_A is a positive map. Let

$$\xi = \left(a_1^{-\frac{1}{6}} a_3^{\frac{1}{6}}, a_2^{-\frac{1}{6}} a_1^{\frac{1}{6}}, a_3^{-\frac{1}{6}} a_2^{\frac{1}{6}} \right)^T.$$

Then,

$$\xi \xi^* = \begin{pmatrix} a_1^{-\frac{1}{3}} a_3^{\frac{1}{3}} & a_2^{-\frac{1}{6}} a_3^{\frac{1}{6}} a_1^{-\frac{1}{6}} a_2^{\frac{1}{6}} & a_1^{-\frac{1}{6}} a_3^{\frac{1}{6}} a_2^{-\frac{1}{6}} a_1^{\frac{1}{6}} \\ a_2^{-\frac{1}{6}} a_3^{\frac{1}{6}} a_1^{-\frac{1}{6}} a_2^{\frac{1}{6}} & a_2^{-\frac{1}{3}} a_1^{\frac{1}{3}} & a_3^{-\frac{1}{6}} a_1^{\frac{1}{6}} a_2^{-\frac{1}{6}} a_3^{\frac{1}{6}} \\ a_1^{-\frac{1}{6}} a_3^{\frac{1}{6}} a_2^{-\frac{1}{6}} a_1^{\frac{1}{6}} & a_3^{-\frac{1}{6}} a_1^{\frac{1}{6}} a_2^{-\frac{1}{6}} a_3^{\frac{1}{6}} & a_3^{-\frac{1}{3}} a_2^{\frac{1}{3}} \end{pmatrix}$$

and

$$\Phi_A(\xi \xi^*) = \begin{pmatrix} a_1^{\frac{2}{3}} a_3^{\frac{1}{3}} + b a_2^{-\frac{1}{3}} a_1^{\frac{1}{3}} & -a_2^{-\frac{1}{6}} a_3^{\frac{1}{6}} & -a_1^{-\frac{1}{6}} a_2^{\frac{1}{6}} \\ -a_2^{-\frac{1}{6}} a_3^{\frac{1}{6}} & a_2^{\frac{2}{3}} a_1^{\frac{1}{3}} + b a_3^{-\frac{1}{3}} a_2^{\frac{1}{3}} & -a_3^{-\frac{1}{6}} a_1^{\frac{1}{6}} \\ -a_1^{-\frac{1}{6}} a_2^{\frac{1}{6}} & -a_3^{-\frac{1}{6}} a_1^{\frac{1}{6}} & a_3^{\frac{2}{3}} a_2^{\frac{1}{3}} + b a_1^{-\frac{1}{3}} a_3^{\frac{1}{3}} \end{pmatrix}.$$

Calculate the determinant of the above matrix

$$\begin{aligned} & \det \Phi_A(\xi \xi^*) \\ &= \left(a_1^{\frac{2}{3}} a_3^{\frac{1}{3}} + b a_2^{-\frac{1}{3}} a_1^{\frac{1}{3}} \right) \left(a_2^{\frac{2}{3}} a_1^{\frac{1}{3}} + b a_3^{-\frac{1}{3}} a_2^{\frac{1}{3}} \right) \left(a_3^{\frac{2}{3}} a_2^{\frac{1}{3}} + b a_1^{-\frac{1}{3}} a_3^{\frac{1}{3}} \right) - 2 \\ & \quad - a_1^{-\frac{1}{3}} a_2^{\frac{1}{3}} \left(a_2^{\frac{2}{3}} a_1^{\frac{1}{3}} + b a_3^{-\frac{1}{3}} a_2^{\frac{1}{3}} \right) - a_3^{-\frac{1}{3}} a_1^{\frac{1}{3}} \left(a_1^{\frac{2}{3}} a_3^{\frac{1}{3}} + b a_2^{-\frac{1}{3}} a_1^{\frac{1}{3}} \right) \\ & \quad - a_2^{-\frac{1}{3}} a_3^{\frac{1}{3}} \left(a_3^{\frac{2}{3}} a_2^{\frac{1}{3}} + b a_1^{-\frac{1}{3}} a_3^{\frac{1}{3}} \right) \\ &= a_1 a_2 a_3 + 3 b a_1^{\frac{2}{3}} a_2^{\frac{2}{3}} a_3^{\frac{2}{3}} + 3 b^2 a_1^{\frac{1}{3}} a_2^{\frac{1}{3}} a_3^{\frac{1}{3}} + b^3 - 2 \\ & \quad - a_2 - b a_1^{-\frac{1}{3}} a_2^{\frac{2}{3}} a_3^{-\frac{1}{3}} - a_1 - b a_1^{\frac{2}{3}} a_2^{-\frac{1}{3}} a_3^{-\frac{1}{3}} - a_3 - b a_1^{-\frac{1}{3}} a_2^{-\frac{1}{3}} a_3^{\frac{2}{3}} \\ &= (a_* + b)^3 - 2 - 3\bar{a} - 3b\bar{a}a_*^{-1} \\ &= a_*^{-1} (6a_* - 3a_*\bar{a} - 3b\bar{a}) = 6a_*^{-1} (a_* - \bar{a}). \end{aligned}$$

In the last line, the assumption that $a_* + b = 2$ was applied. It follows from positivity of Φ_A that $a_* \geq \bar{a}$. Hence, $a_* = \bar{a}$ and consequently all a_i are equal. $\square$

Remark 5.2 The above considerations show that (2.6) and (2.7) do not form a set of sufficient conditions for positivity of the map Φ_A in the case when a_i are distinct numbers. At this stage, it is still an open problem how to generalize Theorems 1.1 and 1.2 for arbitrary numbers a_i, b_j, c_k .

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