



The Short Local Algebras of Dimension 6 with Non-projective Reflexive Modules

Claus Michael Ringel¹

Received: 30 November 2022 / Revised: 14 January 2023 / Accepted: 3 February 2023 /

Published online: 30 May 2023

© The Author(s) 2023

Abstract

Let A be a finite-dimensional local algebra over an algebraically closed field, let J be the radical of A . The modules we are interested in are the finitely generated left A -modules. Projective modules are always reflexive, and an algebra is self-injective iff all modules are reflexive. We discuss the existence of non-projective reflexive modules in case A is not self-injective. We assume that A is short (this means that $J^3 = 0$). In a joint paper with Zhang Pu, it has been shown that 6 is the smallest possible dimension of A that can occur and that in this case the following conditions have to be satisfied: J^2 is both the left socle and the right socle of A and there is no uniform ideal of length 3. The present paper is devoted to showing the converse.

Keywords Short local algebra · Reflexive module · Gorenstein-projective module · Bristle · Atom · Bar · Bristle-bar layout

Mathematics Subject Classification 16G10 · 13D07 · 16E65 · 16G50 · 16L30

1 Introduction

Let k be an algebraically closed field. Let A be a finite-dimensional local k -algebra with radical J . The algebra A is said to be short, provided $J^3 = 0$. Let $e = \dim J/J^2$ and $s = \dim J^2$. If A is short, then the pair (e, s) is called the Hilbert type of A . The modules to be considered in this paper are usually left A -modules, and always finitely generated. In case A is a short local algebra of Hilbert type (e, s) , the local modules of length e and Loewy length at most 2 are of interest; we call them the atoms of the short local algebra A . Several relevant, but standard definitions will be recalled in Sect. 1.4.

✉ Claus Michael Ringel
ringel@math.uni-bielefeld.de

¹ Fakultät für Mathematik, Universität Bielefeld, PO Box 100131, 33501 Bielefeld, Germany

Our aim is to discuss the existence of non-projective reflexive modules over a short local algebra A . Already in 1971, Ramras [2] asked for a characterization of the rings with the property that all reflexive modules are projective.

Of course, if A is self-injective, then all modules are reflexive. Thus, we will assume that A is not self-injective. In cooperation with Zhang Pu, it has been shown that the existence of a non-projective reflexive module implies that $2 \leq s \leq e - 1$, see [8]: thus the dimension of A has to be at least 6, and if A is 6-dimensional, the Hilbert type of A has to be $(3, 2)$. Therefore, as a first test case, it seems reasonable to consider in detail the short local algebras of Hilbert-type $(3, 2)$. This is done in the present note. From now on, A will be a short local algebra of Hilbert type $(3, 2)$. Let us stress that in our case the atoms are the local modules of length 3 and Loewy length 2.

Theorem 1.1 *Let k be an algebraically closed field. Let A be a short local k -algebra of Hilbert type $(3, 2)$ with radical J . The following conditions are equivalent:*

- (i) $J^2 = \text{soc } {}_A A = \text{soc } A_A$ and there is no uniform left ideal with length 3.
- (ii) There is a reflexive atom.
- (iii) There is a non-projective reflexive module.

The implication (iii) \implies (i) has essentially been shown in [8] (but phrased differently), see 7.3 below. Since the implication (ii) \implies (iii) is trivial, it remains to show the implication (i) \implies (ii); this is the target of the present paper. The proof of this implication will be based on a study of the submodules of ${}_A J$ of length 2, 3 and 4. The proof will be given in 9.6 by showing that there are elements $a \in J \setminus J^2$ such that A/Aa is an atom which is both torsionless and extensionless; therefore $Aa = \Omega(A/Aa)$ is a reflexive atom (for any module M , we denote by ΩM the first syzygy module of M , the kernel of a projective cover of M). For the purpose of this paper, an algebra A will be said to be special provided A is a short local algebra of Hilbert type $(3, 2)$ satisfying the condition (i). Thus our aim is to show that any special algebra has reflexive atoms. It seems that similar considerations may show that any special algebra has even Gorenstein-projective atoms, at least if k is large, say uncountable.

1.1 The Use of 3-Kronecker Modules

The modules of Loewy length at most 2 are just the modules annihilated by J^2 , thus the $L(3)$ -modules, where $L(3) = A/J^2$. The $L(3)$ -modules correspond nicely to the $K(3)$ -modules, where $K(3)$ is the path algebra of the 3-Kronecker quiver. We will recall the definition of the 3-Kronecker quiver at the beginning of Sect. 2. The relationship between $K(3)$ -modules and $L(3)$ -modules is furnished by the push-down functor $\pi : \text{mod } K(3) \rightarrow \text{mod } L(3)$, see the beginning of Sect. 6. We say that a $K(3)$ -module W with $\dim W = (3, 2)$ is *special* provided that W is faithful and $\text{End } W = k$, or, equivalently, provided that W is faithful and has no submodule which is simple injective or which is uniform with length 3, see Proposition 2.1. Note that a short local algebra A is special iff ${}_A J = \pi W$ for a special $K(3)$ -module W , see 6.5 and 7.3.

1.2 Bristles and Bars

A bristle (in any length category) is an indecomposable object of length 2. If the submodule B of the module M is a bristle, we call B a subbristle of M . If W is a special $K(3)$ -module, the number of subbristles of W will be called the bristle type of W , it turns out that the bristle type of any special $K(3)$ -module is equal to 1, 2, 3 or ∞ , see 4.2. A 4-dimensional submodule V of a special $K(3)$ -module W will be called a bar provided V is not faithful. Given a special $K(3)$ -module, the subbristles and the bars in W are the obstacles for atoms to be extensionless or torsionless, see Propositions 9.3 and 9.4. Therefore, we introduce the bristle-bar layout of W in Sect. 4.1. As we will see, the shape of the bristle-bar layout only depends on the bristle type of W . For each bristle type, we will present a coefficient quiver for W , as well as the corresponding bristle-bar layout in 4.2.

1.3 Outline

Being interested in a special algebra A with radical J , the main object to look at will be the projective space $\mathbb{P}E$, where $E = J/J^2$. Note that $\mathbb{P}E$ is a natural parameter space for the set of isomorphism classes of atoms, see 9.1. We will use the fact that E is the top of the left module ${}_A J$, as well as the top of the right module J_A . As we have mentioned, the subbristles and the bars in ${}_A J$ and in J_A have to be studied, thus the left and the right bristle-bar layout of the algebra A .

Sections 2–5 will be devoted to the 3-Kronecker quiver $K(3)$. The relationship between $K(3)$ -modules and $L(3)$ -modules is discussed at the beginning of Sect. 6. The proof of Theorem 1.1 is given at the end of Sect. 9. We should stress that Sects. 4 and 5 which provide coefficient quivers for the special $K(3)$ -modules are not essential for the proof of Theorem 1.1.

Section 12 is devoted to commutative algebras where already all four bristle types do occur. And it turns out that for any special commutative algebra, all reflexive atoms are Gorenstein-projective.

1.4 Some Relevant Definitions

For the benefit of the reader, the referees have suggested to recall relevant definitions (further details may be found in [6] as well as in Sect. 2 of [8]). Let M be a module. The module M is said to be torsionless, provided it can be embedded into a projective module. The module M is said to be extensionless, provided $\text{Ext}^1(M, {}_A A) = 0$. A left ${}_A A$ -approximation of M is a map $f : M \rightarrow P$, where P is projective, such that any map $g : M \rightarrow P'$ with P' projective, can be factorized as $g = g'f$ with $g' : P \rightarrow P'$. Such a left ${}_A A$ -approximation f is said to be minimal provided the only direct summand of P containing $f(M)$ is P itself. Any module M has a minimal left ${}_A A$ -approximation f_M , and the cokernel of f_M is denoted by $\mathcal{U}M$ (and called the agemo of M). Of course, M is torsionless iff f_M is a monomorphism, and in this case the canonical exact sequence

$$0 \rightarrow M \xrightarrow{f_M} P \rightarrow \mathcal{U}M \rightarrow 0$$

is said to be an \mathcal{U} -sequence. The modules of the form $\mathcal{U}M$ are extensionless, and any non-projective indecomposable extensionless module is of this form.

The right A -module $M^* = \text{Hom}(M, A)$ is called the A -dual of M . Let $\phi_M : M \rightarrow M^{**}$ be defined by $\phi_M(m)(f) = f(m)$ for $m \in M, f \in M^*$. Clearly, M is torsionless provided that ϕ_M is injective, and M is called reflexive provided that ϕ_M is bijective. It is important to be aware that M is reflexive iff both modules M and $\mathcal{U}M$ are torsionless.

The module M is said to be Gorenstein-projective provided there is an exact complex

$$\dots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \longrightarrow \dots$$

with all modules P_i projective, and which remains exact when we form its A -dual, such that M is the image of f_0 . The Gorenstein projective modules are reflexive, but usually reflexive modules will not be Gorenstein-projective.

Finally, we recall that a module of finite length is said to be uniform provided its socle is simple.

2 $K(3)$ -Modules

The 3-Kronecker quiver $K(3)$ is given by two vertices which we will label 0 and 1, and three arrows $0 \rightarrow 1$. Alternatively, we may start with a 3-dimensional vector space E over k (considered as the arrow space of the quiver) and select, if necessary, a basis of E as the arrows of the quiver. A 3-Kronecker module (or $K(3)$ -module) is a representation M of $K(3)$, that means: it is of the form $M = (M_0, M_1; \phi_M : E \otimes_k M_0 \rightarrow M_1)$, where M_0, M_1 are (finite-dimensional) vector spaces and ϕ_M is a linear map. In case this map is a canonical one, we just will write $M = (M_0, M_1)$. If M, M' are 3-Kronecker modules, a homomorphism $f : M \rightarrow M'$ is a pair $f = (f_0, f_1)$ of linear maps $f_i : M_i \rightarrow M'_i$ (where $i = 0, 1$) such that $\phi_{M'}(1 \otimes f_0) = f_1 \phi_M$. The $K(3)$ -modules form an abelian category denoted by $\text{mod } K(3)$. There are two simple $K(3)$ -modules, namely $S_0 = (k, 0)$ and $S_1 = (0, k)$. The module S_0 is injective, the module S_1 is projective.

If M is a $K(3)$ -module, the pair $\mathbf{dim} M = (\dim M_0, \dim M_1)$ is called the dimension vector of M (it is an element of the Grothendieck group \mathbb{Z}^2 of $\text{mod } K(3)$). We use the bilinear form $[-, -]$ on \mathbb{Z}^2 defined by $[(z_0, z_1), (z'_0, z'_1)] = z_0 z'_0 + z_1 z'_1 - 3z_0 z'_1$, for $z_0, z_1, z'_0, z'_1 \in \mathbb{Z}$. Note that we have $[\mathbf{dim} M, \mathbf{dim} M'] = \dim \text{Hom}(M, M') - \dim \text{Ext}^1(M, M')$, for any pair M, M' of $K(3)$ -modules, see [3]. We are going to define some important $K(3)$ -modules. If x_1, \dots, x_n are elements in a vector space F , we denote by $\langle x_1, \dots, x_n \rangle$ the subspace of F generated by x_1, \dots, x_n .

(a) **Atoms.** For $0 \neq a \in E$, let $\tilde{C}(a) = \tilde{C}(\langle a \rangle) = (k, E/\langle a \rangle)$. This is an atom, any atom is obtained in this way, and $\tilde{C}(a), \tilde{C}(a')$ are isomorphic iff $\langle a \rangle = \langle a' \rangle$. The function $a \mapsto \tilde{C}(a)$ provides a bijection between $\mathbb{P}E$ and the set of isomorphism classes of atoms.

(b) Bristles. If a_1, a_2 are linearly independent in E , then $\tilde{B}(a_1, a_2) = \tilde{B}(\langle a_1, a_2 \rangle) = (\tilde{k}, E/\langle a_1, a_2 \rangle)$. This is a bristle, any bristle is obtained in this way, and $\tilde{B}(a_1, a_2), \tilde{B}(a'_1, a'_2)$ are isomorphic iff $\langle a_1, a_2 \rangle = \langle a'_1, a'_2 \rangle$. The function $\langle a_1, a_2 \rangle \mapsto \tilde{B}(a_1, a_2)$ provides a bijection between the set of lines in $\mathbb{P}E$ and the set of isomorphism classes of bristles.

(c) Atoms and bristles. If $0 \neq a \in E$ and l is a line in $\mathbb{P}E$, then $\langle a \rangle \in l$ iff $\tilde{C}(a)$ maps onto $\tilde{B}(l)$.

2.1 $K(3)$ -Modules Which are Not Faithful

If z is a non-zero element of E , let $\text{mod } K(3)/\langle z \rangle$ be the category of all $K(3)$ -modules annihilated by z . Then, of course, $\text{mod } K(3)/\langle z \rangle$ is equivalent to the category of 2-Kronecker modules. We will use the well-known structure of the category of 2-Kronecker modules. In particular, we need the following facts:

The 2-Kronecker modules with dimension vector $(2, 2)$ without a simple direct summand are either decomposable, then a direct sum of 2 bristles, or else indecomposable, then an extension of a bristle by itself. An indecomposable 2-Kronecker-module has a unique factor module which is a bristle, and this bristle is also the unique subbristle.

There is only one isomorphism class of indecomposable 2-Kronecker modules with dimension vector $(3, 2)$. If W is such a module, any bristle 2-Kronecker module occurs as a submodule of W .

Let W be a $K(3)$ -module. There is an atom which generates W iff W is not faithful. And if W is a faithful $K(3)/\langle a \rangle$ -module, then $\tilde{C}(a)$ is the only atom which generates W .

Proposition 2.1 *Let W be a $K(3)$ -module W with $\mathbf{dim} W = (3, 2)$. The following conditions are equivalent:*

- (i) $\text{End } W = k$.
- (ii) W is indecomposable and has no uniform submodule of length 3.
- (iii) W has no submodule which is simple injective or which is uniform and has length 3.

Proof (i) \implies (ii). We assume that $\text{End } W = k$. Then W is indecomposable. Assume that U is a uniform submodule of W of length 3. We have $\mathbf{dim} U = (2, 1)$ and therefore $\mathbf{dim} W/U = (1, 1)$. We use the homological bilinear form $[-, -]$. We have $[\mathbf{dim} W/U, \mathbf{dim} U] = [(1, 1), (2, 1)] = 0$. Since W is indecomposable, W is a non-trivial extension of U by W/U , therefore $\text{Ext}^1(W/U, U) \neq 0$. It follows that $\text{Hom}(W/U, U) \neq 0$, therefore $\text{End } W \neq k$.

(ii) \implies (iii) is trivial, since a simple injective submodule would be a direct summand.

(iii) \implies (i). We assume that W has no submodule which is simple and injective or which is uniform and has length 3. Since W has no direct summand S_0 , any proper direct decomposition has to be of the form $W = W' \oplus W''$ with $\mathbf{dim} W' = (1, 1)$ and $\mathbf{dim} W'' = (2, 1)$. However, in this case, W'' is uniform of length 3. Thus, we see that W is indecomposable. Assume that $\text{End } W \neq k$. Since W is indecomposable, $\text{End } W$ is a local ring. Since it operates on the vector spaces W_0 (of dimension 3) and W_1 (of

dimension 2), we see that $\text{End } W$ cannot be a division ring. It follows that there is a non-zero endomorphism ϕ of W with $\phi^2 = 0$. Let I be the image of ϕ and K the kernel of ϕ , thus we deal with submodules $I \subseteq K$ of W . Since I is isomorphic to W/K , we must have $\mathbf{dim} I = (1, 1)$ and therefore $\mathbf{dim} K = (2, 1)$. If K is decomposable, then K has a direct summand of the form S_0 , but S_0 is injective and thus also W has a simple injective direct summand. If K is indecomposable, then K is a uniform module of length 3. Both cases are impossible. \square

We will be interested in the faithful 3-Kronecker modules W with $\mathbf{dim} W = (3, 2)$ and $\text{End } W = k$; as we have mentioned, we call them just the special $K(3)$ -modules.

Lemma 2.2 *Let W be a $K(3)$ -module with $\mathbf{dim} W = (3, 2)$ and $\text{End } W = k$. Then the indecomposable submodules of W of length 3 are atoms.*

Proof This follows directly from 2.3. Namely, an indecomposable module of length 3 is either uniform or an atom, but W has no uniform submodule of length 3. \square

If M, N are modules, the largest submodule of M generated by N (thus the sum of the images of all homomorphisms from N to M) is called the trace of N in M .

Proposition 2.3 *Let W be a special $K(3)$ -module. Let $0 \neq a \in E$. The following conditions are equivalent:*

- (i) *The atom $\tilde{C}(a)$ has no factor module which is a subbristle of W .*
- (ii) *The trace of $\tilde{C}(a)$ in W is an atom.*
- (ii') *The trace of $\tilde{C}(a)$ in W has length 3.*

Let us add: If the trace of $\tilde{C}(a)$ in W is an atom, then $\tilde{C}(a)$ is cogenerated by W . The converse is (of course) not true.

Proof (i) implies (ii). We assume that $\tilde{C}(a)$ has no factor module which is a subbristle of W . Using the bilinear form $[-, -]$, we see that there is a non-zero homomorphism $\phi: \tilde{C}(a) \rightarrow W$. The image cannot be of length 2, since otherwise the image would be a subbristle of W . Thus ϕ is a monomorphism. Let D be the image of ϕ . We claim that D is the trace of $\tilde{C}(a)$ in W . Otherwise, there is a homomorphism $\psi: \tilde{C}(a) \rightarrow W$ with $\psi(a) \notin D$. Then $N = \text{Im } \phi + \text{Im } \psi$ has length 4, and is generated by $\tilde{C}(a)$, thus not faithful. It follows that N generates a bristle B which is also a subbristle of N , see 2.2. In particular, B is a subbristle of W . Now $\tilde{C}(a)$ generates N and N generates B , thus $\tilde{C}(a)$ generates B . But this means that B is a factor module of $\tilde{C}(a)$, in contrast to our assumption.

(ii') implies (i). We assume that the trace D of $\tilde{C}(a)$ in W has length 3. We claim that D is indecomposable. Otherwise, D would have a direct summand of the form S_0 or S_1 . But W has no submodule of the form S_0 , since S_0 is injective and W is special, and S_1 is not generated by $\tilde{C}(a)$. Since D is indecomposable and has length 3, Lemma 2.2 asserts that D is an atom. Assume that $\tilde{C}(a)$ has a factor module B which is a subbristle of W . Since D is an atom, $B \not\subseteq D$. But this contradicts the assumption that D is the trace of $\tilde{C}(a)$ in W . \square

3 Bristle and Bar Submodules of Special $K(3)$ -Modules

Let W be a $K(3)$ -module with $\dim W = (3, 2)$. We recall that an indecomposable submodule of W of length 2 is called a subbristle of W . A bar in W is a submodule of W with dimension vector $(2, 2)$ which is not faithful.

A detailed study of the subbristles and bars of a special $K(3)$ -module will be given in Sects. 4 and 5. Section 3 provides only some partial information which will be needed in the proof of Theorem 1.1, namely Propositions 3.7 and 3.8.

3.1 The Structure of a Bar

Let W be a special $K(3)$ -module.

(a) There are three different kinds of bars V in W :

- V is the direct sum of two isomorphic bristles, say of bristles isomorphic to B (then all 3-dimensional submodules are isomorphic to $B \oplus S_0$).
- V is the direct sum of two non-isomorphic bristles, say $V = B_1 \oplus B_2$ (then V contains no further bristle).
- V is indecomposable and contains precisely one bristle.

(b) Any bar contains a bristle.

(c) Any bar has a factor module which is a subbristle of V , thus of W .

(d) All indecomposable submodules of length 3 of a bar are isomorphic (Note that indecomposable submodules of length 3 do not exist iff V is the direct sum of two isomorphic bristles).

Proof This follows directly from the classification of the 2-Kronecker modules, as mentioned in 2.2. Namely, assume that the bar V is annihilated by $0 \neq z \in E$, thus it may be considered as a $K(3)/\langle z \rangle$ -module (thus as a 2-Kronecker module). In this way, V is just a 4-dimensional 2-Kronecker module without a simple direct summand. For the proof of (d), let U be an indecomposable submodule of V of length 3. Now any submodule of V is again a $K(3)/\langle z \rangle$ -module. Since W has no uniform submodule of length 3, we see that U has to be local. Thus U is a projective cover of S_0 considered as a $K(3)/\langle z \rangle$ -module. \square

Lemma 3.1 *Let W be a special $K(3)$ -modules and B, B' different subbristles of W , then $B \cap B' = 0$, and $B \oplus B'$ is a bar.*

Proof If $B \cap B' \neq 0$, then $B + B'$ is a submodule of W with dimension vector $(2, 1)$, thus either uniform or isomorphic to $B \oplus S_1$, impossible. Thus $B \cap B' = 0$ and therefore $B + B' = B \oplus B'$. The annihilator of a bristle is a 2-dimensional subspace of E . The annihilator of $B \oplus B'$ is the intersection of the annihilator of B and the annihilator of B' , thus non-zero. \square

Lemma 3.2 *Let W be a special $K(3)$ -module. Let N, N' be different bars in W . Then, there is a bristle B such that $N \cap N' = B + \text{soc } W$.*

Proof Both N, N' contain $\text{soc } W$. Thus, the dimension vector of $N \cap N'$ is $(1, 2)$. Assume that $N \cap N'$ is indecomposable, say isomorphic to $(k, E/\langle z \rangle)$. Since N is not faithful, and has a submodule of the form $(k, E/\langle z \rangle)$, it follows that N is annihilated by z . Similarly, N' is annihilated by z . Since $W = N + N'$, we see that W is annihilated by z . This contradicts the assumption that W is faithful.

Since $N \cap N'$ is decomposable, and W has no submodule isomorphic to S_0 , it follows that $N \cap N'$ contains an indecomposable module of length 2, a bristle. \square

Lemma 3.3 *Let W be a special $K(3)$ -module. If there are two isomorphic subbristles B, B' , then $B \cap B' = 0$ and $B \oplus B'$ is the only bar in W . The indecomposable direct summands of $B \oplus B'$ are all the subbristles of W , and all subbristles of W are isomorphic to B .*

Proof According to 3.2, we have $B \cap B' = 0$ and $V = B_1 \oplus B_2$ is a bar. Since B and B' are isomorphic, all submodules of V of length 3 are of the form $B \oplus S_1$.

Assume that there is a bristle B'' outside of V . Then $W = V + B''$ is annihilated by $\text{Ann}(B) \cap \text{Ann}(B'')$, and $\text{Ann}(B) \cap \text{Ann}(B'') \neq 0$, thus W is not faithful - a contradiction. This shows that the indecomposable direct summands of V are the subbristles of W . In particular, all subbristles of W are isomorphic to B .

Now, assume that there is a second bar V' . According to 3.3, $V \cap V' = B'' + \text{soc } W$ for some bristle B'' . Since B'' is a subbristle of W , we see that B'' is isomorphic to B . Now V' is not faithful, say annihilated by $0 \neq z \in E$. Then z annihilates B'' , thus B and B' . Since $W = V + V' = B + B' + V'$, we see that z annihilates W , impossible. \square

Lemma 3.4 *Let W be a special $K(3)$ -module. If there are (at least) three bristles B_1, B_2, B_3 , not all isomorphic, then they are pairwise non-isomorphic, they are the only bristles, the modules $B_i \oplus B_j$ with $i \neq j$ are the only bars and $W = B_1 + B_2 + B_3$.*

Proof The bristles are pairwise non-isomorphic, according to 3.4. If $B_1 \cap B_2 \neq 0$, then $B_1 + B_2$ is indecomposable, thus uniform. But this is impossible. Therefore $B_1 \cap B_2 = 0$, thus $B_1 + B_2 = B_1 \oplus B_2$. Similarly, we have $B_1 + B_3 = B_1 \oplus B_3$ and $B_2 + B_3 = B_2 \oplus B_3$. Since B_1, B_2 are non-isomorphic, B_1 and B_2 are the only subbristles of $B_1 \oplus B_2$, therefore $B_3 \not\subseteq (B_1 + B_2)$. It follows that $W = B_1 + B_2 + B_3$.

Assume that V is a bar. Then $V \cap (B_1 \oplus B_2)$ is of the form $B + \text{soc } W$ with B a bristle, say, without loss of generality $B = B_1$. Also $V \cap B_2 \oplus B_3$ is of the form $B' + \text{soc } W$ for some bristle B' , thus B' is either B_2 or B_3 . In the first case, we have $V = B_1 \oplus B_2$, in the second case, we have $V = B_1 \oplus B_3$. Thus, there are just three bars $B_1 + B_2, B_1 + B_3, B_2 + B_3$.

Finally, consider any subbristles B . If $B \neq B_1$, then $B + B_1$ is a bar, thus one of the three bars $B_1 + B_2, B_1 + B_3, B_2 + B_3$. The only subbristles of these bars are the bristles B_1, B_2, B_3 , thus B is one of them. \square

Lemma 3.5 *A special $K(3)$ -module W has at most one indecomposable bar submodule.*

Proof Let W be a special $K(3)$ -module. Assume $N \neq N'$ are indecomposable bars in W . According to 3.3, $N \cap N' = B + \text{soc } W$ for some bristle B .

Since B is a bristle, and N is an indecomposable 2-Kronecker module, the classification of the indecomposable 2-Kronecker modules shows that $N/B \simeq B$. Similarly, $N'/B \simeq B$. We claim that W/B has a direct summand isomorphic to S_0 . Namely, we have $W/B = (N + N')/B = N/B + N'/B$. Since N/B and N'/B are isomorphic, N/B is a direct summand of $N/B + N'/B$ and a direct complement has to be of the form S_0 . Thus $W/B = N/B \oplus U/B$ for some submodule U with $B \subseteq U$ and $U/B \simeq S_0$. If U is decomposable, then $U \simeq B \oplus S_0$, thus S_0 is a submodule of W . Or else U is uniform. Both cases are impossible, according to 2.4. \square

Proposition 3.6 *A special $K(3)$ -module W has at most three isomorphism classes of subbristles and at most three bars.*

Proof If W has subbristles $B \neq B'$ which are isomorphic, then we apply 3.4 in order to see that W has only one isomorphism class of subbristles and only one bar. If W has at least 3 subbristles which are pairwise non-isomorphic, then we apply 3.5 in order to see that W has only 3 subbristles (thus 3 isomorphism classes of subbristles) and only 3 bars.

Thus, we can assume that W has at most 2 subbristles. We claim that W has at most two bars. If W has two subbristles B, B' , then $B \oplus B'$ is a bar, and this is the only decomposable bar. If W has just one subbristle, then there cannot be any decomposable bar. This shows that if W has at most 2 subbristles, there is at most one decomposable bar. And according to 3.6, there is at most one indecomposable bar. \square

Remark 3.7 As we will see later, the number of isomorphism classes of subbristles of a special 3-Kronecker module is the same as the number of bar submodules, see 4.6.

Proposition 3.8 *Let M be a $K(3)$ -module with $\mathbf{dim} M = (3, 2)$. Any subbristle of M is contained in a bar submodule.*

Proof Let B be a subbristle of M , annihilated by linearly independent elements $y, z \in E$. We choose elements c, d of M such that B, c, d generate M . Let $F = \langle y, z \rangle$ and $T = \langle c, d \rangle$. We consider the multiplication map $\mu : F \otimes T \rightarrow \text{soc } M$. This is a linear map, say with kernel K of dimension at least 2 (since $F \otimes T$ has dimension 4). The set of decomposable tensors in $F \otimes T$ is a subvariety \mathcal{Y} isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, thus $\dim \mathcal{Y} = 2$. Let $\mathcal{K} = \mathbb{P}K$, it has dimension 1. Thus \mathcal{K} and \mathcal{Y} are subvarieties of $\mathbb{P}(F \otimes T)$ which intersect non-trivially, since $\dim \mathcal{K} + \dim \mathcal{Y} \geq \dim \mathbb{P}(F \otimes T)$. Thus there are elements $0 \neq e \in F$ and $0 \neq t \in T$ with $\mu(e \otimes t) = 0$. Let $V = B + At + \text{soc } W$. Then V is 4-dimensional, contains B and is annihilated by e (note that $e \in F$ annihilates not only t but also B , and of course $\text{soc } W$). This shows that V is a bar. \square

Addendum *Any special $K(3)$ -module contains a bristle.*

Remark 3.9 An immediate consequence of 3.9 and 3.8 is: Any special $K(3)$ -module contains a bar. This is the essential new information, because the known classification of the 2-Kronecker modules shows that any bar contains a bristle.

Proof of 3.9. It has been shown in [5] that any $K(3)$ -module M with $\mathbf{dim} M = (2, 3)$ has a submodule U with dimension vector $(1, 2)$, thus M has a factor module with

dimension vector $(1, 1)$. Using duality, we see that any $K(3)$ -module with dimension vector $(3, 2)$ has a submodule with dimension vector $(1, 1)$.

For the convenience of the reader, let us copy the proof of [5]. Let $M = (M_1, M_2)$ be a $K(3)$ -module with $\mathbf{dim}(2, 3)$. We first show that there are non-zero elements $m \in M_1$ and $\alpha \in E$ such that $\alpha m = 0$. Let K be the kernel of the map $f : E \otimes_k M_1 \rightarrow M_2$. We have $\dim E \otimes_k M_1 = 3 \cdot 2 = 6$. Since $\dim M_2 = 3$, it follows that $\dim K \geq 3$. The projective space $\mathbb{P}(E \otimes M_1)$ has dimension 5, the decomposable tensors in $E \otimes M_1$ form a closed subvariety \mathcal{V} of $\mathbb{P}(E \otimes M_1)$ of dimension $(3 - 1) + (2 - 1) = 3$. On the other hand, $\mathcal{K} = \mathbb{P}(K)$ is a closed subspace of $\mathbb{P}(E \otimes M_1)$ of dimension 2. It follows that

$$\dim(\mathcal{V} \cap \mathcal{K}) \geq 3 + 2 - 5 \geq 0,$$

thus $\mathcal{V} \cap \mathcal{K}$ is non-empty. As a consequence, we get non-zero elements $m \in M_1, \alpha \in E$ such that $\alpha m = 0$, as required.

Given non-zero elements $m \in M_1$ and $\alpha \in E$ such that $\alpha m = 0$, the element m generates a submodule U' which is annihilated by α , thus $\mathbf{dim} U' = (1, u)$ with $0 \leq u \leq 2$. Since $\dim M_2 \geq 3$, there is a semi-simple submodule U'' of M with dimension vector $(0, 2 - u)$ such that $U' \cap U'' = 0$. Let $U = U' \oplus U''$. This is a submodule of M with dimension vector $\mathbf{dim} U = \mathbf{dim} U' \oplus U'' = (1, 2)$. \square

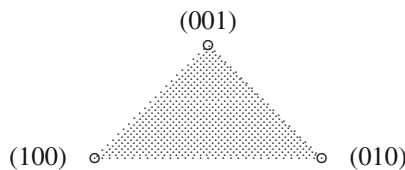
As we have mentioned already in the introduction, the number of subbristles of a special $K(3)$ -module W will be called the bristle type of W .

4 Coefficient Quivers and Bristle-Bar Layouts

We are going to describe coefficient quivers for the special $K(3)$ -modules as well as the corresponding bristle-bar layouts.

The bristle-bar layout of a special 3-Kronecker module W is given by the projective plane $\mathbb{P}(W_0)$ with marks which highlight the elements $\langle a \rangle \in \mathbb{P}W_0$ with Aa being a subbristle of W (say using squares \blacksquare) as well the top lines $t(N) = \{\langle a \rangle \in \mathbb{P}W_0 \mid Aa \subset N\}$, where N is a bar in W (say by drawing $t(N)$ as a solid line). To any bar N , we add the annihilator $\langle z_1, \dots, z_t \rangle$ of N in E (here, z_1, \dots, z_t are elements of E). (Note that this kind of visualization was used already in our joint paper [7] with Zhang Pu.)

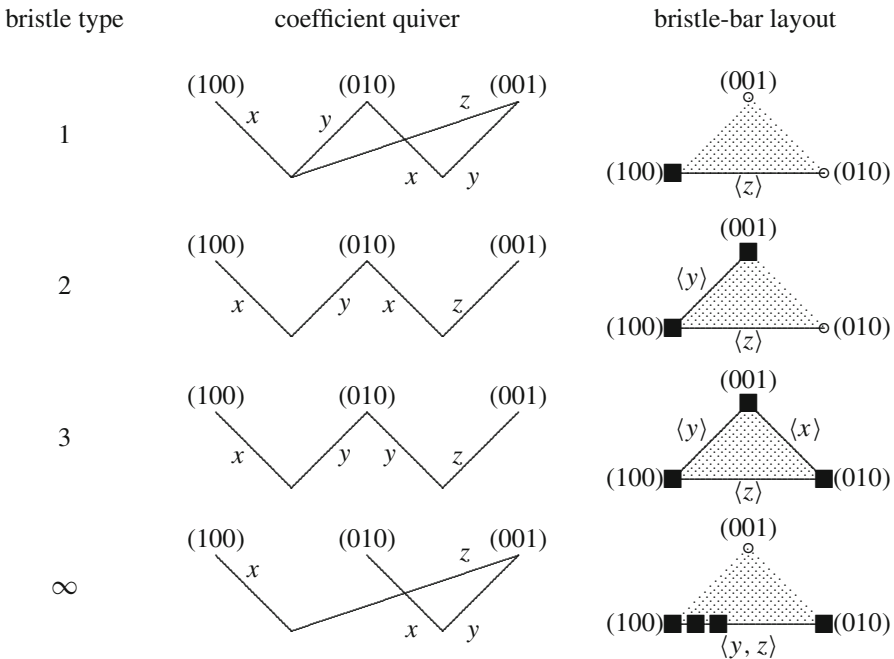
Convention: If $W_0 = k^3$, we draw the plane $\mathbb{P}W_0$ as follows: the left corner is (100) , the right corner is (010) , thus the upper corner is (001) .



4.1 The Special $K(3)$ -Modules

Recall that the bristle type of a special $K(3)$ -module W is the number of subbristles of W . Given a special $K(3)$ -module W , we are going to write down a coefficient quiver, as defined in [4].

Proposition 4.1 *The bristle type of a special $K(3)$ -module W is 1, 2, 3 or ∞ . Depending on its bristle type, W can be described by one of the following coefficient quivers. On the right, we show the corresponding bristle-bar layout.*



The proof of Proposition 4.2 will be given in Sect. 5. In the present Sect. 4, we verify that a Kronecker module with one of the four coefficient quivers presented in the table is special and has the described bristle-bar layout. We also present a bijection between the subspaces $\langle z \rangle$ of E , where $\dim zW = 1$ and the subbristles of W , see 4.5.

Theorem 4.2 *Let W be a $K(3)$ -module as exhibited in 4.2. Let $0 \neq a \in W_0$. If $\langle a \rangle$ is marked by a black square, then a generates a bristle, otherwise it generates an atom.*

Proof It is straightforward to see that the bristles displayed in the layout actually do exist. We only write down how to verify that the remaining non-zero elements $a \in W_0 = k^3$ generate atoms. We identify $W_1 = k^2$, by choosing as first basis element of W_1 the element $x(100)$, as second one the element $x(010)$ if the bristle type is 1, 2 or ∞ and $y(010)$ if the bristle type is 3.

Type 1. First, assume that $\langle a \rangle$ lies on the shown bar, but is different from $\langle 100 \rangle$, say $a = (\alpha, 1, 0)$ for some $\alpha \in k$. Then $xa = (\alpha, 1)$ and $ya = (1, 0)$, thus a generates an atom. Second, assume that $\langle a \rangle$ does not lie on the bar, say $a = (\alpha, \beta, 1)$ for some $\alpha, \beta \in k$. Then $za = (1, 0)$ and $ya = (\beta, 1)$. Again, a generates an atom.

Type 2. The elements (100) and (001) generate bristles. Since these bristles are non-isomorphic, $(\alpha, 0, \gamma)$ with $\alpha \neq 0, \gamma \neq 0$ generates an atom. Finally, let $a = (\alpha, 1, \gamma)$. Then $xa = (\alpha, 1)$ and $za = (1, 0)$ show that a generates an atom. Thus, there are precisely 2 bristles.

Type 3. The elements $(100), (010)$ generate pairwise non-isomorphic bristles, thus the elements $(1, \beta, 0)$ with $\beta \neq 0$ generate atoms. Similarly, the elements $(1, 0, \gamma)$ and $(0, 1, \gamma)$ with $\gamma \neq 0$ generate atoms. Finally, consider $a = (\alpha, 1, \gamma)$ with $\alpha \neq 0$ and $\gamma \neq 0$. Then $xa = (\alpha, 0)$ and $za = (0, \gamma)$ shows that a generates an atom.

Type ∞ . The elements on the bar generate bristles. Let a be outside the bar, thus we can assume $a = (\alpha, \beta, 1)$. Then $za = (1, 0)$ and $ya = (0, 1)$, thus a generates a bristle. □

Theorem 4.3 *Let W be a $K(3)$ -module as exhibited in 4.2. Then W is a special $K(3)$ -module.*

Proof The module W has no submodule which is simple injective, since the non-zero elements of W_0 generate bristles and atoms. The module W has no uniform submodule V of length 3. Namely, $t(V)$ would be a line consisting of pairwise non-isomorphic bristles. Lines of bristles do exist only in type ∞ , but in this case, all the bristles are isomorphic.

According to Proposition 2.1, it follows that $\text{End } W = k$, thus, W is indecomposable. As we have mentioned in 2.2, an indecomposable 3-Kronecker module with dimension vector $(3, 2)$ which is not faithful has infinitely many pairwise non-isomorphic subbristles. Thus W has to be faithful. □

Proposition 4.4 *Let W be a $K(3)$ -module with $\mathbf{dim} W = (3, 2)$.*

- (a) *If W has no simple injective submodule and $z \in E$ satisfies $\dim zW = 1$, then zW is the socle of a subbristle.*
- (b) *If W is special, then we obtain in this way a bijection between the subspaces $\langle z \rangle$ generated by the elements $z \in E$ with $\dim zW = 1$ and the subbristles of W .*

Proof (a) Let $z \in E$ with $\dim zW = 1$. Of course, $U = (0, zW)$ is a submodule of W and W/U is a 3-Kronecker module annihilated by z , thus a 2-Kronecker module for $E/\langle z \rangle$. The dimension vector of W/U is $(3, 1)$. It must have a direct summand isomorphic to S_0 . Thus there is a submodule $V \subseteq W$ with $U \subseteq V$ and V/U isomorphic to S_0 . Of course, $\mathbf{dim} V = (1, 1)$. If V would be decomposable, S_0 would be a submodule of V , thus of W . But this is impossible, since we assume that W has no simple injective submodule. Therefore U is indecomposable, thus a bristle. The socle of U is just zW .

(b) We assume now that W is special. First, let us start with a subbristle B of W . Let $U = \text{soc } B$. Then W/U has B/U as a submodule. Since W/U is isomorphic to S_0 , we see that B/U is a direct summand, thus there is $V \subseteq W$ with $U \subseteq V$ and $W/U = B/U \oplus V/U$. The module V has $\mathbf{dim} V = (2, 1)$, thus it cannot be faithful:

there is $0 \neq z \in E$ with $zV = 0$. But this means that $zW \subseteq U$. Since W is special, W is faithful, thus we must have $zW \neq 0$, therefore $zW = U$.

It remains to be seen that given a bristle B and elements z, z' with $zW = z'W = \text{soc } B$, the subspaces generated by z and by z' are equal. Assume not. Then there is an element $x \in E$ such that x, z, z' is a basis of E . Let $U = zW = z'W$. Since W_0 is 3-dimensional, there is a 2-dimensional subspace X of W_0 with $xX \subseteq U$. We claim that (X, U) is a submodule of W . Namely, since $U = zW = z'W$, we have $zX \subseteq U$ and $z'X \subseteq U$. By assumption, we also have $xX \subseteq U$. Note that $\text{dim}(X, U) = (2, 1)$. Since W is special, W has no submodule with dimension vector $(2, 1)$. \square

Corollary 4.5 *Let W be a special $K(3)$ -module. The number θ of bristles in W is the same as the number of subspaces $\langle z \rangle$ of E with $\text{dim } zW = 1$, namely 1, 2, 3 or ∞ . Recall that this number θ is called the bristle type of W . The number of bars in W is less or equal to θ (the number of subbristles of W).*

Actually, when we have finished the proof of 4.2, we will know that the number of bars in a special Kronecker module W is equal to the number of isomorphism classes of subbristle of W .

Proof of Corollary. The first assertion is just 4.5 (b). For the second assertion, one only has to observe: For any $z \in E$ with $\text{dim } zW = 1$, the submodule $(\text{Ker } z, W_0)$ is a bar in W . \square

Remark 4.6 We have seen in 4.5 that given a special 3-Kronecker module W and $z \in E$ with $\text{dim } zW = 1$, then $zW = \text{soc } B$ for some bristle B . Let us stress that there are two different possibilities, namely, we may have either $zB = 0$ or else $zB \neq 0$ (and then $zB = \text{soc } B$).

Both possibilities occur, as the special modules of bristle type 2 (exhibited in 4.2) show: There are two bristles, namely B generated by (100) and B' generated by (001) . The element $y \in E$ satisfies $yW = \text{soc } B$ and we have $yB = 0$. The element $z \in E$ satisfies $zW = \text{soc } B'$ and we have $zB' \neq 0$.

Theorem 4.7 *Let W be a special $K(3)$ -module as exhibited in 4.2. it Then all bars in W it are shown in the corresponding layout on the right.*

Proof Again, it is clear that the lines marked in the layout are bars. We have to show that there are no additional ones.

Type $\theta = 1, 2, 3$. Here we have θ bristles. According to 4.4, W is a special $K(3)$ -module. Thus, according to 4.6, there are at most t bars. Since t lines are marked as bars, we see that these are the only bars.

Type ∞ . Since the submodules generated by (100) and (010) are isomorphic bristles, Lemma 3.3 asserts that W has only one bar. \square

Remark 4.8 Let W be a 3-Kronecker module as exhibited in 3.2. We see: if the number of subbristles is finite, then the number of subbristles is the same as the number of bars, namely 1, 2, or 3 and that any bar contains a bristle. Also, if a bar contains at least 3 bristles, then it contains infinitely many bristles and there is no other bar. Finally, a line with at least two bristles is a bar.

Altogether, these assertions imply: If there are precisely three bristles, the bars form a triangle (they intersect in pairwise different vertices).

Proof the three bristles cannot lie on a single bar (since otherwise there would exist infinitely many bristles), thus they are the corners of a triangle. This yields 3 bars, so there cannot be a further bar. \square

5 Proof of Proposition 4.2

5.1 Type 1: Precisely One Bristle

Proof Let $B \subset N$, where B is the bristle and N a bar. Since there is only one bristle, N is indecomposable. Let a be a generator of B and assume that z generates the annihilator of N . According to 4.5 (a), we have $zW = \text{soc } B$. Since $zN = 0$, we have $za = 0$. Since B is a bristle and annihilated by z , there is a basis x, y, z such that also $yB = 0$. Now a is not in the kernel of x , thus $d = xa$ is a non-zero element in W_1 . Since N is an indecomposable bar with subbristle B , also N/B is isomorphic to B , thus x maps $N \cap W_0$ bijectively onto W_1 .

Since x has rank 2, its kernel has dimension 1. Let c be a non-zero element in the kernel of x . According to 4.5 (a), the image of z coincides with the socle of B . Thus, replacing, if necessary, c by a non-zero multiple, we can assume that $zc = d$. Note that $d = zc$ and $e = yc$ have to be linearly independent in W_1 , since otherwise c would generate a second bristle.

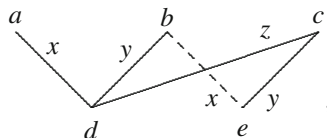
Choose $b' \in N \cap W_0$ such that $yb' = d$ (since $ya = 0$, the elements a, b' are linearly independent). Write xb' as a linear combination of d and e , say $xb' = \delta d + \epsilon e$. Note that $\epsilon \neq 0$, since otherwise b' would generate a bristle. Let $b = -\delta a + b'$. Then

$$xb = -\delta xa + xb' = -\delta d + \delta d + \epsilon e = \epsilon e$$

and

$$yb = -\delta ya + yb' = d.$$

We obtain the following coefficient quiver



where the dashed line indicates that here a non-zero scalar is involved, namely $xb = \epsilon e$. In order to remove the scalar, we replace x by $\epsilon^{-1}x$ and a by ϵa . Then we get the same picture, with the dashed line replaced by a solid one. \square

5.2 Type 2: Precisely Two Bristles

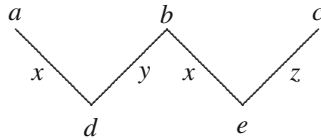
Proof Let $B \neq B'$ be the bristles of W . According to 3.4, they are not isomorphic and $B \cap B' = 0$. The annihilator of $B \oplus B'$ is $\langle y \rangle$ for some non-zero element y , therefore $\dim yW = 1$. According to 4.5, there is an element $z \in E$ such that also $\dim zW = 1$, with y, z being linearly independent. The submodule $\text{Ker } z$ is a bar. Note that $\text{Ker } y \neq \text{Ker } z$, since otherwise $B \oplus B'$ would be annihilated by y and z , and then B and B' would be isomorphic bristles. According to 3.3, the intersection $\text{Ker } z \cap \text{Ker } y$ contains a bristle. Without loss of generality, $B \subseteq \text{Ker } z \cap \text{Ker } y$ and then B' is not contained in $\text{Ker } z$.

Since B' is not contained in $\text{Ker } z$, we see that zB' is the socle of B' , therefore $zW = \text{soc } B'$. On the other hand, using again 4.5, yW is the socle of a bristle different from B' , thus $yW = \text{soc } B$.

Choose $x \in E$ such that x, y are linearly independent and annihilate B' . Since z does not annihilate B' , we see that x, y, z is a basis of E . We know that yW and zW are both one-dimensional. Since there are only 2 bristles, 4.5 asserts that we have $\dim xW = 2$, thus $xW = W_1$.

The bristle B is annihilated by y, z , the bristle B' is annihilated by x, y . Since these are the only bristles, no non-zero element in W_0 is annihilated by x and z . Thus $(\text{Ker } z) \cap W_0$ is mapped under x bijectively to W_1 .

Let c be a generator of B' and $e = zc$. Choose $b \in (\text{Ker } z) \cap W_0$ such that $xb = e$. Let $d = yb$. Then d is a non-zero element of the socle of B . Let a be a generator of B with $xa = d$. Altogether we have obtained the following coefficient quiver:

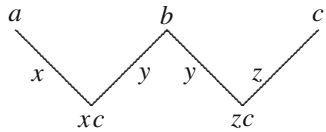


□

5.3 Type 3: At Least Three Pairwise Non-isomorphic bristles

Proof Let B_1, B_2, B_3 be pairwise non-isomorphic subbristles of the special $K(3)$ -module W . According to 3.2, we have $B_i \cap B_j = 0$ for $i \neq j$. Let x', y, z' be non-zero elements of E such that $x'(B_2 + B_3) = y(B_1 + B_3) = z'(B_1 + B_3) = 0$. Since B_1 and B_3 are non-isomorphic, $B_1 + B_3 = B_1 \oplus B_3$ contains only the bristles B_1, B_3 , thus $B_1 + B_2 + B_3 = W$. If we would have $y \in \langle x', z' \rangle$, then y would annihilate $W = B_1 + B_2 + B_3$, but we assume that W is faithful. Thus x', y, z' is a basis of E . Let $a \in B_1 \setminus \text{soc } B_1, b \in B_2 \setminus \text{soc } B_2, c \in B_3 \setminus \text{soc } B_3$. Since $B_1 + B_3 = B_1 \oplus B_3$, the elements $x'a, z'c$ form a basis of W_1 . Therefore $yb = \alpha x'a + \gamma z'c$ for some (non-zero) elements $\alpha, \gamma \in k$. Let $x = \alpha x'$ and $z = \gamma z'$, so that $yb = xa + zc$. We

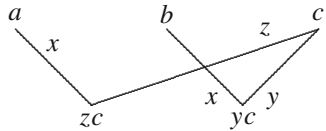
get the following coefficient quiver:



□

5.4 Type ∞ . At Least Two Isomorphic Bristles

Proof Let N be a bar which is the direct sum of two isomorphic bristles, say bristles isomorphic to B . Let $B = (k, E/\langle y, z \rangle)$, where x, y, z is a basis of E . Then $\text{Ker } x$ is not contained in N , since otherwise we get an embedding of S_0 into W . Let $c \in \text{Ker } x \setminus N$. Then c does not generate a bristle (since otherwise W would not be faithful), thus yc, zc is a basis of W_1 . There are elements $a, b \in N_0$ with $xa = zc$ and $xb = yc$. Altogether we get



□

Remark 5.1 We have seen: In case the bristle type of W is not 1, then W is a tree module. Let us add: A special $K(3)$ -module with precisely one bristle is not a tree module.

Proof There are only two possibilities for the coefficient quiver of a tree module with dimension vector $(3, 2)$, namely



In both cases, we see that there are at least 2 bristles. (Actually, only the first case can occur, since in the second case, W contains a submodule with dimension vector $(2, 1)$, and such a submodule is either uniform or has S_0 as a submodule.) □

Remark 5.2 All the coefficient quivers which occur are without parameters. This is clear for tree modules. In the remaining case (type 1), there is just one bar, and this bar contains a bristle. Thus the submodule U given by the cycle is not a bar. The fact that U is faithful implies immediately that any parameter can be deleted.

6 $L(3)$ -Modules

We want to adapt Propositions 3.7 and 3.8, as well as 2.5 to $L(3)$ -modules. As we have mentioned in the introduction, $L(3)$ is the 4-dimensional local k -algebra with radical

square zero, thus $L(3) = k\langle x, y, z \rangle / (x, y, z)^2$. The radical of $L(3)$ will be denoted by E . The dimension vector $\mathbf{dim}M$ of an $L(3)$ -module M is the pair of numbers $(\dim M/EM, \dim EM)$.

6.1 Atoms and Bristles

The notation to be used corresponds to the notation for $K(3)$ -modules introduced in Sect. 2.1; the precise correspondence will be stated in Sect. 6.3.

If $0 \neq a \in E$, then $C(a) = C(\langle a \rangle) = {}_A A / \langle a \rangle$ is an atom, any atom is obtained in this way, and $a \mapsto C(a)$ provides a bijection between $\mathbb{P}E$ and the set of isomorphism classes of atoms. If a_1, a_2 are linearly independent in E , then $B(a_1, a_2) = B(\langle a_1, a_2 \rangle) = {}_A A / \langle a_1, a_2 \rangle$ is a bristle, any bristle if obtained in this way, and $\langle a_1, a_2 \rangle$ provides a bijection between the set of lines in $\mathbb{P}E$ and the set of isomorphism classes of bristle. If $0 \neq a \in E$, and l is a line in \mathbb{E} , then $\langle a \rangle \in l$ iff $C(a)$ maps onto $B(l)$.

The relation between $K(3)$ and $L(3)$ is given by the push-down functor $\pi : \text{mod } K(3) \rightarrow \text{mod } L(3)$ which will be described in two different (of course related) ways (partly, we follow the presentation in the appendix A of [8]).

6.2 The Push-Down Functor π

This is the functor

$$\pi : \text{mod } K(3) \rightarrow \text{mod } L(3)$$

which sends the $K(3)$ -module $M = (M_0, M_1, \phi : E \otimes M_0 \rightarrow M_1)$ to the $L(3)$ -module $\pi M = (M_0 \oplus M_1, \begin{bmatrix} 0 & \phi \\ 0 & 0 \end{bmatrix})$. The decisive property of π is the following: If M, M' are 3-Kronecker modules without non-zero simple direct summands, then we have

$$\text{Hom}_{L(3)}(\pi(M), \pi(M')) = \pi \text{Hom}_{K(3)}(M, M') \oplus \text{Hom}_{L(3)}(\pi(M), \pi(M'))_{ss};$$

here, we denote by $\text{Hom}_{L(3)}(X, X')_{ss}$ the set of $L(3)$ -homomorphisms with semisimple image, for arbitrary $L(3)$ -modules X, X' .

The functor π provides a bijection between the isomorphism classes of the indecomposable $K(3)$ -modules M different from S_1 and the isomorphism classes of the indecomposable $L(3)$ -modules, such that $\mathbf{dim}M = \mathbf{dim}\pi M$. (Of course, also πS_1 is defined; but $\mathbf{dim}S_1 = (0, 1)$, whereas $\mathbf{dim}\pi S_1 = (1, 0)$.)

The $L(3)$ -module ${}_{L(3)}L(3)$ is just $\pi(k, E)$. Of course, the radical of $\pi(k, E)$ is $0 \oplus E = E$. Let us stress that in this way, the arrow space E of the quiver $K(3)$ is identified with the radical of $L(3)$.

There is an alternative way to define π . The $K(3)$ -modules may be seen as the R -modules, where R is the path algebra of the quiver $K(3)$, thus $R = \begin{bmatrix} k & 0 \\ E & k \end{bmatrix}$ is the ring of all matrices of the form $\begin{bmatrix} \lambda & 0 \\ e & \lambda' \end{bmatrix}$, with $\lambda, \lambda' \in k$ and $e \in E$. Here, the $K(3)$ -module $(M_0, M_1; \phi_M)$ corresponds to the set $\begin{bmatrix} M_0 \\ M_1 \end{bmatrix}$ of column matrices, and

the module multiplication is just matrix multiplication using the map ϕ_M . By abuse of notation, we denote the ring R also by $K(3)$ (this explains our convention to call the $K(3)$ -modules just $K(3)$ -modules). We may consider $L(3)$ as the subring of $K(3)$ given by the matrices of the form $\begin{bmatrix} \lambda & 0 \\ e & \lambda \end{bmatrix}$ with $\lambda \in k$, and $e \in E$. Then, the functor π is the restriction functor, which sends the $K(3)$ -module M to the $L(3)$ -module often denoted by $M|_{L(3)}$ (the map $K(3) \rightarrow \text{End}_k M$ given by the module multiplication is restricted to $L(3)$). In particular, in this interpretation, the underlying vector spaces of M and πM are the same. This interpretation is particularly helpful when we compare the submodule structure of M and of πM , see 6.3 and 6.4.

6.3 Socle Properties

Theorem 6.1 *Let M be a $K(3)$ -module. Always, we have $|\text{soc } M| = |\text{soc } \pi M|$. If M is indecomposable and not simple, then $\text{soc } M = M_1 = J(\pi M) = \text{soc } \pi M$.*

Proof It is enough to look at indecomposable $K(3)$ -modules. If M is simple, then also πM is simple. Now assume that $M = (M_0, M_1, \phi_M)$ is indecomposable and not simple. Since M is not isomorphic to S_0 , we have $\text{soc } M = M_1$. On the other hand, πM is an indecomposable $L(3)$ -module of Loewy length 2, therefore $\text{soc } \pi M = J(\pi M)$ and $J(\pi M)$ is the image of ϕ_M , thus $J(\pi M) = M_1$. □

Corollary 6.2 *The $K(3)$ -module M is uniform iff the $L(3)$ -module πM is uniform.*

Atoms *The functor π provides a bijection between the isomorphism classes of the atom $K(3)$ -modules and the isomorphism classes of the atom $L(3)$ -modules with*

$$\pi \tilde{C}(a) = C(a),$$

where $0 \neq a \in E$.

Bristles *The functor π provides a bijection between the isomorphism classes of the bristle $K(3)$ -modules and the isomorphism classes of the bristle $L(3)$ -modules with*

$$\pi \tilde{B}(a_1, a_2) = B(a_1, a_2),$$

where a_1, a_2 are linearly independent in E .

6.4 Submodules

Given a $K(3)$ -module M , its $K(3)$ -submodules are, of course, also $L(3)$ -submodules, but usually, there are $L(3)$ -submodules of πM which are not $K(3)$ -submodules. There is the following general observation:

Lemma 6.3 *Let M be a $K(3)$ -module. Then any $L(3)$ -submodule N of πM with $M_1 \subseteq N$ is a $K(3)$ -submodule of M .*

Proof We note that $K(3)$ is generated as a vector space by $L(3)$ and the matrix $p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and we have $pM = M_1$. Thus any $L(3)$ -submodule N of M with $M_1 \subseteq N$ is a $K(3)$ -submodule. □

6.5 Special $K(3)$ -Modules, Special $L(3)$ -Modules

An $L(3)$ -module M will be said to be *special* provided $\dim M = (3, 2)$, M is faithful, and has no simple direct summand and no uniform submodule of length 3.

Lemma 6.3 *Let W be a $K(3)$ -module with $\dim W = (3, 2)$. Then W is a special $K(3)$ -module iff πW is a special $L(3)$ -module.*

Proof According to condition (ii) in 2.3, W is a special $K(3)$ -module iff W is faithful, indecomposable and has no uniform submodule of length 3.

Let M be an arbitrary $K(3)$ -module. As we have mentioned, M is indecomposable iff πM is indecomposable. Also, $M = (M_0, M_1)$ is faithful iff there is no non-zero element $e \in E$ with $eM_0 = \phi_M(e \otimes M) = 0$, iff there is no non-zero element $e \in E$ with $e(\pi M) = 0$ iff πM is faithful. Also, if U is a uniform submodule of M , then U is an $L(3)$ -submodule of πM , and U is uniform also as an $L(3)$ -module.

Conversely, assume that V is a uniform $L(3)$ -submodule of πM . Let $V = \pi U$. The embedding $v : V \rightarrow \pi M$ is of the form $v = \pi u + v'$, where $u : U \rightarrow M$ is $K(3)$ -linear and $v' : V \rightarrow \pi M$ is an $L(3)$ -homomorphism with semisimple image. Then $v' : V \rightarrow \pi M$ vanishes on $\text{soc } V$. Thus $v = \pi u + v'$ shows that u does not vanish on $\text{soc } U$. Since U is uniform, u is a monomorphism, thus M has a uniform submodule of length 3. □

Following [8], an $L(3)$ -module M is said to be *solid* provided $M = \pi W$ for some $K(3)$ -module W with $\text{End } W = k$.

Corollary 6.4 *An $L(3)$ -module M is special iff M has dimension vector $(3, 2)$, is faithful and solid.*

We are interested in bristle and bar submodules of these special $L(3)$ -modules.

6.6 Bristle Submodules

Let M be a $K(3)$ -module. The bristle $K(3)$ -submodules are bristle $L(3)$ -submodules of πM and for any bristle $L(3)$ -submodule U' of πM , there is a bristle $K(3)$ -submodule U of M such that $\text{soc } U' = \text{soc } U$. All bristle $L(3)$ -submodules U' and U'' of πM with $\text{soc } U' = \text{soc } U''$ are isomorphic.

The following formulation provides more details: Let $R = K(3)$ and $A = L(3) \subset K(3)$. Let $B = Rb$ be a subbristle of ${}_R M$, then B is a subbristle of πM with $B \cap M_0 \neq 0$. For any element $x \in M_1$, $A(b + x)$ is a subbristle of πM , and the R -submodule generated by $A(b + x)$ is $Rb + Rx$ (thus a bristle iff $Rx \subseteq Rb$). We have $A(b + x) = A(b + x')$ iff $x - x' \in B \cap M_1$; also, $A(b + x) \cap M_0 \neq 0$ iff $x \in B \cap M_1$. Conversely, any subbristle of πM is of the form $A(b + x)$, where Rb is an R -bristle and $x \in M_1$.

Bristle modules are defined for any algebra, bar modules not. Similar to the case of a $K(3)$ -module with dimension vector $(3, 2)$, we consider here an $L(3)$ -module M with dimension vector $(3, 2)$ and call a submodule N of M a bar, provided $\dim N = (2, 2)$ and N is not faithful.

6.7 Bar Submodules of Special Modules

Theorem 6.5 *Let W be a special $K(3)$ -module. Any bar $L(3)$ -submodule of πW is actually a $K(3)$ -submodule.*

Proof This follows directly from 6.4. □

Proposition 6.6 *A special $L(3)$ -module has at most 3 isomorphism classes of subbristles and at most 3 bar submodules. Any subbristle of a special $L(3)$ -module is contained in a bar submodule.*

Remark 6.7 One has to be aware that there is a decisive difference between special $K(3)$ -modules and special $L(3)$ -modules: Whereas a special $K(3)$ -module may have only finitely many subbristles (and then 1, 2, or 3), a special $L(3)$ -modules always has infinitely many subbristles, see 3.9 and 6.6.

Proof of Proposition 6.6 Let W be a special $K(3)$ -module. The bar $K(3)$ -submodules are the bar $L(3)$ -submodules of πW . The bristle $K(3)$ -submodules are bristle $L(3)$ -submodules of πW and for any bristle $L(3)$ -submodule U of πW , there is a bristle $K(3)$ -submodule U' of W such that U and U' are isomorphic $K(3)$ -modules and $\text{soc } U = \text{soc } U'$. Altogether we see that 6.8 is a direct consequence of 3.7 and 3.8. □

If M is a special $L(3)$ module, the number of simple modules which occur as the socle of a subbristle of M will be called the *bristle type* of M . If M is a special $K(3)$ -module, then the bristle type of M is equal to the bristle type of πM .

Finally, we show that also the analog of 2.5 holds true. Note that the assertion (ii') has to be modified, see 6.11.

Proposition 6.8 *Let M be a special $L(3)$ -module. Let $0 \neq a \in E$. The following conditions are equivalent:*

- (i) *The atom $C(a)$ has no factor module which is a subbristle of M .*
- (ii) *The trace of $C(a)$ in M is an atom.*
- (ii') *The trace of $C(a)$ in M is indecomposable and has length 3.*

Proof Let $M = \pi W$, where W is a special $K(3)$ -module.

(i) \implies (ii). We assume that $C(a)$ has no factor module which is a subbristle of M . If $\phi : \tilde{C}(a) \rightarrow W$ would be a $K(3)$ -homomorphism whose image is a bristle, then $\pi(\phi)$ would be an $L(3)$ -homomorphism whose image is a bristle. Thus, we see that $\tilde{C}(a)$ has no factor module which is a subbristle of W , this is condition (i) in 2.5. It follows that the trace D of $\tilde{C}(a)$ in W is an atom. Of course, $\pi(D)$ is a submodule of M which is an atom. First of all, this means that there is a homomorphism $\psi : \tilde{C}(a) \rightarrow W$ with image D . As a consequence, $\pi(\psi) : C(a) \rightarrow M$ has image in πD . Second, any homomorphism $\tilde{C}(a) \rightarrow W$ maps into D . Let $f : C(a) \rightarrow M$ be a homomorphism. According to 6.2, we can write $f = \pi(\psi') + f'$, where $\psi' : \tilde{C}(a) \rightarrow W$ and where $f' : C(a) \rightarrow M$ has semisimple image. As we have mentioned, ψ' maps into $\pi(D)$. Since $\pi(D)$ is an atom, its socle has length 2, and therefore coincides with the socle of M . Since the image of f' is semisimple, this image is contained in $\text{soc } M = \text{soc } \pi D$, thus in πD . This shows that πD is the trace of $C(a)$ in M .

(ii') \implies (i). We assume that the trace N of $C(a)$ in M has length 3 and is indecomposable. Since M has no uniform submodule of length 3, it follows that N is an atom. There is $f : C(a) \rightarrow M$ whose image is N . According to 6.2, we write $f = \pi(\phi) + f'$, with a $K(3)$ -homomorphism $\phi : \tilde{C}(a) \rightarrow W$ and an $L(3)$ -homomorphism $f' : C(a) \rightarrow M$ with semisimple image. The image of f' lies in $\text{soc } M$. Since the image of f does not lie in $\text{soc } M$, the image of ϕ cannot lie in $\text{soc } W$. Therefore the image of ϕ has length 2 or 3. Now, the image of ϕ cannot be a bristle, since otherwise also the image of $\pi(\phi)$ would be a bristle. This shows that the image of ϕ has length 3. Let D be the trace of $\tilde{C}(a)$ in W . Then $\text{Im}(\phi) \subseteq D$, thus $|D| \geq 3$. On the other hand, the trace D of $\tilde{C}(a)$ in W is mapped under π into the trace N of $C(a)$ in M . But $\pi(D) \subseteq N$ implies that $|D| \leq |N| = 3$. This shows that the trace D of $\tilde{C}(a)$ in W has length 3. According to 2.5, it follows that $\tilde{C}(a)$ has no factor module which is a subbristle of W .

Assume that $C(a)$ has a factor module B which is a subbristle of M . Since there is an $L(3)$ -epimorphism $C(a) \rightarrow B$, there is also a $K(3)$ -epimorphism $\tilde{C}(a) \rightarrow \tilde{B}$, where $\pi(\tilde{B}) = B$. Since B is a subbristle of M , \tilde{B} is a subbristle of W , thus $\tilde{C}(a)$ has a factor module which is a subbristle of W , a contradiction. \square

Remark 6.9 If M is a special $L(3)$ -module and $0 \neq a \in E$, the trace of $C(a)$ in M has length 3 or 4. In both cases, the trace of $C(a)$ in M may be decomposable.

Proof Let W be a special $L(3)$ -module and $M = \pi W$. Since $L(3)$ is a local algebra and $C(a)$ is non-zero, the socle $\text{soc } M$ of M is contained in the trace D of $C(a)$ in M . Also, there is a non-zero homomorphism $\phi : \tilde{C}(a) \rightarrow W$. Since the image of $\pi(\phi) : C(a) \rightarrow M$ is not contained in $\text{soc } M$, we see that $\text{soc } M$ is a proper submodule of D . Thus, $|D| \geq 3$. On the other hand, D is not faithful, thus D is a proper submodule of M , therefore $|D| \leq 4$.

Here is an example where the trace has length 3 and is decomposable: Let W be the special $K(3)$ -module of bristle type 1, as exhibited in Proposition 4.2. The module $\tilde{C}(y)$ is the local module annihilated by y , it has as factor module the bristle $B = B(y, z)$ annihilated by y and z . Now B is the submodule of W generated by (100) , and this is the trace of $\tilde{C}(y)$ in W . The trace D of $C(y)$ in $M = \pi W$ is $\pi B + \text{soc } M$. Thus D has length 3 and is the direct sum of πB and a simple module.

Here is an example where the trace has length 4 and is decomposable: Let W be the special $K(3)$ -module of bristle type 2, as exhibited in Proposition 4.2. The trace of the module $\tilde{C}(y)$ in W is the direct sum of two bristles, thus also the trace of $C(y)$ in $M = \pi W$ is the direct sum of two bristles. \square

7 The Special Algebras

Lemma 7.1 Let A be a short local algebra with radical J and assume that $J^2 \neq 0$. The following conditions are equivalent.

- (i) ${}_A J$ is faithful as an A/J^2 -module.
- (i') If $z \in J$ with $zJ = 0$, then $z \in J^2$.
- (i'') $\text{soc } J_A \subseteq J^2$.

- (ii) $J^2 = \text{soc } J_A$.
- (iii) J_A has no simple direct summand.

Proof The equivalence of (i) and (i') is just the definition of faithfulness. (i') implies (i''): The socle of a right module M_A are just the elements $m \in M$ with $mJ = 0$.

(i'') implies (ii): Since A is short, we have $J^2 \subseteq \text{soc } J_A$.

(ii) implies (iii): Since $J^2 \neq 0$, we know that J_A is a right module of Loewy length 2. A right module M_A of Loewy length 2 has no simple direct summand iff $MJ = \text{soc } M_A$.

(iii) implies (i'). Assume there is $z \in J \setminus J^2$ with $zJ = 0$. Then $\langle z \rangle$ is a simple direct summand of J_A . □

Lemma 7.2 *Assume that A is a short local algebra of Hilbert type $(3, 2)$ with $J^2 = \text{soc } {}_A A = \text{soc } A_A$. Then the following conditions are equivalent:*

- (i) ${}_A A$ has no uniform submodule of length 3.
- (ii) A has no serial module of length 3.
- (i*) A_A has no uniform submodule of length 3.
- (ii*) A^{op} has no serial module of length 3.

Proof (i) \implies (ii): Assume that there exists a serial module M of length 3. Since M is local, its projective cover is ${}_A A$ and ΩM is a left ideal of length 3. The socle of ΩM is contained in $\text{soc } {}_A A$, thus of length 1 or 2. If the socle of ΩM would be of length 2, then $\text{soc } M = \text{soc } {}_A A = J^2$, and then $M = AA/\Omega M$ has Loewy length at most 2, a contradiction.

(ii) \implies (i): Assume that ${}_A A$ has a uniform submodule U of length 3. Then $J^2 = \text{soc } {}_A A$ is not contained in U , therefore ${}_A A/U$ is a module of length 3 which is of Loewy length 3. Of course, a module of length 3 and Loewy length 3 is serial.

The equivalence of (ii) and (ii*) is given by k -duality. □

Proposition 7.3 *Let A be a short local algebra of Hilbert type $(3, 2)$. The following conditions are equivalent.*

- (i) $J^2 = \text{soc } {}_A A = \text{soc } A_A$ and there is no uniform left ideal with length 3.
- (i') $J^2 = \text{soc } {}_A A = \text{soc } A_A$ and there is no uniform right ideal with length 3.
- (ii) ${}_A J$ is a special $L(3)$ -module.
- (ii') J_A is a special $L(3)$ -module.
- (iii) ${}_A J$ is a solid A -module, J_A is a solid A^{op} -module.

Recall that we call an algebra special provided it satisfies the equivalent conditions mentioned in Proposition 7.3. It is shown in [8] that for a short local algebra with Hilbert type $(3, 2)$ the existence of a non-projective reflexive module implies that both ${}_A J$ and J_A are solid, thus that A is special. This is the implication (i) \implies (iii) in the main theorem of the present paper.

The importance of condition (ii) lies in the fact that this condition refers only to properties of A as a left A -module.

Proof of Proposition 7.3 The equivalence of (i) and (i') is shown in 7.2.

The equivalence of (i) and (ii) is seen as follows: To say that ${}_A J$ is special means that ${}_A J$ is faithful as an A/J^2 -module, that ${}_A J$ has no simple direct summand and no uniform submodule of length 3. According to 7.1, the condition $J^2 = \text{soc } J_A$ is equivalent to the condition that ${}_A J$ is faithful as an A/J^2 -module. Second, $J^2 = \text{soc } {}_A J$ is equivalent to the condition that ${}_A J$ has no simple direct summand.

Since the conditions (i) and (i') are equivalent, also (ii) and (ii') are equivalent. According to Corollary 6.5, (ii) implies that ${}_A J$ is solid. Since (ii) and (ii') are equivalent, we see that these conditions imply that also J_A is solid. Thus (ii) implies (iii).

Finally, let us show that (iii) implies (ii). We assume that both ${}_A J$ and J_A are solid. Since J_A is solid, it has no simple direct summand, thus, according to 7.1, ${}_A J$ is faithful as an A/J^2 -module (that means, as an $L(3)$ -module). Since ${}_A J$ is also solid, Corollary 6.5 asserts that ${}_A J$ is special. □

Let us mention an important property of these special algebras.

Lemma 7.4 *Let A be a special algebra. Then any left or right ideal of A of length 3 or 4 is a two-sided ideal.*

Proof Since the opposite of a special algebra is a special algebra, we only have to look at left ideals.

Since A is short, $J^2 \subseteq \text{soc } {}_A J \cap \text{soc } J_A$. Since ${}_A J$ is solid, it has no simple direct summand, thus $\text{soc } {}_A J \subseteq J^2$. Since ${}_A J$ is faithful as an A/J^2 -module, 7.1 asserts that J_A has no simple direct summand, thus also $\text{soc } J_A \subseteq J^2$. Altogether, we see that $J^2 = \text{soc } {}_A A = \text{soc } A_A$.

Since A is special, we know that ${}_A J$ has no uniform submodule of length 3. It follows that any left ideal of A of length at least 3 contains $\text{soc } {}_A J = J^2$. Thus, let U be a left ideal of A included in J . We want to see that $Ua \subseteq U$ for any $a \in A$. Now $a = \lambda + a'$ for some $\lambda \in k$ and $a' \in J$. Of course, $U\lambda \subseteq U$. And $Ua' \subseteq J^2 \subseteq U$. □

Since for a special algebra A the radical J considered as a left module is a special $L(3)$ -module, and any proper left ideal of A is contained in J , we can apply Proposition 6.6 and obtain:

Proposition 7.5 *A special algebras has at most 3 isomorphism classes of bristle modules which occur as left ideals and at most 3 left ideals which are bars. Any bristle left ideal is contained in a left ideal which is a bar.* □

If A is a special algebra with radical J such that ${}_A J = \pi W$, then we call the bristle type of W the left bristle type of A ; it is the number of simple left ideals of A which occur as socle of a bristle left ideal. Similarly, the right bristle type of A is the number of simple right ideals of A which occur as socle of a bristle right ideal.

8 The Correspondence Between Bristle Left Ideals and Right Bars

Lemma 8.1 *Let A be a special algebra. If B is a left module which is a bristle, let $l(B) = \{a \in A \mid aB = 0\}$, this is the left annihilator of B . Then $\Omega({}_A B) = l(B)$, and this is a 4-dimensional ideal of A and $B = A/l(B)$. In this way, we obtain a bijection between the isomorphism classes of bristles and the 4-dimensional ideals of A .*

Proof Let M be a local module and $f : {}_A A \rightarrow M$ be a projective cover, thus $\Omega({}_A M) = \text{Ker } f$. Of course, $l(M) = \{a \in A \mid aM = 0\} \subseteq \text{Ker } f = \Omega({}_A M)$, since for $a \in l(M)$ we have $f(a) = f(a \cdot 1) = af(1) = 0$. Actually, $l(M)$ is the largest two-sided ideal of A which is contained in $\text{Ker } f$ (namely, if I is a two-sided ideal of A , then $I(A/I) = 0$).

Now, if $M = B$ is a left bristle, then $\Omega({}_A B)$ is 4-dimensional and any 4-dimensional left ideal of A is a two-sided ideal, see 7.4. This shows that $\Omega({}_A B) = l(B)$. It follows that $B \simeq {}_A A/l(B)$. Thus, if B, B' are bristles with $l(B) = l(B')$, then B and B' are isomorphic.

Of course, conversely, isomorphic modules have the same annihilator. Thus, if B, B' are isomorphic left modules, then $l(B) = l(B')$. □

Lemma 8.2 *Let A be a special algebra. Under this bijection the isomorphism classes of the bristles which occur as left ideals correspond bijectively to the right bars.*

Proof Let $B = Aa$ be a left bristle. Then $\Omega({}_A B) = l(B)$ is annihilated from the right by a , thus not faithful as right A/J^2 -module. Conversely, assume that N_A is a bar of J_A , say annihilated by $a \in J \setminus J^2$ (from the right). Consider the map ${}_A A \rightarrow {}_A A$ given by right multiplication with a ; its image is Aa , its kernel is $\Omega({}_A Aa)$. Since $a \in J \setminus J^2$, Aa has length at least 2. Since $Na = 0$, we see that N is contained in the kernel. But N is of dimension 4, thus Aa has length 2, and $N = \Omega({}_A Aa)$.

Remark 8.3 The bijection is between isomorphism classes of bristles and individual ideals. In particular, this has to be noticed in case we consider left ideals which are bristles and right bars, thus on the one hand we deal with left ideals of dimension 2 (better: with the corresponding isomorphism classes) and on the other hand with ideals of dimension 4.

Applying Lemma 8.1 to A^{op} , we similarly get a bijection between the isomorphism classes of right bristles and the 4-dimensional ideals of A , where a right bristle B is sent to $r(B) = \{s \in A \mid Ba = 0\}$, whereas a 4-dimensional ideal I of A is sent to $B = A/I$, considered as a right B -module.

But note, since $L(3)$ is commutative, any left bristle B may be considered also as a right bristle and conversely; also, the left annihilator $l(B)$ coincides with the right annihilator $r(B)$ and one obtains in this way just the 4-dimensional ideals.

Lemma 8.4 *Let A be a special algebra. Under the bijection between the bristle right modules and the 4-dimensional ideals of A the isomorphism classes of the bristles which occur as right ideals correspond bijectively to the left bars.* □

9 Proof of Theorem 1.1

We assume that A is a special algebra. We fix a complement E of J^2 in J , thus E is a subspace of J with $J = J^2 \oplus E$. Of course, we may identify E with J/J^2 (thus with the radical of $L(3) = A/J^2$).

9.1 The Projective Plane $\mathbb{P}E$

The projective plane $\mathbb{P}E$ as index set for the atoms. If $a \in J \setminus J^2$, let $U(a) = Aa + J^2$, thus we have $C(a) = {}_A A/U(a)$. Similarly, the set of lines in $\mathbb{P}E$ is index set for the bristles. If $\langle a \rangle$ is an element of $\mathbb{P}E$, and l is a line in $\mathbb{P}E$, then $\langle a \rangle \in l$ iff $B(l)$ is a factor module of $C(a)$.

As we have seen in Lemma 7.4, any left or right ideal of A of length 3 or 4 is a two-sided ideal. Thus, the left bristles may be considered also as right bristles, the left atoms as right atoms.

9.2 The Lines $g(B)$ and $t(N)$

If N is a 4-dimensional ideal of A , let $t(N) = \{\langle a \rangle \mid a \in N \cap E\}$. Clearly, $t(N)$ is a line in $\mathbb{P}E$ (it is the line joining a and a' , where a, a' are elements of E such that $N = Aa + Aa'$).

If B is a bristle, we write $g(B)$ for the set of all elements $\langle a \rangle \in \mathbb{P}E$, where $a \in E$, such that B is a factor module of $C(a)$. Of course, $g(B)$ is a line in $\mathbb{P}E$, and all lines in $\mathbb{P}E$ are obtained in this way, see 6.1.

Lemma 9.1 *Lemma 9.1 Let A be a special algebra. If B is a bristle, then*

$$g(B) = t(\Omega_A B).$$

Proof First, consider $a \in E$ with $\langle a \rangle \in g(B)$. Thus there is an epimorphism $\phi : C(a) \rightarrow B$. Since $C(a)$ is local, the projective cover of $C(a)$ is ${}_A A$ and there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & Aa + \text{soc } A & \longrightarrow & {}_A A & \longrightarrow & C(a) \longrightarrow 0 \\ & & \mu \downarrow & & \parallel & & \downarrow \phi \\ 0 & \longrightarrow & \Omega_A B & \longrightarrow & {}_A A & \longrightarrow & B \longrightarrow 0 \end{array}$$

with an inclusion map μ . The inclusion map μ shows that $a \in \Omega_A B \cap E$ is a non-zero element and thus $\langle a \rangle \in t(\Omega_A B)$.

Conversely, if $0 \neq a \in \Omega_A B \cap E$ (so that $\langle a \rangle \in t(\Omega_A B)$), then $Aa + \text{soc } A \subseteq \Omega_A B$, and we get the left part of the commutative diagram seen above (with μ the inclusion map). Completing it on the right, we obtain the required epimorphism $C(a) \rightarrow B$. \square

Proposition 9.2 *Let A be a special algebra. Let $0 \neq a \in E$. The following conditions are equivalent:*

- (i) *The element $\langle a \rangle$ in $\mathbb{P}E$ does not belong to a left bar, and aA is not a right bristle.*
- (ii) *The atom $C(a)$ is extensionless (and $\Omega C(a)$ is an atom).*

Remark 9.3 The bracket in (ii) adds a general fact: If M is an indecomposable non-projective module which is extensionless, then ΩM is indecomposable, again, see [8]. Of course, $C(a)$ is indecomposable and non-projective. Thus, if $C(a)$ is extensionless,

then $\Omega C(a)$ is indecomposable. Since $C(a) = {}_A A/U(a)$, with $U(a) = Aa + J^2$, we have $\Omega C(a) = U(a)$. If $U(a)$ is indecomposable, then it is an atom.

Proof of 9.2 (i) implies (ii). First, let us assume that a does not belong to a left bar. Then Aa is not a bristle (see 7.5), thus an atom. Let D be the trace of $U(a)$ in ${}_A J$, thus $U(a) \subseteq D$. Since D is generated by Aa and Aa is not faithful as an $L(3)$ -module, also D is not faithful. Since ${}_A J$ is faithful as an $L(3)$ -module, D is a proper submodule of ${}_A J$. If D has length 4, then D is a left bar. But a is not contained in a left bar, D has length 3 and therefore $D = U(a)$. Since aA is not a bristle, it follows that $aA = U(a)$. As a consequence, the embedding $Aa = U(a) \rightarrow {}_A J$ is a left ${}_A A$ -approximation, of course a minimal one. This shows that $0 \rightarrow Aa \rightarrow {}_A A \rightarrow C(a) \rightarrow 0$ is an \mathcal{U} -sequence, thus $C(a)$ is extensionless.

(ii) implies (i). Assume that $C(a)$ is extensionless. Then $\Omega C(a) = U(a)$ is indecomposable, thus equal to Aa , and $0 \rightarrow Aa \rightarrow {}_A A \rightarrow C(a) \rightarrow 0$ is an \mathcal{U} -sequence.

Assume that a belongs to a left bar N . Since Aa is a local module of length 3, and N is not faithful as an $L(3)$ -module, Aa generates the module N , thus there is a homomorphism $\phi : Aa \rightarrow N$ with image not contained in Aa . Since the inclusion map $\mu : Aa \rightarrow {}_A A$ is a left ${}_A A$ -approximation, there is $\phi' : {}_A A \rightarrow {}_A A$ with $\phi = \phi'\mu$. Now ϕ' is the right multiplication by some element $b \in A$, therefore $\phi(a) = \phi'\mu(a) = ab \in aA$. Thus $\phi(Aa) = A\phi(a) \subseteq AaA = U(a) = Aa$. This contradiction shows that a does not belong to a left bar.

Finally, we show that $U(a) = aA$. Let $x \in \text{soc } U(a)$. The map $\phi : Aa \rightarrow {}_A A$ with $\phi(a) = x$ factors through the inclusion map $\mu : Aa \rightarrow {}_A A$, thus there is $\phi' : {}_A A \rightarrow {}_A A$ with $\phi = \phi'\mu$. Again, ϕ' is the right multiplication by some element $b \in A$, therefore $x = \phi(a) = \phi'\mu(a) = ab \in aA$. This shows that $\text{soc } U(a) \subseteq aA$, therefore $aA = U(a)$. In particular, aA is not a bristle. □

Proposition 9.4 *Let A be a special algebra. Let $0 \neq a \in E$. The following conditions are equivalent:*

- (i) *The element $\langle a \rangle$ in $\mathbb{P}E$ does not belong to a right bar.*
- (i') *The atom $C(a)$ has no factor module which is a left subbristle.*
- (ii) *The trace of $C(a)$ in ${}_A A$ is an atom.*

Let us add: If the trace of $C(a)$ in ${}_A A$ is an atom, then $C(a)$ is torsionless. The converse is of course not true.

Proof of Proposition. The equivalence of (i) and (i') follows from Lemma 9.2. which asserts that $g(B) = t(\Omega B)$. For the equivalence of (i') and (ii), see 6.10 with $M = {}_A J$. □

Corollary 9.5 *If a does not belong to a left bar nor to a right bar, then $C(a)$ is both extensionless and torsionless. Thus, $\Omega C(a)$ is a reflexive atom.*

Proof Assume that a does not belong to a left bar nor to a right bar. Then aA is not a right bristle, since otherwise a belongs to a right bar, see 7.5. According to Proposition 9.2, $C(a)$ is extensionless and $\Omega C(a) = U(a)$ is an atom. According to Proposition 9.4, $C(a)$ is torsionless. Also, $\mathcal{U}U(a) = \mathcal{U}\Omega C(a)$, thus $\mathcal{U}U(a) = C(a)$. This shows that $\mathcal{U}U(a)$ is torsionless. Since both $U(a)$ and $\mathcal{U}U(a)$ are torsionless, $U(a)$ is reflexive. □

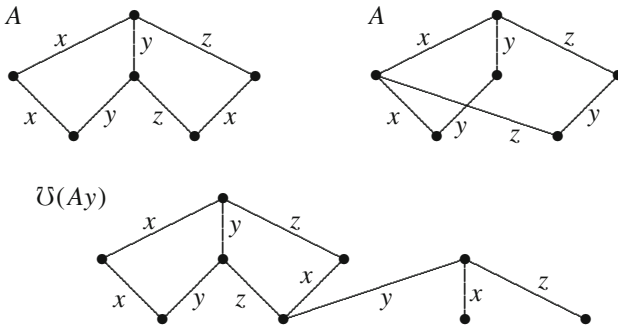
9.6 Proof of Theorem 1.1

We take an element $0 \neq a \in E$ which does not belong to a left bar nor to a right bar. Such elements do exist, since there are only finitely many left bars and finitely many right bars, and k is algebraically closed.

Since $\langle a \rangle$ does not belong to a right bar, aA is not a right bristle. According to 9.5, $\Omega C(a)$ is a reflexive atom. □

Remark 9.6 Let A be a short local algebra with radical J . Let U be an atom submodule of J . Even if the trace of U in ${}_A J$ is an atom, the embedding $U \rightarrow J$ does not have to be a left ${}_A A$ -approximation.

Example 9.7 Let A be the algebra shown in the upper row on the left (with A^{op} on the right). Let $U = Ay$. In the lower row, we depict $\bar{U}\Omega C(y) = \bar{U}(Ay)$.



Note that for all pairs $(\beta, \gamma) \neq 0$, the right ideal $(\beta y + \gamma z)A$ is a bristle, thus $C(\beta y + \gamma z)$ is not extensionless (The atom $C(\alpha x + \beta y + \gamma z)$ is extensionless iff $\alpha \neq 0$ and $\gamma \neq 0$) The picture shows the case $\alpha = 0, \beta = 1, \gamma = 0$.

As we have seen in Proposition 9.2 and Remark 9.6 the fact that $C(a) = A/Aa$ is not extensionless depends not only on the left module structure of the radical J , but also on the right module structure of J .

Given a special algebra A with radical $J = \pi W$, where W is a (special) $K(3)$ -module, and U an atom in W , there is a short local algebra A' with the “same” radical πW (where “same” refers to the left-module structure) such that πU , considered as an A' -module, is Gorenstein-projective. Namely:

Proposition 9.8 Let $\dim W = (3, 2)$. Let U be a submodule of W with $\dim U = (1, 2)$ such that also the trace of U in W is an atom.

Assume that the annihilator of U in E is generated by $x \in E$. There is a short local algebras A with an element $x \in J$ such that ${}_A J = W$ and $U = Ax$.

If A is a short local algebra with an element $x \in J$ such that ${}_A J = W$ and $U = Ax$, then x is a (left and right) Conca element of A (thus, U is a Gorenstein projective A -module with $\Omega U \simeq U$).

Recall that if A is a short local algebra, then $x \in A$ is said to be a (left and right) Conca element provided $x^2 = 0$ and $xJ = Jx = J^2$ (see [1] and also [8]).

Proof Let a, b, c be a generating set of W such that x generates U . Let A be generated by x, y, z , with radical J such that $x \cdot 1 = a, y \cdot 1 = b, z \cdot 1 = c$. □

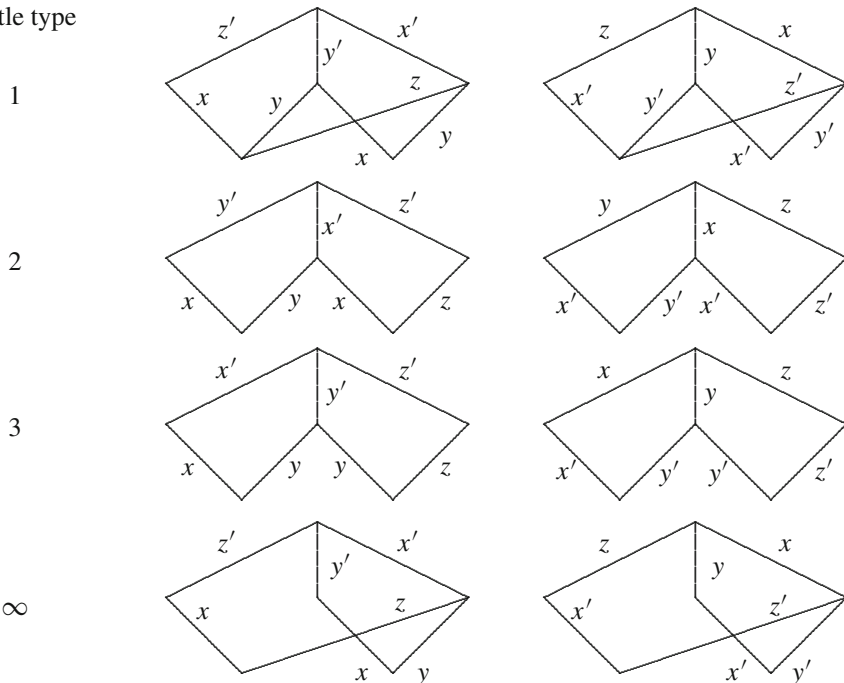
10 The Left Layout and the Right Layout of a Special Algebra

Lemma 10.1 *The right layout of a special algebra is isomorphic to the left layout.*

Proof The following table shows in the left column the various special algebras A and the corresponding opposite algebra A^{op} in the right column. We assume that A is a special algebra with radical πW , where W is a $K(3)$ -module with coefficient quiver as displayed in Sect. 4. Note that this means that we work with a fixed basis x, y, z of E , and also with a fixed basis (say $(100), (010), (001)$) of W_0 . In our identification of ${}_A J$ with πW , the elements $(100), (010), (001)$ of πW will correspond (in some order) to a second basis x', y', z' of E , the order which we choose is seen in the pictures in the left column.

It is straightforward to observe that indeed the opposite algebra has the presentation as shown in the right column. In particular, its radical (and this is the right module J_A) has a coefficient quiver which is obtained from the coefficient quiver we started with by replacing x, y, z by x', y', z' .

Bristle type

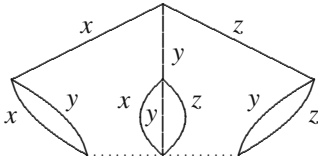


□

As we have seen, the left layout and the right layout have the same shape. Of course, as subsets of $\mathbb{P}E$ they may be different. Let us present an example with 3 left subbristles and 3 right subbristles, all pairwise different.

Example 10.2 A special algebra A with six isomorphism classes of bristles which are left or right subbristles. The algebra A is defined by the following relations:

$$zx = 0, x^2 = yx, xy = y^2 = zy, xz = 0, yz = z^2, y^2 + x^2 + z^2.$$



One easily checks that A is a special algebra. Also:

- Ax is annihilated by z and $x - y$;
- Ay is annihilated by $x - y$ and $y - z$;
- Az is annihilated by x and $y - z$.
- $(x - y)A$ is annihilated by x and y ;
- $(y - z)A$ is annihilated by y and z ;
- $(x - y + z)A$ is annihilated by x and z .

Therefore: the left ideals

$$Ax, Ay, Az$$

are bristles, and the right ideals

$$(x - y)A, (y - z)A, (x - y + z)A$$

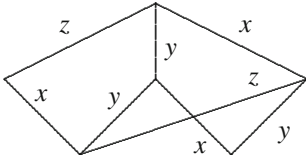
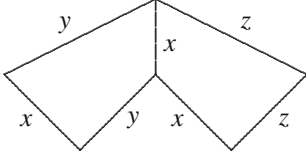
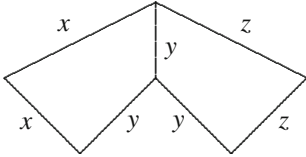
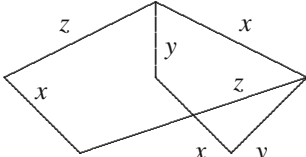
are also bristles, thus there are 6 isomorphism classes of bristles which occur as left or right ideals. Also, there are 6 four-dimensional ideals which are left bars or right bars, namely

$$Ax + Ay, Ax + Az, Ay + Az; \\ (x - y)A + (y - z)A, (x - y)A + zA, (y - z)A + xA.$$

11 Commutative Algebras

Proposition 11.1 *Let M be a special $L(3)$ -module. There is a commutative algebra A such that ${}_A J = M$ as A/J^2 -module.*

Proof Here are such algebras (on the right, we write down the defining relations, in order to obtain A from $k[x, y, z]$):

Bristle type		Relations
1		$yz, x^2, y^2 - xz, z^2$
2		$xz, yz, y^2, x^2 - z^2$
3		$xy, xz, yz, x^2 + y^2 + z^2$
∞		xy, x^2, y^2, z^2

When dealing with a commutative special algebra, a bar is a 4-dimensional ideal which is not faithful as an $L(3)$ -module.

Proposition 11.2 *Let A be a commutative special algebra and $a \in E$. The following conditions are equivalent:*

- (i) *The element $\langle a \rangle$ does not belong to a bar.*
- (ii) *The atom $C(a)$ is extensionless.*
- (iii) *The trace of $C(a)$ in ${}_A A$ is an atom (in particular, $C(a)$ is torsionless).*
- (iv) *The atom $C(a)$ is Gorenstein-projective.*
- (v) *The atom $C(a)$ is reflexive.*

If these conditions are satisfied, then $C(a)$ is Ω -periodic with period at most 2.

Concerning the condition (iii): Assume that the trace of $C(a)$ is an atom Y . Since Y is local, there is a homomorphism $\phi : C(a) \rightarrow {}_A A$ with image Y . Since both $C(a)$ and Y have length 3, ϕ is an isomorphism. This shows that $C(a)$ is torsionless. Remark 11.6 will show that the converse is not true: $C(a)$ may be torsionless, whereas its trace in ${}_A A$ is not an atom.

Proof of 11.2 The equivalence of (i) and (ii) follows from Proposition 9.2, the equivalence of (i) and (iii) follows from Proposition 9.4.

Let us show that the equivalent conditions (i)–(iii) imply (iv). Thus, assume that a does not belong to a bar. According to (ii), $C(a)$ is extensionless, and therefore $U(a) = \Omega C(a)$ is indecomposable, thus equal to Aa . According to (iii), the trace of $C(a)$ in ${}_A A$ is an atom, say equal to Ab for some $b \in E$.

For any $x \in A$, let $\rho_x : {}_A A \rightarrow {}_A A$ be the right multiplication by x . Then $Ab = U(b)$ is the image of ρ_b and Aa is the image of ρ_a . Look at the composition

$${}_A A \xrightarrow{\rho_a} {}_A A \xrightarrow{\rho_b} {}_A A.$$

The image of ρ_a is $Aa = U(a)$. The image Ab of ρ_b is the trace of $C(a)$ in ${}_A A$. It follows that $U(a)$ is the kernel of ρ_b . In particular, we have $ba = 0$ and the sequence exhibited above is exact.

Since A is commutative, we also have $ab = 0$. Therefore, also the sequence

$${}_A A \xrightarrow{\rho_b} {}_A A \xrightarrow{\rho_a} {}_A A$$

is a complex. Now, the image Ab of ρ_b has length 3. Also the image Aa of ρ_a has length 3, thus the kernel of ρ_a has length 3, and therefore this sequence is also exact. Altogether, we see that $U(b) = Ab = \Omega Aa$. It follows that $C(a) = Ab$ is Ω -periodic with period at most 2. This proves the additional assertion.

If we consider the complex P_\bullet

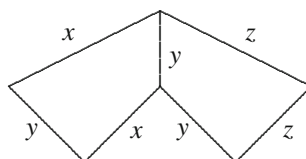
$$\dots \longrightarrow P_2 \xrightarrow{\rho_b} P_1 \xrightarrow{\rho_a} P_0 \xrightarrow{\rho_b} P_{-1} \longrightarrow \dots$$

with all modules $P_i = {}_A A$ and using alternatively ρ_a and ρ_b , we obtain an exact complex. The A -dual of P_\bullet is again P_\bullet , since the A -dual of $\rho_x : {}_A A \rightarrow {}_A A$ is just the left multiplication by x . Thus shows that $C(a)$ is the image in an exact complex P_\bullet whose A -dual is also exact, thus $C(a)$ is Gorenstein-projective.

Of course, (iv) implies (ii), and also (iv) implies (v).

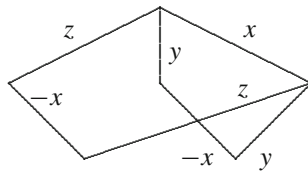
(v) implies (iii). Assume that $C(a)$ is reflexive. Then $C(a)$ and $\mathcal{U}C(a)$ both are torsionless and we have an \mathcal{U} -sequence $0 \rightarrow C(a) \rightarrow {}_A A^t \rightarrow \mathcal{U}C(a) \rightarrow 0$, for some $t \geq 1$. If $t \geq 2$, then the Loewy length of the cokernel of $C(a) \rightarrow {}_A A^t$ would be 3, but this cokernel is $\mathcal{U}C(a)$ and $\mathcal{U}C(a)$ has Loewy length 2, since it is torsionless (and has no non-zero projective direct summand). Therefore $t = 1$. Since the embedding $C(a) \rightarrow {}_A A$ is a left ${}_A A$ -approximation, the image of $C(a)$ in ${}_A A$ has to be the trace of $C(a)$ in ${}_A A$, thus the trace of $C(a)$ in ${}_A A$ is an atom. □

Remark 11.3 In condition (iii) of 11.2, it is not enough to assume that the atom $C(a)$ is torsionless: The module $C(a)$ may be torsionless, without being Gorenstein projective. As an example, take the following commutative algebra A .



Here, the atoms $C(x)$ and $C(z)$ both are torsionless, but $\Omega C(x)$ and $\Omega C(z)$ are decomposable. Thus, for $a = x$ and $a = z$, the module $C(a)$ is torsionless, but not extensionless and, as a consequence, not Gorenstein-projective. The traces of $C(x)$ and $C(z)$ in ${}_A A$ have length 4 (the trace of $C(x)$ is decomposable, the trace of $C(z)$ is indecomposable). \square

Remark 11.4 In case W is a special $K(3)$ -module of bristle type ∞ , there is also an anti-commutative short local algebra A with ${}_A J = \pi W$ as A/J^2 -module:

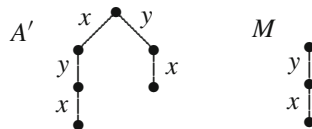


12 More About 6-Dimensional Local Algebras

Let A be a local algebra which is not self-injective and has a non-projective reflexive module. We have mentioned in the introduction that in case A is short the dimension of A has to be at least 6. Using Proposition 4.2, one may construct many isomorphism classes of short local algebras of dimension 6 which are not self-injective, but have non-projective reflexive modules. Let us stress that there do exist also 6-dimensional local algebras which are not self-injective, with non-projective reflexive modules, and which are not short.

6-dimensional local algebras with non-projective reflexive modules need not to be self-injective or short.

Example 12.1 Let $A' = k\langle x, y \rangle / (x^2, y^2, yxy)$ and $M = A'x = A'/A'x$.



The module M is Ω -periodic with period 1, and Gorenstein projective, thus reflexive.

Acknowledgements This paper is a continuation of our joint work with Zhang Pu from SJTU Shanghai devoted to the study of modules over short local algebras. Unfortunately, due to the severe Corona restrictions, the cooperation had to be suspended. The investigations presented here strongly rely on the previous joint papers and the author has to acknowledge his dependence on Pu’s insight into the realm of Gorenstein-projective modules. In addition, the author wants to thank two referees for spotting misprints and giving helpful advice.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest There is no conflict of interests.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Avramov, L.L., Iyengar, S.B., Şega, L.M.: Free resolutions over short local rings. *J. Lond. Math. Soc.* **78**, 459–476 (2008)
2. Ramras, M.: Betti numbers and reflexive modules, Ring theory (Proc. Conf., Park City, Utah.: Academic Press. New York **1972**, 297–308 (1971)
3. Ringel, C.M.: Representations of k -species and bimodules. *J. Algebra* **41**, 269–302 (1976)
4. Ringel, C.M.: Exceptional modules are tree modules. *Lin. Alg. Appl.* **275–276**, 471–493 (1998)
5. Ringel, C.M.: The elementary 3-Kronecker modules. [arXiv:1612.09141](https://arxiv.org/abs/1612.09141)
6. Ringel, C.M., Zhang, P.: Gorenstein-projective and semi-Gorenstein-projective modules. *Algebra Number Theory* **14–1**, 1–36 (2020). <https://doi.org/10.2140/ant.2020.14.1>
7. Ringel, C.M., Zhang, P.: Gorenstein-projective and semi-Gorenstein-projective modules II. *J. Pure Appl. Algebra* (2020). <https://doi.org/10.1016/j.jpaa.2019.106248>
8. Ringel, C.M., Zhang, P.: Gorenstein-projective modules over short local algebras. *J. Lond. Math. Soc.* **106(2)**, 528–589 (2022). <https://doi.org/10.1112/jlms.12577>