Primary Decomposition of Powers of the Prime Ideal of a Numerical Semigroup Ring

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To Le Tuan Hoa on his sixtieth birthday

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Abstract

Let $R = k[t^{n_1}, \dots, t^{n_s}] = k[x_1, \dots, x_s]/P$ be a numerical semigroup ring and let $P^{(n)} = P^n R_P \cap R$ be the symbolic power of P and $R_s(P) = \bigoplus_{i \geq 0} P^{(i)} t^i$ the symbolic Rees ring of P . It is hard to determine symbolic powers of P ; there are even non-Noetherian symbolic Rees rings for 3-generated semigroups. We determine the primary decomposition of powers of P for some classes of 3-generated numerical semigroups.

Keywords Primary decomposition · Symbolic power · Numerical semigroup ring

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1 Introduction

If I is an ideal in a Noetherian ring R , then the subring $R(I) = R[It]$ of $R[t]$ is called the Rees ring of I . This was introduced by Rees, in his proof of the Artin-Rees lemma. If P is a prime ideal, then the primary decomposition of P^n always contains a P -primary component, it is $R = k[t^{n_1}, \dots, t^{n_s}] = k[x_1, \dots, x_n]/P$ and is called the symbolic n th power of P . Cowsik [1] asked if the symbolic Rees algebra $R_s(P) = R[Pt, P^{(2)}t^2, P^{(3)}t^3, \dots]$ is always Noetherian. This was shown not to be true by Roberts [7, 8]. There are even counterexamples when $R = k[t^{n_1}, t^{n_2}, t^{n_3}] = k[x, y, z]/P$. Goto-Nishida-Watanabe [2] showed that for $n \geq 4$ then $k[t^{7n-3}, t^{(5n-2)n}, t^{8n-3}]$ does not have a finitely generated symbolic Rees algebra if $\text{char}(k) = 0$. Their smallest counterexample is $k[t^{25}, t^{72}, t^{29}] = k[x, y, z]/(x^{11} - yz^7, y^3 - x^4z^4, z^{11} - x^7y^2)$. Hochster [4] has shown that $P^{(n)} = P^n$ for all n if $k[x_1, \dots, x_s]/P$ is a complete intersection. Huneke [5] has shown that if P is a prime ideal of height 2 in a 3-dimensional ring, then the opposite holds.

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2 Numerical Semigroup Rings

If $R = k[t^{n_1}, \dots, t^{n_s}]$, we map $k[x_1, \dots, x_s]$ to $k[t]$, by $x_i \mapsto t^{n_i}$. Then $R \simeq k[x_1, \dots, x_s]/P$, and P is a prime ideal since R is a domain. In the case of numerical semigroup rings, R is 1-dimensional, so $P^{(n)} = P^n$ or $P^{(n)} \cap Q$, where Q is $(x_1, \dots, x_s)$ -primary. If $P^{(n)} = P^n$ for all n , i.e., if the semigroup ring is a complete intersection, then $R_s(P) = R(P)$ is Noetherian and P^n is P -primary. Kunz [6] has shown that the semigroup ring of S is Gorenstein if and only if the semigroup S is symmetric, i.e., if for some D , we have $n \in S$ if and only if $D - n \notin S$.

3 3-Generated Numerical Semigroups

In the sequel, we mean numerical semigroup when we write semigroup. A 3-generated semigroup is Gorenstein if and only if it is a complete intersection. This is so, since if we factor with a generator, we get a ring of embedding dimension 2. The factor ring is Gorenstein (complete intersection) if and only if the semigroup ring is Gorenstein (complete intersection). For a ring of embedding dimension 2, the concepts are equivalent. If the semigroup is generated by 3 elements, and is not a complete intersection, then $R = k[t^{n_1}, t^{n_2}, t^{n_3}] \simeq k[x, y, z]/P$ where P is generated by the three 2×2 -minors of a matrix (the relation matrix)

$$\begin{pmatrix} x^{a_1} & y^{b_1} & z^{c_1} \\ z^{c_2} & x^{a_2} & y^{b_2} \end{pmatrix}.$$

Herzog and Ulrich [3] have shown that $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $a_1 = a_2$, $b_1 \leq b_2$, $c_1 \geq c_2$ (or a permutation). Huneke [5] has shown that if P is a 2-dimensional prime in a 3-dimensional ring, then $P^{(2)}/P^2$ is generated by one element Δ . Schenzel [9] has, in the case of a 3-generated semigroup, determined Δ . The result is, if $a_1 \leq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, in particular this holds if $R_s(P) = R[Pt, P^{(2)}t^2]$ according to the result of Herzog and Ulrich,

$$\Delta = \begin{vmatrix} x^{a_1} & y^{b_1} & z^{c_1} \\ z^{c_2} & x^{a_2} & y^{b_2} \\ y^{b_1} & x^{a_2-a_1}y^{b_1-b_2}z^{c_2} & y^{a_1}z^{c_1-c_2} \end{vmatrix}.$$

He also showed that $(x^{a_1}, y^{b_2}, z^{c_2})\Delta \in P^2$.

In the other case, $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, there is a similar result:

$$\Delta = \begin{vmatrix} x^{a_1} & y^{b_1} & z^{c_1} \\ z^{c_2} & x^{a_2} & y^{b_2} \\ x^{a_1-a_2} & y^{b_1-b_2}z^{c_1} & x^{a_1}z^{c_1-c_2} \end{vmatrix}$$

and $(x^{a_2}, y^{b_2}, z^{c_2})\Delta \in P^2$. If $R_s(P) \neq R[Pt, P^{(2)}t^2]$, we can only determine the primary decomposition of P^2 , but if $R_s(P) = R[Pt, P^{(2)}t^2]$, we will construct the primary decomposition of P^n for all n in the following cases: if the semigroup is generated by an arithmetic sequence, or if it is generated by $a < b < c$ with $c - a \leq 4$, or if the multiplicity is ≤ 4 .

Theorem 1 Suppose that $R = k[t^a, t^b, t^c] = k[x, y, z]/P$ is not a complete intersection. Then $P^2 = ((\Delta) + P^2) \cap ((z^{c_2}) + P^2)$ is a primary decomposition. If furthermore $R_s(P) = R[Pt, P^{(2)}t^2]$, then $P^{2n} = (P^{(2)})^n \cap ((z^{nc_2}) + P^{2n})$ and $P^{2n+1} = P(P^{(2)})^n \cap ((z^{nc_2}) + P^{2n})$.

Proof Since $P^{(2)} = (\Delta) + P^2$ and since $(z^{c_2}) + P^2$ is (x, y, z) -primary, it suffices to note that $(z^{c_2}) \cap (\Delta) = z^{c_2} \Delta \subseteq P^2$ ([9, Theorem 10.3]) to see the first statement. If $R_s(P) = R[Pt, P^{(2)}t^2]$, then $P^{(2n)} = \sum_{i=0}^n (P^{(2)})^i P^{2n-2i} = (P^{(2)})^n$ since $P^2 \subseteq P^{(2)}$. In the same way, we see that $P^{(2n+1)} = P P^{(2n)}$. Finally, $(z^{nc_2}) \cap P^{(2n)} = (z^{nc_2}) \cap ((\Delta) + (P^2)^n) \subseteq P^{2n}$ since $z^{c_2} \Delta \subseteq P^2$.

The remaining part is a search for examples when $R_s(P) = R[Pt, P^{(2)}t^2]$. $\square$

3.1 Arithmetic Sequences

Now suppose that the semigroup is generated by $m, m+d, m+2d$, $\gcd(m, m+d, m+2d) = 1$. The semigroup is symmetric (so the semigroup ring is a complete intersection) if m is even and d odd. Otherwise, the relation matrix is

$$\begin{pmatrix} x^{k+d} & y & z \\ z^k & x & y \end{pmatrix}.$$

Thus, according to the theorem by Herzog and Ulrich $R_s(P) = R[Pt, P^{(2)}t^2]$. We use this result also below to see when $R_s(P) = R[Pt, P^{(2)}t^2]$.

Theorem 2 *Let $R = k[t^m, t^{m+d}, t^{m+2d}] = k[x, y, z]/P$ be nonsymmetric. Then the primary decomposition is $P^{2n} = ((\Delta) + P^2)^n \cap ((z^n) + P^2)^n$ and $P^{2n+1} = (P(\Delta) + P^2)^n \cap ((z^n) + P^2)^n$.*

3.2 Semigroups Generated by $a < b < c, c - a \leq 4$

If the semigroup is not generated by an arithmetic sequence, the generators are $m, m+1, m+3$ or $m, m+1, m+4$ or $m, m+2, m+3$ or $m, m+3, m+4$.

If the semigroup is generated by $m, m+1, m+3$, it is symmetric if $m \equiv 0 \pmod{3}$.

If $m = 3k+1$ the relation matrix is

$$\begin{pmatrix} x^k & y & z \\ z^k & x^2 & y^2 \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$ for all k .

If $m = 3k+2$ the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^2 & z \\ z^k & x^2 & y \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$ unless $k = 1$.

If the semigroup is generated by $m, m+1, m+4$, it is symmetric if $m \equiv 0 \pmod{4}$ (and if $m = 5$).

If $m = 4k+1, k \geq 2$, the relation matrix is

$$\begin{pmatrix} x^{k-1} & y & z \\ z^k & x^3 & y^3 \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$.

If $m = 4k+2, k \geq 2$, the relation matrix is

$$\begin{pmatrix} x^k & y^2 & z \\ z^k & x^3 & y^2 \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $k \geq 3$ and $k = 1$.

If $m = 4k + 3$, the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^3 & z \\ z^k & x^3 & y \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ only if $k = 1$ or $k = 2$.

If the semigroup is generated by $m, m+2, m+3$, it is symmetric if $m \equiv 0 \pmod{3}$ (and if $m = 4$).

If $m = 3k + 1$, the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^2 & z^2 \\ z^{k-1} & x & y \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$ for all k .

If $m = 3k + 2$ the relation matrix is

$$\begin{pmatrix} x^k & y^2 & z^2 \\ z^k & x & y \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$ for all k .

If the semigroup is generated by $m, m+3, m+4$, it is symmetric if $m \equiv 0 \pmod{4}$ (and if $m = 6$ or $m = 9$).

If $m = 4k + 1, k \geq 2$, the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^3 & z^3 \\ z^{k-1} & x & y \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$.

If $m = 4k + 2, k \geq 2$, the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^2 & z^3 \\ z^{k-1} & x & y^2 \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $k \geq 4$.

If $m = 4k + 3$ the relation matrix is

$$\begin{pmatrix} x^{k+1} & y & z^3 \\ z^k & x & y^3 \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ only if $k = 3$.

For $m = 5$ the relation matrix is

$$\begin{pmatrix} x^2 & y^2 & z \\ z & x^3 & y \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$.

Theorem 3 *If the semigroup is generated by $a < b < c$, $c - a \leq 4$, not symmetric, and a, b, c not an arithmetic sequence, then $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if the generators are 5, 6, 8 or 6, 7, 10 or 15, 18, 19 or 7, 8, 11 or 11, 12, 15 or 15, 18, 19 or $4k + 2, 4k + 3, 4k + 6, k \geq 3$ or $4k + 2, 4k + 5, 4k + 6, k \geq 4$.*

3.3 Semigroups of Multiplicity 3

We note that if the semigroup is of multiplicity ≤ 4 , then Huneke [5] has shown that the symbolic Rees ring is Noetherian. Suppose that the semigroup is generated by $3, 3k+1, 3l+$

2. In order to have a 3-generated semigroup, we must have $l \leq 2k$ and $k \leq 2l + 1$. The semigroup is never symmetric. The relation matrix is

$$\begin{pmatrix} x^{2l-k+1} & y & z \\ z & x^{2k-l} & y \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$.

3.4 Semigroups of Multiplicity 4

If a 3-generated semigroup has multiplicity 4 and is not symmetric, it has generators $4, 4k + 1, 4l + 3$. If $k > l$ the relation matrix is

$$\begin{pmatrix} x^{3l-k+2} & y & z^2 \\ z & x^{2k-2l-1} & y \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $5l - 3k + 3 \leq 0$. If $k \leq l$, the relation matrix is

$$\begin{pmatrix} x^{2l-2k+1} & y & z \\ z & x^{3k-l} & y^2 \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $3l - 5k + 1 \geq 0$.

Theorem 4 *If the semigroup has multiplicity 3, then $R_s(P) = R[Pt, P^{(2)}t^2]$. If the multiplicity is 4 and not symmetric, it is generated by $4, 4k + 1, 4l + 3$, and $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $k > l$ and $5l - 3k + 3 \leq 0$ or if $k \leq l$ and $3l - 5k + 1 \geq 0$.*

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