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An Almost Tight Lower Bound for the Scheduling Problem to Meet Two Min-Sum Objectives

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Abstract In this note, we provide an almost tight lower bound for the scheduling problem to meet two min-sum objectives considered by Angel et al. in Oper. Res. Lett. 35(1): 69–73, 2007.

Keywords Bi-criteria · Scheduling · Approximation algorithm

Mathematics Subject Classification (2010) 90C27 · 68W25

1 Previous results

Angel et al. [1] recently investigated the following bi-criteria scheduling problem $1 \parallel \{\sum C_j, \sum w_j C_j\}$ via the *simultaneous approximation* approach invented by Stein and Wein [2] and obtained the following result:

Theorem 1 (Angel et al. [1]) *For the bi-criteria schedule problem* $1 \parallel (\sum C_j, \sum w_j C_j)$ *with n jobs*, (i) *there exists a* $(1+\frac{1}{r}, 1+r)$*-approximation schedule for any* $r > 0$; *and* (ii) *there exists an instance such that no* $(1+\frac{1}{r}, 1+\frac{r-1}{2r+1})$*-approximation schedule exists for* $r > 1$.

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The lower bound result above was improved later by Yan [3]:

Theorem 2 (Yan [3]) *For the bi-criteria schedule problem* $1 \parallel (\sum C_j, \sum w_j C_j)$ *with* n *jobs, there exists an instance such that no* $(1 + \frac{1}{r}, 1 + \frac{r-1}{1.5+\sqrt{2r}})$*-approximation schedule exists for any* $r > 1$.

Note that the second term of the lower bound results in Theorems 1 and 2 are respectively in the order of $\Omega(1)$ and $\Omega(\sqrt{r})$.

In this note, we improve the lower bound further to obtain an almost tight lower bound up to a constant factor, namely in the order of $\Omega(r)$.

2 Our Results

Theorem 3 *For the bi-criteria schedule problem* $1 \parallel (\sum C_j, \sum w_j C_j)$ *with* n *jobs, there exists an instance such that no* $(1 + \frac{1}{r}, 1 + \frac{1}{2}r - \epsilon)$*-approximate schedule exists for any* $r > 0$ *and* $\epsilon > 0$.

Proof Let k be a positive integer such that $k > \frac{1}{r}$, for any given $r > 0$. Consider the following instance: there are $n > k$ jobs with processing times

$$p_1 = \cdots = p_{n-1} = 1,$$
$$p_n = 1 + \frac{n(n+1)/2}{rk - 1},$$

and with weights $w_1 = \cdots = w_{n-1} = 0$ and $w_n = 1$. Let π_ℓ $(\ell = 1, \cdots, n)$ be the schedule with job order corresponding to the permutation such that job p_n is on the ℓth position, namely $\pi_\ell = (1, \cdots, \ell - 1, n, \ell, \cdots, n-1)$. By the choice of the processing times p's, evidently π_n and π_1 are the optimal schedules for the two objectives $\sum C_j$ and $\sum w_j C_j$, respectively. For schedule π_ℓ, we have

$$f(\ell) := \sum_{j=1}^{n} C_j(\pi_\ell) = \ell - 1 + \frac{n(n-1)}{2} + p_n(n - \ell + 1),$$
$$g(\ell) := \sum_{j=1}^{n} w_j C_j(\pi_\ell) = \ell - 1 + p_n.$$

Note that f and g are strictly respectively decreasing and increasing functions of ℓ.

By the choice of the processing times p's, for the first objective we have

$$\frac{f(n-k)}{f(n)} = \frac{\sum_{j=1}^{n} C_j(\pi_{n-k})}{\sum_{j=1}^{n} C_j(\pi_n)} = 1 + \frac{1}{r},$$

implying that for all $1 \leqslant \ell < n - k$:

$$\frac{f(\ell)}{f(n)} > 1 + \frac{1}{r}.$$

Therefore for each schedule π_ℓ ($\ell = n-k+1, \cdots, n$), we have that

$$\frac{f(\ell)}{f(n)} = \frac{\sum_{j=1}^{n} C_j(\pi_\ell)}{\sum_{j=1}^{n} C_j(\pi_n)} < 1 + \frac{1}{r},$$

because $f(\ell)$ is strictly decreasing with ℓ.

However, for these schedules, the smallest approximation ratio with respect to the second objective is equal to

$$\min_{\ell=n-k+1,\cdots,n} \frac{g(\ell)}{g(1)} = \frac{g(n-k+1)}{g(1)} = \frac{n-k+p_n}{p_n} = 1 + \frac{1}{\frac{1}{n-k} + \frac{n(n+1)/2}{(n-k)(rk-1)}} := R(n),$$

where the first equality follows from that $g(\ell)$ is increasing with ℓ. The last quantity $R(n)$ is a concave function of n and achieves its maximum when $n^* = k + \sqrt{k^2 + (2r+1)k - 2}$. Although n^* may not be an integer, we can find a lower bound of $R(n^*)$ as follows

$$\begin{aligned} R(n^*) \geqslant R(\lceil n^* \rceil) \geqslant R(n^*+1) &= 1 + \frac{1}{\frac{1}{n^*+1-k} + \frac{(n^*+1)(n^*+2)/2}{(n^*+1-k)(rk-1)}} \\ &= 1 + \frac{1}{\frac{1}{n^*+1-k} + \frac{(n^*/k+1)(n^*/k+2/k)/2}{(n^*/k+1/k-1)(r-1/k)}}, \end{aligned}$$

which is an increasing function of k, asymptotically attaining its supreme $1 + \frac{1}{2}r$, when $k \to \infty$. Therefore there exists k large enough (and hence n^*) such that, for any $\epsilon > 0$:

$$R(n^*) \geqslant R(\lceil n^* \rceil) \geqslant 1 + \frac{1}{2}r - \epsilon. \qquad \square$$

Together with the upper bound result in Theorem 1 [1], we actually have

Corollary 1 *For the bi-criteria schedule problem* $1 \parallel (\sum C_j, \sum w_j C_j)$ *with n jobs, any* $(1 + \frac{1}{r}, 1 + ar - \epsilon)$*-approximation schedule must satisfy* $\frac{1}{2} \leqslant a \leqslant 1$, *where* $r, \epsilon > 0$.

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