A Note on Obata's Rigidity Theorem

Guoqiang Wu · Rugang Ye

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Abstract In this note we present various extensions of Obata's rigidity theorem concerning the Hessian of a function on a Riemannian manifold. They include general rigidity theorems for the generalized Obata equation, and hyperbolic and Euclidean analogs of Obata's theorem. Besides analyzing the full rigidity case, we also characterize the geometry and topology of the underlying manifold in more general situations.

Keywords Obata rigidity · Hessian · Sphere theorems · Warped product metric

Mathematics Subject Classification 53C24

1 Introduction

Obata's rigidity theorem [2] as stated below is well known.

Theorem 1.1 (*Obata*) *Let (M, g) be a connected complete Riemannian manifold of dimension $n \geq 1$, which admits a nonconstant smooth solution of Obata's equation*

$$\nabla dw + wg = 0. \quad (1.1)$$

Then (M, g) is isometric to the n -dimensional round sphere $\mathbf{S}^n$.

G. Wu

Department of Mathematics, University of Science and Technology of China, Hefei, China
e-mail: cumtwgq@mail.ustc.edu.cn

R. Ye (✉)

Department of Mathematics, UCSB, Santa Barbara, CA 93106, USA
e-mail: yer@math.ucsb.edu

This theorem has various important geometric applications. For example, it is a main tool for establishing the rigidity part of Lichnerowicz–Obata theorem [1, 2] regarding the first eigenvalue of the Laplacian under a positive lower bound for the Ricci curvature. Another example is that it leads to uniqueness of constant scalar curvature metrics in a conformal class of metrics containing Einstein metrics [3, 8]. On the other hand, Obata’s equation (1.1) stands out as an important and interesting geometric equation for its own sake.

In this note, we discuss various extensions of Obata’s rigidity theorem. First we obtain general rigidity theorems and differentiable sphere theorems for the generalized Obata equation

$$\nabla dw + f(w)g = 0 \quad (1.2)$$

with a given smooth function f . Indeed, in Sect. 2 we first construct a class of Riemannian manifolds $M_{f,\mu}$ for a given function f and a value μ . Then we derive a Jacobi field formula from the Eq. (1.2), from which the desired rigidity results with $M_{f,\mu}$ as model manifolds easily follow. The material in this section is adapted from the lecture notes [4]. A critical point of w is assumed to exist in this section. In Sect. 3 we formulate some natural conditions on f and show that they imply the existence of critical points of w .

In Sect. 4 we present a derivation of global warped product structures implied by the Eq. (1.2). Previous works on this subject have been done by Brinkmann in [5] and Cheeger and Colding in [6]. See that section for further discussions.

In Sect. 5 we handle the hyperbolic case of the generalized Obata equation, i. e., the equation $\nabla dw - wg = 0$, and obtain hyperbolic versions of Obata’s theorem. In Sect. 6 similar results are obtained for the Euclidean cases of (1.2). Our main rigidity results here involve a condition on the dimension of the solution space and improve Theorem 1.3 in [7] substantially. Our results are actually optimal. In this context, besides analyzing the full rigidity case which requires the said dimension to be no less than a critical bound, we also characterize the geometry and topology of the underlying manifold when this dimension is lower.

In the last section, we extend our results to the following more general formulation of (1.2)

$$\nabla dw + f(w, \cdot)g = 0 \quad (1.3)$$

with a given smooth f defined on $I \times M$ for an interval I and the general equation

$$\nabla dw + zg = 0, \quad (1.4)$$

where w and z are two smooth functions on M , which is equivalent to

$$\nabla dw - \frac{\Delta w}{n}g = 0, \quad (1.5)$$

where $n = \dim M$. Another equivalent statement is that the Hessian of w has only one eigenvalue everywhere. (For the general equation $\nabla dw = wq$ with an arbitrarily given

smooth symmetric 2-tensor field q and a smooth function w we refer to [7], in which warped product rigidity is derived from this equation under a natural dimensional condition regarding its solution space.)

All manifolds in this note are assumed to be smooth. We would like to mention that the theory in this paper has been extended to a non-complete set-up, see [9]. For more background and additional previous treatments of the subject we refer to the papers [10–14]. Finally, we would like to acknowledge relevant discussions with T. Colding, G. Wei, and W. Wylie.

2 General Rigidity Theorems I

Before proceeding, we would like to note that Obata's equation can be transformed to the equation $\nabla dw + cwg = 0$ for an arbitrary positive constant c by a rescaling of the metric. This leads to an obvious rescaled extension of Obata's theorem. The same holds true for the various extensions of Obata's equation in this paper, whenever the metric rescaling leads to a more general equation in question.

2.1 Rigidity Theorems

Theorem 2.1 *Let (M, g) be a connected complete Riemannian manifold of dimension $n \geq 2$ which admits a nonconstant smooth solution w of the generalized Obata equation (1.2) for a smooth function $f(s)$. Assume that w has at least one critical point p . Then M is diffeomorphic to $\mathbf{R}^n$ or $\mathbf{S}^n$. Moreover, (M, g) is isometric to $M_{f,\mu}$ with $\mu = w(p)$.*

The manifolds $M_{f,\mu}$ will be constructed below. Theorem 2.1 leads to the following differentiable sphere theorem.

Theorem 2.2 *Let (M, g) be an n -dimensional connected compact Riemannian manifold. Assume that it admits a nonconstant smooth solution of the generalized Obata equation for some smooth function f . Then M is diffeomorphic to $\mathbf{S}^n$.*

Note that Obata's theorem easily follows from Theorem 2.1, as it is easy to show that the solution w in Eq. (1.1) must have a maximum point.

2.2 Construction of $M_{f,\mu}$

Let $f(s)$ be a smooth function defined on an interval $I = (A, B)$, $[A, B)$, $(A, B]$ or $[A, B]$ (A is allowed to be $-\infty$ and B is allowed to be ∞), and let $\mu \in I$ satisfy $f(\mu) \neq 0$. Let u be the unique maximally extended smooth solution of the initial value problem

$$u'' + f(u) = 0, u(0) = \mu, u'(0) = 0. \quad (2.1)$$

By the uniqueness of u we infer that u is an even function. On the other hand, there holds $u''u' + f(u)u' = 0$, which implies

$$u'^2 = -2h(u), \quad (2.2)$$

where h is the antiderivative of f such that $h(\mu) = 0$. Let T be the supremum of t such that u is defined on $[0, t]$ and $u' \neq 0$ in $(0, t]$. We define for $n \geq 2$

$$g = dt^2 + f(\mu)^{-2}u'^2 g_{\mathbb{S}^{n-1}} \quad (2.3)$$

on $(0, T) \times \mathbb{S}^{n-1}$. Set $\phi = -f(\mu)^{-1}u'$. Then $\phi(0) = 0$, $\phi'(0) = 1$. Moreover, ϕ is an odd function because u is even. It follows that g extends to a smooth metric on the n -dimensional Euclidean open ball $B_T(0)$ where $B_T(0) - \{0\}$ is identified with $(0, T) \times \mathbb{S}^{n-1}$. (If $T = \infty$, then $B_T(0) = \mathbb{R}^n$.) There are three cases to consider.

Case 1 $h(s) \neq 0$ for all $s > \mu$ if $f(\mu) < 0$, and $h(s) \neq 0$ for all $s < \mu$ if $f(\mu) > 0$.

In this case we make the following divergence assumption:

$$\int_J (-h)^{-1/2} ds = \infty, \quad (2.4)$$

where $J = (\mu, B)$ if $f(\mu) < 0$ and $J = (A, \mu)$ if $f(\mu) > 0$. To proceed, consider the function $s = u(t)$ on $[0, T)$. We have $t = u^{-1}(s)$. By Eq. (2.2) there holds

$$\frac{dt}{ds} = \pm(-2h(s))^{-1/2} \quad (2.5)$$

and hence

$$t = t(s) = \pm \int_{\mu}^s (-2h(\tau))^{-1/2} d\tau. \quad (2.6)$$

It follows that $T = \int_J (-2h)^{-1/2} ds = \infty$ and the manifold $(\mathbb{R}^n, g)$ is complete. We denote it by $M_{f,\mu}$. A pair (f, μ) satisfying the above conditions will be called of noncompact type I.

Case 2 $h(v) = 0$ for some v , where $v > \mu$ if $f(\mu) < 0$ and $v < \mu$ if $f(\mu) > 0$. We assume that v is the nearest such number from μ .

Case 2.1 There holds $f(v) = 0$. We say that (f, μ) is of noncompact type II.

Since $h'(v) = f(v) = 0$ it then follows that $\int_J (-h)^{-1/2} ds = \infty$, where $J = (\mu, v)$ or (v, μ) . As before, we infer that $T = \infty$. The complete manifold $(\mathbb{R}^n, g)$ is again denoted by $M_{f,\mu}$.

Case 2.2 There holds $f(v) \neq 0$.

Lemma 2.3 Assume $f(v) \neq 0$. Then T is contained in the domain of u and $u(T) = v$.

Proof We present the case $f(\mu) < 0$, while the case $f(\mu) > 0$ is similar. Since $h' = f$, the condition $f(v) \neq 0$ implies $\int_{\mu}^v (-h)^{-1/2} ds < \infty$. By the definition of T there holds $u(t) < v$ for all $0 \leq t < T$. Hence

$$T \leq \int_{\mu}^v (-2h(s))^{-1/2} ds < \infty. \quad (2.7)$$

On the other hand, we have $|u''| \leq |f(u)| \leq \max\{|f(s)| : \mu \leq s \leq v\}$ on $[0, T)$. Consequently, T is in the domain of u . By the definition of T we then infer $u(T) = v$. $\square$

Next we assume in addition to $f(v) \neq 0$ the coincidence condition $f(\mu) = -f(v)$. (The pair (f, μ) will be called of compact type.) Then the metric g smoothly extends to $\mathbf{S}^n$, where $\mathbf{S}^n - \{p, -p\}$ ($p \in \mathbf{S}^n$) is identified with $(0, T) \times \mathbf{S}^{n-1}$. The Riemannian manifold $(\mathbf{S}^n, g)$ is also denoted by $M_{f,\mu}$.

Note that the above arguments also provide a formula for the solution u . Indeed, $u(t)$ is the inverse of the function $t(s)$ given by (2.6).

Lemma 2.4 *In the above construction of $M_{f,\mu}$, the function $w = u(t)$ on $(0, T) \times \mathbf{S}^{n-1}$ smoothly extends to $M_{f,\mu}$ and satisfies the generalized Obata equation (1.2) with the given f .*

Proof The evenness of u implies that u is a smooth function of t^2 . But t is the distance to the origin. In geodesic coordinates x^i there holds $t^2 = |x|^2$ and hence $w = u(t)$ extends smoothly across the origin. The situation in a second critical point is similar. On the other hand, a calculation similar to the proof of Lemma 5.9 below shows that w satisfies the generalized Obata equation on $(0, T) \times \mathbf{S}^{n-1}$, and hence on $M_{f,\mu}$. $\square$

Examples 2.5 In the following examples the domain of f is $\mathbf{R}$.

- (1) $f(s) = s^{2m}$ for a natural number m , $h(s) = (2m+1)^{-1}(s^{2m+1} - 1)$, and $\mu = 1$. This is of noncompact type I.
- (2) $f(s) = 1$, $h(s) = s - 1$, and $\mu = 1$. This is also of noncompact type I. Note that $M_{1,1}$ is the Euclidean space $\mathbf{R}^n$.
- (3) $f(s) = s^3 - s$, $h(s) = \frac{1}{4}s^4 - \frac{1}{2}s^2$, $\mu = -\sqrt{2}$, and $v = 0$. This is of noncompact type II.
- (4) $f(s) = s^{2m-1}$ for a natural number m , $h(s) = (2m)^{-1}(s^{2m} - 1)$, $\mu = 1$, and $v = -1$. This is of compact type. Note that $M_{s,1}$ is the round sphere $\mathbf{S}^n$. Indeed, there holds in this case $u = \cos t$, $u' = -\sin t$ and hence $g = dt^2 + \sin^2 t \cdot g_{\mathbf{S}^{n-1}}$.
- (5) $f(s) = \cos s$, $h(s) = \sin s$, $\mu = 0$, and $v = \pi$. This is of compact type.

2.3 Calculation of Jacobi Fields

Lemma 2.6 *Let w be a nonconstant smooth solution of the generalized Obata equation (1.2) on a connected complete Riemannian manifold (M, g) of dimension n and for a smooth function $f(s)$. Let p_0 be a critical point of w . Set $\mu = w(p_0)$. Then*

$f(\mu) \neq 0$. Consequently, p_0 is a nondegenerate local extremum point of w . Moreover, if γ is a unit-speed geodesic starting at p_0 , then there hold

$$w(\gamma(t)) = u(t) \quad (2.8)$$

and

$$\nabla w \circ \gamma = u' \gamma', \quad (2.9)$$

where u is the solution of (2.1). In particular, γ (with critical points of w deleted) consists of reparametrizations of gradient flow lines of w .

Proof Along each unit-speed geodesic $\gamma(t)$ there holds

$$\frac{d^2}{dt^2} w(\gamma(t)) + f(w(t)) = 0. \quad (2.10)$$

We also have $\frac{d}{dt} w(\gamma(t))|_{t=0} = 0$. Hence the formula (2.8) holds true. The claim $f(\mu) \neq 0$ follows, because otherwise $w(\gamma(t)) \equiv \mu$ and hence $w \equiv \mu$ on M . The fact $f(\mu) \neq 0$ and the Eq. (1.2) imply that p_0 is a nondegenerate local extremum point of w .

To see (2.9) we write $\nabla w \circ \gamma = X + \phi \gamma'$, where X is normal. The generalized Obata equation is equivalent to

$$\nabla_{\mathbf{v}} \nabla w + f(w) \mathbf{v} = 0 \quad (2.11)$$

for all tangent vectors $\mathbf{v}$. We deduce that X is parallel and $\phi' + f(u) = 0$. Hence we have $X \equiv 0$ and $\phi' - u'' = 0$. The last equation and the initial values of ϕ and u' imply $\phi = u'$. $\square$

Next we present a calculation of Jacobi fields.

Proposition 2.7 *Assume the same as in Lemma 2.6. Let Y be a normal Jacobi field along γ such that $Y(0) = 0$ and V denote the parallel transport of $Y'(0)$ along γ . Then there holds*

$$Y = -f(\mu)^{-1} u' V. \quad (2.12)$$

Proof Set $\mathbf{u} = \gamma'(0)$, $\mathbf{v} = Y'(0)$ and $\gamma(t, s) = \exp_{p_0}(t(\mathbf{u} + s\mathbf{v}))$. Then

$$Y(t) = \frac{\partial \gamma}{\partial s} |_{s=0}. \quad (2.13)$$

By Eq. (2.9) we deduce

$$\nabla w(\gamma(t, s)) = |\mathbf{u} + s\mathbf{v}|^{-1} u'(t|\mathbf{u} + s\mathbf{v}|) \frac{\partial \gamma}{\partial t}. \quad (2.14)$$

Hence

$$\begin{aligned}\nabla_Y \nabla w &= \frac{\partial}{\partial s} (|\mathbf{u} + s\mathbf{v}^{-1}| \cdot u'(t|\mathbf{u} + s\mathbf{v}|))|_{s=0} \frac{\partial \gamma}{\partial t} \\ &\quad + |\mathbf{u} + s\mathbf{v}|^{-1} u'(t|\mathbf{u} + s\mathbf{v}|) \nabla_{\frac{\partial}{\partial t}} Y.\end{aligned}\quad (2.15)$$

Since $\frac{\partial}{\partial s} |\mathbf{u} + s\mathbf{v}| = |\mathbf{u} + s\mathbf{v}|^{-1} \mathbf{u} \cdot \mathbf{v} = 0$, we infer

$$\nabla_Y \nabla w = u' \nabla_{\frac{\partial}{\partial t}} Y. \quad (2.16)$$

It follows that

$$u' \nabla_{\frac{\partial}{\partial t}} Y + f(u)Y = 0. \quad (2.17)$$

Setting $Y = \phi V$ we deduce $u'\phi' + f(u)\phi = 0$, i. e., $u'\phi' - u''\phi = 0$. It follows that $\phi = Cu'$ for a constant C . Since $\phi(0) = 0$ and $\phi'(0) = 1$ we derive $\phi = -f(\mu)^{-1}u'$. $\square$

2.4 Proof of Theorem 2.1

Proof of Theorem 2.1 We can assume $n \geq 2$. The formula (2.12) implies

$$d\exp_{p_0}|_{t\mathbf{u}}(\mathbf{v}) = -\frac{f(\mu)^{-1}u'}{t}V(t) \quad (2.18)$$

which leads to

$$\exp_{p_0}^* g = dr^2 + f(\mu)^{-2}u'^2 g_{S^{n-1}} \quad (2.19)$$

for all $r > 0$. By Lemma 2.6, each unit-speed geodesic γ starting at p_0 is a reparametrization of a gradient flow line of w (before reaching a critical point of w). Hence they cannot meet each other before reaching a critical point of w . Moreover, no γ can intersect itself before reaching a critical point of w . By Eq. (2.9), they reach a critical point precisely at the first positive zero of u' . If u' has no positive zero, then we conclude that $\exp_{p_0}$ is a diffeomorphism from $T_{p_0}M$ onto M . Next let T be the first positive zero of u' . Then $\exp_{p_0}$ is a diffeomorphism from $B_T(0)$ onto $B_T(p_0)$ (both are open balls). By Eq. (2.18), $\exp_{p_0}$ maps $\partial B_T(0)$ onto a critical point p_1 . Employing the exponential map $\exp_{p_1}$ we then infer that $\exp_{p_0}$ extends to a smooth diffeomorphism from S^n onto M . Finally, from the construction of $M_{f,\mu}$ and the completeness of g it is easy to see that (f, μ) is one of the types in that construction and (M, g) is isometric to $M_{f,\mu}$. $\square$

3 General Rigidity Theorem II

The main purpose of this section is to present natural conditions on f which allow us to remove the condition of critical points in Theorem 2.1 and, for example, imply a differentiable sphere theorem without assuming compactness of the manifold.

Definition 3.1 Let $f(s)$ be a smooth function on an interval I .

- (1) We say that f is *coercive*, if the following holds true. Let h be a somewhere negative antiderivative h of f . Then h has zeros. Moreover, there holds $f(\mu)^2 + f(\nu)^2 \neq 0$ if $h(\mu) = h(\nu) = 0$ and $h < 0$ on (μ, ν) . If the maximal zero μ exists and $h < 0$ on $I \cap (\mu, \infty)$, or if the minimal zero μ exists and $h < 0$ on $I \cap (-\infty, \mu)$, we also assume $f(\mu) \neq 0$. (A special case is that $f(\mu) \neq 0$ for each zero μ of h .)
- (2) We say that f is *degenerately coercive*, if it is coercive and the following holds true. Let h be an antiderivative h of f . If $h(\mu) = h(\nu) = 0$ and $h < 0$ on (μ, ν) , then $f(\mu)f(\nu) = 0$.
- (3) We say that f is *nondegenerately coercive*, if the following holds true. Let h be a somewhere negative antiderivative h of f . Then $h^{-1}((-\infty, 0))$ is a disjoint union of bounded intervals whose endpoints are contained in the domain of f . Moreover, there holds $f(\mu) \neq 0$ for each such endpoint μ .

Examples 3.2 In the following examples, the domain of the function is $\mathbf{R}$. The function $f(s) = s^{2m-1}$ for a natural number m is nondegenerately coercive. The function $f(s) = \pm s^{2m}$ for a natural number m is degenerately coercive. The functions $f(s) = 1$ and $f(s) = 1 + \frac{1}{2} \cos s$ are degenerately coercive. The function $f(s) = \cos s$ is nondegenerately coercive. The function $f(s) = s^3 - s$ is degenerately coercive. It is easy to construct many more examples.

Theorem 3.3 Let (M, g) be a connected complete Riemannian manifold of dimension $n \geq 2$ which admits a nonconstant smooth solution of the generalized Obata equation (1.2) for a coercive function f . Then $M = M_{f,\mu}$ for some μ . In particular, M is diffeomorphic to $\mathbf{S}^n$ or $\mathbf{R}^n$.

Theorem 3.4 Let (M, g) be a connected complete Riemannian manifold of dimension n which admits a nonconstant smooth solution of the generalized Obata equation (1.2) for a degenerately coercive function f . Then M is diffeomorphic to $\mathbf{R}^n$. Moreover, if $n \geq 2$, then (M, g) is isometric to $M_{f,\mu}$ for some μ .

Theorem 3.5 Let (M, g) be a connected complete Riemannian manifold of dimension n which admits a nonconstant smooth solution of the generalized Obata equation (1.2) for a nondegenerately coercive function f . Then M is diffeomorphic to $\mathbf{S}^n$. Moreover, if $n \geq 2$, then (M, g) is isometric to $M_{f,\mu}$ for some μ .

In the case of Obata's equation, we have $f(s) = s$ and hence $h(s) = \frac{1}{2}s^2 - C$. It is easy to see that f is nondegenerately coercive. Hence Obata's rigidity theorem is included in Theorem 3.5 as a special case.

Lemma 3.6 *Let u be a nonconstant smooth solution of the equation*

$$u'' + f(u) = 0 \quad (3.1)$$

on $\mathbf{R}$ for some smooth function f . Then the following hold true.

- (1) *u is symmetric with respect to each of its critical points.*
- (2) *If f is coercive, then u has at least one critical point.*
- (3) *If f is degenerately coercive, then u has precisely one critical point.*
- (4) *If f is nondegenerately coercive and $u'(t_0) = 0$ for some t_0 , then $u'(t_1) = 0$ for some $t_1 > t_0$. (It follows that u is a periodic function.)*

Proof (1) Let t_0 be a critical point of u . Then $u(t_0 - t) = u(t_0 + t)$ follows from the uniqueness of the solution of (2.1).

(2) Let f be coercive. Assume that u has no critical point. Then $u(\mathbf{R})$ is an open interval (μ_1, μ_2) . By (2.2) there holds $h \neq 0$ on (μ_1, μ_2) . There holds for the inverse $t = t(s)$ of $u(t)$

$$t = \pm \int_c^s (-2h(s))^{-1/2} ds + t_0 \quad (3.2)$$

with $c = u(t_0)$ for some t_0 in the domain of u . Since u is defined on $\mathbf{R}$ we deduce

$$\int_c^{\mu_2} (-2h(s))^{-1/2} ds = \infty, \quad \int_{\mu_1}^c (-2h(s))^{-1/2} ds = \infty. \quad (3.3)$$

It follows that $h(\mu_i) = 0$ whenever μ_i is finite, $i = 1, 2$. It is impossible for both μ_1 and μ_2 to be infinite, otherwise $u(\mathbf{R}) = \mathbf{R}$ and then $h(u(t)) = 0$ for some t , and hence $u'(t) = 0$. By the coercivity assumption we then deduce that $h(\mu_i) = 0$ and $f(u_i) \neq 0$ for some i , say $i = 2$. But then

$$\int_c^{\mu_2} (-2h(s))^{-1/2} ds < \infty \quad (3.4)$$

as $h'(\mu_2) = f(\mu_2) \neq 0$. This is a contradiction. The case $i = 1$ is similar.

(3) Let f be degenerately coercive. Assume that u has more than one critical points. Since u is nonconstant, we translate the argument to achieve the following: u has critical points 0 and $t_0 > 0$, such that $u' \neq 0$ on $(0, t_0)$. Since u is nonconstant, there holds $f(\mu) \neq 0$, where $\mu = u(0)$. It follows that $f(v) = 0$, where $v = u(t_0)$. It follows then that

$$t_0 = \int_I (-2h(s))^{-1/2} ds = \infty, \quad (3.5)$$

which is a contradiction, where $I = (\mu, v)$ or (v, μ) .

(4) Let f be nondegenerately coercive and t_0 a critical point of u . Translating the argument we can assume $t_0 = 0$. Then u is an even function. Hence it suffices to

find one more critical point of u . Set $\mu = u(0)$. There holds $h(\mu) = 0$. By (2.2), h is nonpositive on the interval $u(\mathbf{R})$. By the nondegenerately coercive assumption, we then infer that $u(\mathbf{R}) \subset [a, b]$ for some finite a and b such that $h < 0$ on (a, b) and $h(a) = h(b) = 0$. Obviously, there holds $\mu = a$ or b . We consider the former case, while the latter is similar. Assume that 0 is the only critical point of u . Then $u(\mathbf{R}) = u([0, \infty)) = [\mu, c)$ for some $c \leq b$. It follows that

$$\int_{\mu}^c (-2h(s))^{-1/2} ds = \infty. \quad (3.6)$$

If $c < b$, then there holds $|h| > \delta$ in a neighborhood of b for some $\delta > 0$. Consequently we infer

$$\int_{\mu}^c (-2h(s))^{-1/2} ds < \infty, \quad (3.7)$$

contradicting (3.6). If $c = b$, we also derive (3.7) as $h'(b) = f(b) \neq 0$. $\square$

Lemma 3.7 *Let w be a nonconstant smooth solution of the generalized Obata equation with some f on a complete Riemannian manifold (M, g) . If f is coercive, then w has at least one critical point.*

Proof Choose a point $p \in M$ with $\nabla w(p) \neq 0$. Let γ be the unit-speed geodesic such that $\gamma(0) = p$ and $\gamma'(0)$ is the unit vector in the direction of $\nabla w(p)$. Set $u(t) = w(\gamma(t))$. Then there holds $u'' + f(u) = 0$. By Lemma, u has at least one critical point t_0 . Following the arguments in the proof of Lemma we deduce $\nabla w(\gamma(t)) = \phi \gamma'$ with $\phi = u' + c$. But $u'(0) = |\nabla w(p)| = \phi(0)$. Hence $\phi = u'$. It follows that $\nabla w(\gamma(t_0)) = 0$. $\square$

Proof of Theorem 3.3 This follows from Lemma 3.7 and Theorem 2.1. $\square$

Proof of Theorems 3.4 and 3.5 They follow from Lemma 3.6, Lemma 3.7, and the proof of Theorem 2.1. In particular, the case of the exceptional dimension $n = 1$ in Theorem 3.3 follows from 2) of Lemma 3.6, because a solution of (2.1) on $\mathbf{S}^1$ leads to a periodic function $u(t)$. $\square$

4 Warped Product Structures

In [5], using a result of partial differential equations and calculation in coordinates, Brinkmann derived from the Eq. (1.2) a local warped product structure for the metric. In [6], Cheeger and Colding derived from the Eq. (1.4) a global warped product structure for the metric in terms of calculation of differential forms. In this section, we present a slightly different derivation of global warped product structures based on the Eq. (1.2). Our approach uses the given solution w as a global coordinate, which is motivated by the arguments in [5]. This leads us to using the flow of the vector field $|\nabla w|^{-2} \nabla w$ in the construction. In comparison, the arguments in [6] implicitly involve the vector field $|\nabla w|^{-1} \nabla w$. (The latter vector field also enters into our argument in an auxiliary

and different way, see the proof of Lemma 4.3.) The results in this section will be extended to the general Eq. (1.4) in Sect. 7, where a special technical point regarding it will be handled. (Some lemmas in this section are formulated for the general Eq. (1.4).)

The formulation in this section is particularly convenient for the applications in the subsequent sections.

Lemma 4.1 *Let w and z be two smooth functions on a Riemannian manifold (M, g) satisfying the Eq. (1.4). Let N be a connected component of a level set of w . Assume that N contains no critical point of w . Then $|\nabla w|$ and z are constants on N . Moreover, the shape operator of N (with the normal direction given by ∇w) is given by $|\nabla w|^{-1} z_N Id$, where z_N is the constant value of z on N . In particular, N is totally geodesic precisely when $z_N = 0$.*

Proof The Eq. (1.4) is equivalent to

$$\nabla_{\mathbf{u}} \nabla w + z \mathbf{u} = 0 \quad (4.1)$$

for all tangent vectors $\mathbf{u}$. We infer for $\mathbf{u}$ tangent to N

$$\nabla_{\mathbf{u}} |\nabla w|^2 = 2 \nabla_{\mathbf{u}} \nabla w \cdot \nabla w = -2z \mathbf{u} \cdot \nabla w = 0. \quad (4.2)$$

Hence $|\nabla w|$ is a constant on N . Next we have

$$\nabla_{\frac{\nabla w}{|\nabla w|^2}} \nabla w + z \frac{\nabla w}{|\nabla w|^2} = 0 \quad (4.3)$$

and hence

$$z = -\frac{1}{2} \nabla_{\frac{\nabla w}{|\nabla w|^2}} |\nabla w|^2. \quad (4.4)$$

Let $F(t, p)$ denote the flow lines of $\nabla w / |\nabla w|^2$ starting on N , where p is an initial point and t the time, with the initial value of t being μ , the value of w on N . There holds

$$\frac{d}{dt} w(F(t, p)) = \nabla w \cdot \frac{\nabla w}{|\nabla w|^2} = 1. \quad (4.5)$$

Hence $w(F(t, p)) = t$. By the established fact that $|\nabla w|$ is a constant on each component of any level set of w we infer that $|\nabla w|(F(t, p))$ is independent of $p \in N$. Now $\nabla_{\frac{\nabla w}{|\nabla w|^2}} |\nabla w|^2$ is equal to $\frac{d}{dt} |\nabla w|^2(F(t, p))|_{t=\mu}$, and hence independent of $p \in N$. We deduce that z is a constant on N .

Finally we have

$$\nabla_{\mathbf{u}} \frac{\nabla w}{|\nabla w|} = -|\nabla w|^{-1} z \mathbf{u} \quad (4.6)$$

for tangent vectors $\mathbf{u}$ of N . $\square$

Lemma 4.2 *Let w and z be two smooth functions on a Riemannian manifold (M, g) satisfying the Eq. (1.4). Then the flow lines of $\nabla w/|\nabla w|$ in the domain $\{\nabla w \neq 0\}$ are unit-speed geodesics.*

Proof By (4.1) we deduce

$$\begin{aligned}\nabla_{\nabla w} \frac{\nabla w}{|\nabla w|} &= -|\nabla w|^{-1} z \nabla w - |\nabla w|^{-3} (\nabla_{\nabla w} \nabla w \cdot \nabla w) \nabla w \\ &= -|\nabla w|^{-1} z \nabla w + |\nabla w|^{-1} z \nabla w = 0.\end{aligned}\quad (4.7)$$

The claim of the lemma follows. $\square$

Lemma 4.3 *Assume that w is a nonconstant solution of the generalized Obata equation (1.2) on a Riemannian manifold (M, g) for a given smooth function f . Let μ be a value of w and N a connected component of $w^{-1}(\mu)$. Let α denote the value of $|\nabla w|$ on N , which is a constant by Lemma 4.1. Then there holds for $p \in M$*

$$|\nabla w|^2(p) = h(w(p)), \quad (4.8)$$

where $h(s) = \alpha^2 - 2 \int_{\mu}^s f(\tau) d\tau$, as long as there is a gradient flow line γ of w such that $\gamma(t)$ converges to a point of N in one direction and it converges to p in the other direction.

Proof By a reparametrization we can assume that γ is a flow line of $\nabla w/|\nabla w|$, and hence a unit-speed geodesic by Lemma 4.2. Set $u = w(\gamma(t))$. Then $u'^2 = |\nabla w|^2$. As in Sect. 2, we have $u'' + f(u) = 0$ and hence $((u')^2 - h(u))' = 0$. (Note that h here is different from h in Sect. 2.) We infer $(u')^2 = h(u) + C$ or $|\nabla w|^2 = h(w) + C$. Evaluating at a point of N we deduce $C = 0$. $\square$

Lemma 4.4 *Let w be a nonconstant smooth solution of the generalized Obata equation (1.4) on a Riemannian manifold (M, g) for a given smooth function f . Let N be a connected component of $w^{-1}(\mu)$ for some μ . Assume that $\nabla w \neq 0$ on N . As above, let $F(s, p)$ be the flow lines of $\nabla w/|\nabla w|^2$ starting on N with the initial time being μ . Assume that F is smoothly defined on $I \times N$ for a time interval I . Then there holds*

$$F^*g = \frac{ds^2}{h(s)} + \frac{h(s)}{\alpha^2} g_N = \frac{ds^2}{h(s)} + \frac{h(s)}{h(\mu)} g_N, \quad (4.9)$$

where α denotes the value of $|\nabla w|$ on N , h is the same as in Lemma 4.3, and g_N is the induced metric on N .

Proof Let $\mathbf{u} \in T_p N$ for some $p \in N$. Set $F_{\mathbf{u}} = dF_{(s,p)}(\mathbf{u})$ and $F_s = \frac{\partial F}{\partial s}$. Using (2.11) and (4.8) we calculate

$$\begin{aligned}\frac{\partial}{\partial s} |F_{\mathbf{u}}|^2 &= 2 \nabla_{F_{\mathbf{u}}} F_s \cdot F_{\mathbf{u}} = 2 \nabla_{F_{\mathbf{u}}} \frac{\nabla w}{|\nabla w|^2} \cdot F_{\mathbf{u}} \\ &= -2f(w) |\nabla w|^{-2} |F_{\mathbf{u}}|^2 = \frac{h'(s)}{h(s)} |F_{\mathbf{u}}|^2.\end{aligned}\quad (4.10)$$

Similarly, there holds

$$\begin{aligned}\frac{\partial}{\partial s}|F_s|^2 &= 2\nabla_{\frac{\partial}{\partial s}} \frac{\nabla w}{|\nabla w|^2} \cdot F_s = -2f(w)|\nabla w|^{-2}|F_s|^2 + 4f(w)|\nabla w|^{-2}|F_s|^2 \\ &= 2f(w)|\nabla w|^{-2}|F_s|^2 = -\frac{h'(s)}{h(s)}|F_s|^2.\end{aligned}\quad (4.11)$$

Integrating then leads to

$$\begin{aligned}|F_{\mathbf{u}}|^2(p, s) &= \frac{h(s)}{\alpha^2}|F_{\mathbf{u}}|^2(p, \mu), \\ |F_s|^2(p, s) &= \frac{\alpha^2}{h(s)}|F_s|^2(p, \mu) = \frac{1}{h(s)}.\end{aligned}\quad (4.12)$$

□

Theorem 4.5 *Let (M, g) be a connected complete Riemannian manifold which admits a nonconstant smooth solution w of the generalized Obata equation (1.2) for some smooth f . Let I denote the interior of the image I_w of w . Let $\mu \in I$. Set $N = w^{-1}(\mu)$ and $\Omega = w^{-1}(I)$. Then (N, g_N) is connected and complete with the induced metric g_N and there is a diffeomorphism $F : I \times N \rightarrow \Omega$ such that $w(F(s, p)) = s$ for all (s, p) . The pullback metric F^*g is a warped product metric given by the formula (4.9). Furthermore, there holds $M = \bar{\Omega}$ and $\partial\Omega$ consists of at most two points. Each point is either a unique global maximum point or a unique global minimum point of w .*

Conversely, if (N, g_N) is a Riemannian manifold and $h(s)$ a positive smooth function on an interval I , then the function $w = s$ on $I \times N$ satisfies the generalized Obata equation with $f = -\frac{1}{2}h'$, where $I \times N$ is equipped with the metric $h^{-1}ds^2 + \frac{h}{\alpha^2}g_N$.

We would like to remark that this theorem can be used to replace some arguments in Sects. 2 and 3. This can, for example, be seen from the proofs of Theorems 5.5 and 5.6 below. But the approach adopted in these two sections is more concise.

An immediate consequence of Theorem 4.5 and Theorem 2.1 is the following result.

Theorem 4.6 *Let (M, g) be a connected complete Riemannian manifold which admits a nonconstant smooth solution w of the generalized Obata equation (1.2) for some smooth f . Then either (M, g) is isometric to $M_{f,\mu}$ for some f and μ , or isometric to a warped product $\mathbf{R} \times_{\phi} (N, g_N)$ for a complete connected Riemannian manifold (N, g_N) and a positive smooth function ϕ on $\mathbf{R}$. In the former case, M is diffeomorphic to $\mathbf{R}^n$ if w has precisely one critical point, and it is diffeomorphic to $\mathbf{S}^n$ if w has two critical points.*

Proof of Theorem 4.5 Given the above results, the main point here is to construct the diffeomorphism F in details, which requires some care in the case of a noncompact M and a possibly noncompact N . Let N_0 be a nonempty connected component of N and let $F(s, p)$ be the same flow lines as given in Lemma 4.4, with N replaced

by N_0 . By the completeness of g , the induced metric g_{N_0} is complete. The formula $w(F(s, p)) = s$ follows from a simple integration along the flow lines. Let J_p be the interval of values of w along the maximally defined $F(s, p)$ for $p \in N_0$. By Lemma 4.1, $|\nabla w|$ at $F(s, p)$ depends only on s . This fact and the completeness of g imply that $J = J_p$ is independent of p . Let Ω be the image of $F(s, p)$ for $p \in N_0$ and $s \in J$. Then $F : J \times N_0 \rightarrow \Omega$ is a diffeomorphism.

Let $p \in \partial\Omega$. Then we have $F(s_k, p_k) \rightarrow p$ for some $s_k \in J$ and $p_k \in N_0$. There holds $s_k \rightarrow s^* \equiv w(p)$. We can assume that s_k is a monotone sequence. Let $\sigma(s)$ denote the value of $|\nabla w|$ at $F(s, p)$. If $\nabla w(p) \neq 0$, then $F(s, p_k)$ is defined on an open interval J' containing s^* as long as k is large enough. Fix such a k_0 . Consider the case $s^* > \mu$, while the case $s^* < \mu$ is similar. The length of the curve $\gamma_k(s) = F(s, p_k)$, $\mu \leq s \leq s^*$ is given by $L = \int_{\mu}^{s^*} \sigma^{-1}$. This integral is finite because $\sigma(s)$ is smooth and positive on $[\mu, s^*]$. It follows that $\text{dist}(p, p_k) \leq L + 1$ for k large. By the completeness of (M, g) a subsequence of p_k converges to a point $q \in N_0$. There holds $F(s, q) \rightarrow p$ as $s \rightarrow s^*$. But $F(s, q)$ is not defined at s^* , otherwise we would have $p \in \Omega$. We infer that p is a critical point of w , and hence a nondegenerate local extremum point of w (Lemma 2.6). It follows that the level sets of w around p are connected spheres filling a ball. (This is also clear from the proof of Theorem 2.1.) Since $F(s, q)$ passes through them, we infer that a neighborhood U of p satisfies $U - \{p\} \subset \Omega$. Note that the image of N_0 under $F(s, \cdot)$ for s close to s^* is one of the said spheres.

Obviously, the above conclusion implies that $\bar{\Omega}$ is open. Hence $\bar{\Omega} = M$. We also infer $M = \bar{\Omega} = \Omega \cup S$, where S consists of at most two critical points of w . If $p \in S$, then $w(p)$ is an endpoint of $J = I$. Moreover, $w(p_1) \neq w(p_2)$ if S contains two points p_1 and p_2 . All these also imply $N_0 = N$. Finally, the claimed warped product formula (4.9) follows from Lemma 4.4. $\square$

5 Hyperbolic Versions

In this section we consider the following hyperbolic case of the generalized Obata equation

$$\nabla dw - wg = 0. \quad (5.1)$$

We first have the following immediate consequence of Theorem 2.1.

Theorem 5.1 *Let (M, g) be a connected complete Riemannian manifold which admits a nonconstant solution of Eq. (5.1) with critical points. Then it is isometric to $\mathbf{H}^n$.*

Note, however, that the function $f(s) = -s$ in (5.1) is not coercive. Indeed, we can take the antiderivative $h(s) = -\frac{1}{2}s^2$. Then 0 is the only zero of h and $h(0) = f(0) = 0$. More to the point, the solution $u = \sinh t$ of the equation

$$u'' - u = 0 \quad (5.2)$$

has no critical point. Hence Theorems 2.1 and 3.3 are not applicable here. We consider instead the dimension of the solution space for (5.1). First we would like to mention the following recent result ([7, Theorem 1.3]).

Theorem 5.2 (He, Petersen and Wylie) *Let (M, g) be a simply connected complete Riemannian manifold of dimension n . Assume that the dimension of the space of smooth solutions of the equation*

$$\nabla dw + \tau wg = 0 \quad (5.3)$$

is at least $n + 1$, where $\tau = -1$ or 0 . Then (M, g) is isometric to $\mathbf{H}^n$ if $\tau = -1$, and isometric to $\mathbf{R}^n$ if $\tau = 0$.

We obtain the following two theorems and their Euclidean analogs which improve this result substantially and are indeed optimal.

Definition 5.3 For a Riemannian manifold (M, g) and a smooth function $f(s)$ let $W_f(M, g)$ denote the space of smooth solutions of the generalized Obata equation (1.2) on (M, g) . We set $W_h(M, g) = W_{-s}(M, g)$.

Theorem 5.4 *Let (M, g) be a connected complete Riemannian manifold of dimension $n \geq 2$. Set $W_h = W_h(M, g)$. Then $\dim W_h \geq n$ iff (M, g) is isometric to $\mathbf{H}^n$. Consequently, if $\dim W_h \geq n$, then $\dim W_h = n + 1$.*

Theorem 5.5 *Let (M, g) be a connected complete Riemannian manifold of dimension $n \geq 2$. Set $W_h = W_h(M, g)$. Then $\dim W_h = n - 1$ iff (M, g) has constant sectional curvature -1 and is diffeomorphic to $\mathbf{R}^{n-1} \times \mathbf{S}^1$ (equivalently, $\pi_1(M) = \mathbf{Z}$). More precisely, $\dim W_h = n - 1$ iff (M, g) is isometric to $\mathbf{H}_{\cosh}^{n-1}(\mathbf{S}^1(\rho))$ or $\mathbf{H}_{\cosh}^{n-2}(\mathbf{H}_{\exp}(\mathbf{S}^1(\rho)))$ for some $\rho > 0$. (The former contains a closed geodesic, while the latter does not.)*

The following theorem characterizes lower dimensions of $W_h(M, g)$.

Theorem 5.6 *Let (M, g) be a connected complete Riemannian manifold of dimension n and $1 \leq k \leq n - 1$. Then $\dim W_h(M, g) \geq k$ iff M is isometric to $\mathbf{H}_{\cosh}^k(N, g_N)$ or $\mathbf{H}_{\cosh}^{k-1}(\mathbf{H}_{\exp}(N, g_N))$ for a connected complete Riemannian manifold (N, g_N) of dimension $n - k$.*

The definition of the manifolds involving cosh and exp in the above theorems is given below.

Definition 5.7 Consider a Riemannian manifold (N, g_N) .

- (1) The cosh warping $\mathbf{H}_{\cosh}(N, g_N)$ of (N, g_N) is defined to be the warped product $\mathbf{R} \times_{\cosh}(N, g_N)$. More precisely, it is defined to be $(\tilde{N}, g_{\tilde{N}})$, where $\tilde{N} = \mathbf{R} \times N$ and $g_{\tilde{N}} = dr^2 + \cosh^2 r \cdot g_N$.
- (2) The exponential warping $\mathbf{H}_{\exp}(N, g) = (\tilde{N}, g_{\tilde{N}})$ is defined by $\tilde{N} = \mathbf{R} \times N$, $g_{\tilde{N}} = dr^2 + e^{2r} g_N$.
- (3) $\mathbf{H}_{\phi}^k(N, g_N)$ denotes the k -fold iteration of the ϕ warping, where $\phi = \cosh$ or $\exp$.

More explicitly, $\mathbf{H}_{\cosh}^k(N, g_N) = (\tilde{N}, g_{\tilde{N}})$ with $\tilde{N} = \mathbf{R}^k \times N$ and

$$g_{\tilde{N}} = dr_1^2 + \cosh^2 r_1 dr_2^2 + \cdots + \cosh^2 r_1 \cdots \cosh^2 r_{k-1} dr_k^2 + \cosh^2 r_1 \cdots \cosh^2 r_k \cdot g_N. \quad (5.4)$$

Note that $\mathbf{H}_{\cosh}^k(\mathbf{H}^m) = \mathbf{H}^{m+k}$, $\mathbf{H}_{\cosh}^{n-1}(\mathbf{R}) = \mathbf{H}^n$ and $\mathbf{H}_{\cosh}^{n-1}(\mathbf{S}^1(\rho))$ is hyperbolic and diffeomorphic to $\mathbf{R}^{n-1} \times \mathbf{S}^1$, where $\mathbf{S}^1(\rho)$ denotes the circle of radius ρ . The last manifold contains a closed geodesic of length $2\pi\rho$. Furthermore, $\mathbf{H}_{\cosh}(N, g_N)$ is hyperbolic if (N, g_N) is hyperbolic, and $\mathbf{H}_{\exp}(N, g_N)$ is hyperbolic if (N, g_N) is flat.

Lemma 5.8 *Let (N, g_N) be a Riemannian manifold and ϕ a positive smooth function on an interval I . Define $g = dr^2 + \phi^2(r)g_N$ on $I \times N$. For a vector field X on $I \times N$ which is tangent to N we write $X = \sum a^\alpha \epsilon_\alpha$, where ϵ_α is a local orthonormal frame of (N, g_N) . Let ∇^N be the Levi-Civita connection of g_N . Then there hold w. r. t. g*

$$\nabla_{\mathbf{v}} X = \nabla_{\mathbf{v}}^N X - (X \cdot \mathbf{v})\phi'\phi^{-1} \frac{\partial}{\partial r} \ \& \ \nabla_{\frac{\partial}{\partial r}} X = \sum (\phi a^\alpha)' \epsilon_\alpha, \quad (5.5)$$

where $\mathbf{v}$ is tangent to N . Consequently, there holds $\nabla_{\frac{\partial}{\partial r}}(\psi X) = (\phi\psi)'X$, if X is independent of r . On the other hand, there hold for $\mathbf{v}$ tangent to N

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0 \ \& \ \nabla_{\mathbf{v}} \frac{\partial}{\partial r} = \phi'\phi^{-1} \mathbf{v}. \quad (5.6)$$

It follows in particular that each function $w = \int \phi(r)$ satisfies the generalized Obata (1.2) equation with $f(s) = -\phi'(r(s))$, where $r(s)$ is the inverse of $\phi'(r)$.

Proof These formulas are well known and follow from easy calculations. Using them it is easy to derive $\nabla dw = \phi'(r)g$, hence the claim regarding w follows. (This has already been observed in [6]. $\square$)

Lemma 5.9 *Let (N, g) be a Riemannian manifold and w_0 a smooth solution of the Eq. (5.3) on (N, g) . Then the functions $\sinh r$ and $\cosh r \cdot w_0$ are solutions of (5.3) on $\mathbf{H}_{\cosh}(N, g)$. Consequently, $\dim W_h(\mathbf{H}_{\cosh}(N, g)) = \dim(N, g) + 1$.*

Proof Set $\phi(r) = \cosh r$. Consider the function $w = \cosh r \cdot w_0 = \phi(r)w_0$. We have

$$\nabla w = \phi'w_0 \frac{\partial}{\partial r} + \phi^{-1} \nabla^N w_0. \quad (5.7)$$

By Lemma 5.8 we then deduce

$$\nabla_{\frac{\partial}{\partial r}} \nabla w = \phi''w_0 \frac{\partial}{\partial r} + (\phi\phi^{-1})' \nabla^N w_0 = w \frac{\partial}{\partial r} \quad (5.8)$$

and for $\mathbf{v}$ tangent to N

$$\begin{aligned} \nabla_{\mathbf{v}} \nabla w &= \phi'(\mathbf{v}w_0) \frac{\partial}{\partial r} + (\phi')^2 \phi^{-1} w_0 \mathbf{v} + \phi^{-1} \nabla_{\mathbf{v}}^N \nabla^N w_0 - (\mathbf{v}w_0) \phi' \frac{\partial}{\partial r} \\ &= (\phi')^2 \phi^{-1} w_0 \mathbf{v} + \phi^{-1} w_0 \mathbf{v} = w \mathbf{v}. \end{aligned} \quad (5.9)$$

The first claim of the lemma follows. Using (5.6) we also deduce that $\sinh r$ is a solution of (5.3). (This also follows from Lemma 5.8.) $\square$

The following lemma is an immediate consequence.

Lemma 5.10 *The functions $\sinh r_1, \cosh r_1 \sinh r_2, \cosh r_1 \cosh r_2 \sinh r_3, \dots, \cosh r_1 \cdots \cosh r_{k-1} \sinh r_k$ and $\cosh r_1 \cdots \cosh r_k \cdot w_0$ (c.f. (5.4)) on $\mathbf{H}_{\cosh}^k(N, g_N)$ are solutions of the Eq. (5.3), where w_0 is an arbitrary solution of (5.3) on (N, g_N) .*

Proof of Theorem 5.6 The “if” part is provided by Lemma 5.9. Now we prove the “only if” part. We set $W = W_h(M, g)$ and assume $\dim W \geq k$. The Eq. (5.3) is linear, hence W is a vector space. As in [7] we choose $p_0 \in M$ and consider the evaluation map $\Phi_{p_0} : W \rightarrow \mathbf{R} \times T_{p_0}M$, $\Phi_{p_0}(w) = (w(p_0), \nabla w(p_0))$. By Lemma 2.6, its kernel is trivial. Hence it is injective, as observed in [7]. Let $T_{p_0}M$ stand for $\{0\} \times T_{p_0}M$. We have $\dim(\text{im } \Phi + T_{p_0}M) \leq n + 1$. Set $\dim(\text{im } \Phi \cap T_{p_0}M) = l$. Then $\dim(\text{im } \Phi + T_{p_0}M) = l + (k - l) + (n - l) = k + n - l$. It follows that

$$k + n - l \leq n + 1, \quad (5.10)$$

whence $l \geq k - 1$.

(1) First assume $k \geq 2$. Then $l \geq 1$. Choose $w_0 \in W$ such that $\Phi(w_0)$ is nonzero and belongs to $\text{im } \Phi \cap T_{p_0}M$. Then $w_0(p_0) = 0, \nabla w_0(p_0) \neq 0$. Set $N = w_0^{-1}(0)$. We apply Theorem 4.5. There holds $\mu = 0$. We choose w_0 such that $\alpha = |\nabla w_0(p_0)| = 1$. Then we have

$$h(s) = 1 - 2 \int_0^s (-\tau) d\tau = 1 + s^2. \quad (5.11)$$

On the other hand, the formula (4.9) implies $|\nabla w_0|^2 = |\nabla s|^2 = h(s) = 1 + s^2 \geq 1$. It follows that $I = \mathbf{R}$ and F is a diffeomorphism from $\mathbf{R} \times N$ onto M . Setting $r = \sinh^{-1} s$ we deduce

$$F^*g = dr^2 + \cosh^2 r \cdot g_N. \quad (5.12)$$

Moreover, $\sinh^{-1}$ maps $\mathbf{R}$ diffeomorphically onto $\mathbf{R}$. It follows that (M, g) is isometric to $\mathbf{H}_{\cosh}^k(N, g_N)$.

By Lemma 4.1 (or Eq. (5.12)), N is totally geodesic. Hence the restriction of each function in W to N satisfies the Eq. (5.3) on (N, g_N) . Set $E_0 = \{(a, \mathbf{v}) \in \mathbf{R} \times T_p M : \mathbf{v} \perp \nabla w_0(p)\}$ and $W_0 = \{w : w \in \Phi_p^{-1}(E_0)\}$. Then $\dim W_0 = k - 1$. Note that $\nabla w(p_0)$ is tangent to N for each $w \in W_0$. Applying the injectivity of the evaluation map for (N, g_N) at p_0 we infer that the restriction to N maps W_0 injectively into $W_h(N, g_N)$. It follows that $\dim W_h(N, g_N) \geq k - 1$. If $k - 1 \geq 2$, we can repeat the above argument. By induction, we deduce that (M, g) is isometric to $\mathbf{H}_{\cosh}^{k-1}(N, g_N)$ for a connected complete manifold (N, g_N) of dimension $n - k + 1$. Moreover, $\dim W_h(N, g_N) \geq 1$.

(2) Next we consider the case $k \geq 1$. Note that the case in (1) is reduced to this case at the last stage. Choose a function $w_0 \in W_h(M, g)$ and a point $p_0 \in M$ such that

$(w(p_0), \nabla w(p_0)) \neq (0, 0)$. Then w is nonconstant, otherwise it would be zero. If $\nabla w(p) = 0$, we can apply Theorem 5.1 to infer that (M, g) is isometric to $\mathbf{H}^n = \mathbf{H}_{\cosh}(\mathbf{H}^{n-1})$. If $w(p) = 0$, we can apply the above argument in (1) to deduce that (M, g) is isometric to $\mathbf{H}_{\cosh}(N, g_N)$ for a connected complete (N, g_N) . Finally we consider the case $\mu \equiv w(p) \neq 0, \nabla w(p) \neq 0$. As before, we choose w such that $|\nabla w(p)| = 1$. We again apply Theorem 4.5. There holds

$$h(s) = 1 + 2 \int_{\mu}^s \tau d\tau = s^2 + 1 - \mu^2. \quad (5.13)$$

There holds $|\nabla w|^2 \geq 1 - \mu^2$. If $\mu^2 < 1$, then F is a diffeomorphism from $\mathbf{R} \times N$ onto M . Setting $\sigma = \sqrt{1 - \mu^2}$ and $r = \sinh^{-1}(t/\sigma)$ we deduce

$$F^*g = dr^2 + \sigma^2 \cosh^2 r \cdot g_N. \quad (5.14)$$

Replacing g_N by $\sigma^2 g_N$ we then infer that (M, g) is isometric to $\mathbf{H}_{\cosh}(N, g_N)$. If $\mu = 1$, we set $s = e^r$ as long as $s > 0$ and deduce

$$F^*g = dr^2 + e^{2r} g_N. \quad (5.15)$$

Since $(\mathbf{R} \times N, dr^2 + e^{2r} g_N)$ is connected and complete, we infer that $s > 0$ everywhere and F coupled with the function e^r maps $\mathbf{R} \times N$ diffeomorphically onto M . It follows that (M, g) is isometric to $\mathbf{H}_{\exp}(N, g_N)$. If $\mu = -1$, we set $s = -e^r$ and arrive at the same conclusion. If $\mu^2 > 1$, we set $\sigma^2 = \mu^2 - 1$ and $r = \cosh^{-1}(s/\sigma)$ to deduce

$$F^*g = dr^2 + \sigma^2 \sinh^2 r \cdot g_N. \quad (5.16)$$

There holds $|\nabla w|^2 = \sigma^2 \sinh^2 r$. We infer that, as $r \rightarrow 0$, each geodesic in the r direction converges to a critical point of w . By Theorem 5.4 we conclude that (M, g) is isometric to $\mathbf{H}^n = \mathbf{H}_{\cosh}(\mathbf{H}^{n-1})$.

(3) Combining the above two cases we then arrive at the claim of the theorem. $\square$

Proof of Theorem 5.5 By Theorem 5.6 we infer that (M, g) is isometric to the manifold $\mathbf{H}_{\cosh}^{n-2}(\mathbf{H}_{\phi}(N, g_N))$ with $\phi = \cosh$ or $\exp$. Then N is 1-dimensional, and hence is isometric to either $\mathbf{R}$ or $\mathbf{S}^1(\rho)$ for some $\rho > 0$. Moreover, $\mathbf{H}_{\phi}(N, g_N)$ is hyperbolic in either case of ϕ . It follows that $\mathbf{H}_{\cosh}^{n-2}(\mathbf{H}_{\phi}(N, g_N))$ is hyperbolic. $\square$

Proof of Theorem 5.4 We follow the arguments in (1) of the proof of Theorem 5.6 and deduce that (M, g) is isometric to $\mathbf{H}_{\cosh}^{n-1}(N, g_N)$, where (N, g_N) is either $\mathbf{R}$ or $\mathbf{S}^1(\rho)$ for some $\rho > 0$. We also deduce that $\dim W_h(N, g_N) \geq 1$. But $W_h(\mathbf{S}^1(\rho)) = \{0\}$. Indeed, each $w \in W_h(\mathbf{S}^1(\rho))$ can be given by the formula $w(\theta) = A \cosh(\theta + \theta_0)$ for some A and θ_0 . Then $A = 0$ because w must be 2π -periodic. It follows that (M, g) is isometric to $\mathbf{H}_{\cosh}^{n-1}(\mathbf{R}) = \mathbf{H}^n$. $\square$

6 Euclidean Versions

In this section we consider the following Euclidean analog of Obata's equation

$$\nabla dw = 0. \quad (6.1)$$

Let $W_e(M, g)$ denote the space of smooth solutions of (6.1) on (M, g) .

Theorem 6.1 *Let (M, g) be an n -dimensional connected complete Riemannian manifold. Then $\dim W_e(M, g) \geq n$ iff (M, g) is isometric to $\mathbf{R}^n$ or $\mathbf{R}^{n-1} \times \mathbf{S}^1(\rho)$ for some $\rho > 0$.*

Next we characterize the general situation $\dim W_e(M, g) \geq k$, $2 \leq k \leq n$. Note $\dim W_e(M, g) \geq 1$ because of the presence of nonzero constant solutions.

Theorem 6.2 *Let (M, g) be an n -dimensional connected complete Riemannian manifold. Then $\dim W_e(M, g) \geq k$ for $2 \leq k \leq n$ iff (M, g) is isometric to $\mathbf{R}^{k-1} \times N$ where N is a connected complete Riemannian manifold.*

Proof We argue as in (1) of the proof of Theorem 5.6. Choose w and apply Theorem 4.5 in the same way as there. Now the function h is given by $h(s) = 1$ and hence

$$F^*g = ds^2 + g_N. \quad (6.2)$$

It follows that (M, g) is isometric to $\mathbf{R} \times N$ with the product metric $ds^2 + g_N$. We apply the restriction to N and induction as before to arrive at the desired conclusion. $\square$

Proof of Theorem 6.1 This theorem is an immediate consequence of Theorem 6.3. $\square$

Another Euclidean version of the Obata equation is the following one.

$$\nabla dw + g = 0. \quad (6.3)$$

The following theorem is a special case of Theorem 3.4. (It should be known.)

Theorem 6.3 *Let (M, g) be a connected complete Riemannian manifold of dimension $n \geq 1$, which admits a smooth solution of the Eq. (6.3). Then (M, g) is isometric to the Euclidean space $\mathbf{R}^n$.*

7 Rigidity Theorems for the General Eqs. (1.3)–(1.5)

As mentioned in Sect. 4, a warped product analysis of the Eq. (1.4) has been presented in [6]. The focus in this section is on a delicate aspect of this equation concerning the relation between z and w around critical points of w . We first formulate a few lemmas.

Lemma 7.1 *Let (M, g) be a Riemannian manifold and w and z two smooth functions on M satisfying (1.4). Let p_0 be a critical point of w and γ a unit-speed geodesic with $\gamma(0) = p_0$. Then there holds*

$$\nabla w \circ \gamma = \frac{dw(\gamma(t))}{dt} \gamma'. \quad (7.1)$$

Hence the parts of γ where $w(\gamma(t))' \neq 0$ are gradient flow lines of w .

Proof The argument in the proof of Lemma 2.6 can easily be adapted to the general Eq. (1.4). $\square$

Lemma 7.2 *Assume the same as in the above lemma. Assume in addition that $\dim M \geq 2$ and M is connected. If w has two sufficiently close critical points, then it is a constant function. Consequently, if w is a nonconstant function, then its critical points are isolated.*

Proof Let p_0 be a critical point of w and $B_r(p_0)$ a convex geodesic ball. Assume that there is another critical point p_1 of w in $B_r(p_0)$. Let γ be the geodesic passing through p_0 and p_1 . For $p \in B_r(p_0) - \gamma$, let γ_1 be the shortest geodesic from p_0 to p , and γ_2 the shortest geodesic from p_1 to p . Then they meet at p nontangentially. By Lemma 7.1, we deduce that p is a critical point of w . By continuity, every point in $B_r(p_0)$ is a critical point of w . The connectedness of M then implies that every point of M is a critical point of w , and hence w is a constant. $\square$

Lemma 7.3 *Assume the same as in Lemma 7.1 and $\dim M \geq 2$. Then each isolated critical point of w is nondegenerate. Equivalently, $z(p_0) \neq 0$ at each isolated critical point p_0 of w .*

Proof Let p_0 be an isolated critical point of w . By Lemma 7.1, the geodesics starting at p_0 (with p_0 deleted) are reparametrizations of gradient flow lines of w until they reach critical points of w . It follows that small geodesic spheres with center p_0 are perpendicular to the gradient of w and hence are level sets of w . Consider a unit-speed geodesic $\gamma(t)$ with $\gamma(0) = p_0$. Set $u(t) = w(\gamma(t))$. By the just derived fact and the connectedness of small geodesic spheres, $u(t)$ is independent of the choice of γ . By the arguments in the proof of Theorem 2.1 we infer that the metric formula (2.19) holds true. If $u''(0) = 0$, then the metric would be degenerate at p_0 . Indeed, the $(n-1)$ -dimensional volume of $\partial B_r(p_0)$ would be bounded from above by cr^{n-1} . We conclude that $u''(0) \neq 0$, which implies that $\nabla dw|_{p_0}$ is nonsingular. $\square$

Theorem 7.4 *Let w and z be two smooth functions on a connected complete Riemannian manifold (M, g) of dimension $n \geq 2$, such that (1.4) holds true. Then there is a unique smooth function f on the image of w such that $z = f(w)$. If w is nonconstant and has at least one critical point, then M is diffeomorphic to either $\mathbf{R}^n$ or $\mathbf{S}^n$. Moreover, (M, g) is isometric to $M_{f,\mu}$ for some μ , where f is determined by the relation $z = f(w)$. If w has no critical point, then (M, g) is isometric to the warped product $\mathbf{R} \times_\phi (N, g_N)$ for a connected complete Riemannian manifold (N, g_N) and a positive smooth function ϕ on $\mathbf{R}$.*

Proof The case of w being a constant is trivial. So we assume that w is a nonconstant function. Based on the above lemmas, it is clear that the proof of Theorem 4.5 can be carried over to yield the same conclusions as there, with the formula (4.9). We set $f(s) = z(F(s, p))$ for a fixed $p \in N$, and $s \in I$, the interior of the image I_w of w . Then $z = f(w)$ on $\Omega = F(I \times N)$. Obviously, f is smooth on I . It is also clear from the properties of F that f extends continuously to I_w . Let $\mu \in \partial I_w \cap I_w$. We claim that f is smooth at μ . Let $p_0 \in M$ be the unique critical point of w such that $w(p_0) = \mu$. Let γ be a unit-speed geodesic with $\gamma(0) = p_0$ and set $u(t) = w(\gamma(t))$ as before. There holds $u(0) = \mu$, $u'(0) = 0$. By Lemma 7.3, $u''(0) \neq 0$. Since u is independent of the choice of γ , it is an even function. Hence $u = \mu + at^2 + Q(t^2)$ with $a \neq 0$, where Q is a smooth function with $Q(0) = Q'(0) = 0$. By the implicit function theorem we deduce $t^2 = H(u)$ for a smooth function H and small t and u . On the other hand, $v(t) = z(\gamma(t))$ is also an even function because $z = f(w)$. It follows that $v = G(t^2)$ for a smooth function G . We arrive at $v = G(H(u))$. Clearly, there holds $f(s) = G(H(s))$, and hence f is smooth at μ .

With the above conclusion, the remaining part of the theorem follows from Theorem 2.1. $\square$

Remark In general, the above function f is not smooth at critical values of w if the dimension of M is 1. Consider, for example, $M = \mathbf{R}$, $w = x^3$ and $z = -3x^2$. Then w and z satisfy (1.4) and $f(s) = -3s^{2/3}$.

An obvious equivalent formulation of the above result is the following theorem.

Theorem 7.5 *Let w be a smooth function on a connected complete Riemannian manifold (M, g) of dimension $n \geq 2$, such that (1.5) holds true. Then there is a unique smooth function f on the image of w such that $-\frac{1}{n}\Delta w = f(w)$. If w is nonconstant and has at least one critical point, then M is diffeomorphic to either $\mathbf{R}^n$ or $\mathbf{S}^n$. Moreover, (M, g) is isometric to $M_{f,\mu}$ for some μ , where f is determined by the relation $-\frac{1}{n}\Delta w = f(w)$. If w has no critical point, then (M, g) is isometric to the warped product $\mathbf{R} \times_\phi (N, g_N)$ for a connected complete Riemannian manifold (N, g_N) and a positive smooth function ϕ on $\mathbf{R}$.*

With the help of Theorem 7.4, all the results in Sects. 3 and 4 concerning the generalized Obata equation (1.2) extend to the more general Eq. (1.3). We formulate this in a combined theorem as follows.

Theorem 7.6 *The theorems in Sects. 3 and 4 continue to hold if the Eq. (1.2) is replaced by the Eq. (1.3), and the conditions on $f(s)$ are assumed to hold for $f(s, p)$ for each fixed p .*

To illustrate the more precise details, we also state an individual case explicitly as one example.

Theorem 7.7 *Let (M, g) be a connected complete Riemannian manifold and f a smooth function on $I \times M$ for an interval I . Assume that $f(\cdot, p)$ is nondegenerately coercive for each $p \in M$ and that there is a nonconstant solution of (1.3) on (M, g) . Then M is diffeomorphic to $\mathbf{S}^n$. Moreover, if $\dim M \geq 2$, then (M, g) is isometric to $M_{f_0,\mu}$, where $f_0 = f(\cdot, p_0)$ and $\mu = w(p_0)$ for a critical point p_0 of w .*

Finally, we state an easy consequence of the above results which provides a different angle of view.

Theorem 7.8 *Let (M, g) be a connected complete Riemannian manifold and f a smooth function on $I \times M$ for an interval I . Assume that there is a smooth solution w of the Eq. (1.3). Then $f(s, \cdot)$ is a constant function on M for each value s of w . Consequently, no solution w of (1.3) can exist for a generic f .*

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