On the Space of Conics on Complete Intersections

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Abstract We get sharp degree bound for generic smoothness and connectedness of the space of lines and conics in low degree complete intersections which generalizes the old work about Fano scheme of lines on hypersurfaces. As a consequence, we prove that for a Fano complete intersection X with index ≥ 2 , the 1-Griffiths group generated by algebraic 1-cycles homologous to 0 modulo algebraic equivalence is trivial, which is a conjecture for general rationally connected varieties.

Keywords Rational curves · Fano varieties · Cycles

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1 Introduction

Definition 1.1 For a variety X in $\mathbb{P}^n$, we define the Fano scheme of lines ([1]) $F(X)$ to be the Hilbert scheme of lines $\text{Hilb}_{t+1}(X)$ on X , and $C(X)$ to be the Hilbert scheme of conics (degree 2 rational curve, union of two lines, or double of one line) $\text{Hilb}_{2t+1}(X)$ on X .

The following result was first proved in [1], which was simplified with (general) sharp bound by Professor Kollár in his book [5].

Theorem 1.2 *Let X be a hyper-surface of degree d . Then,*

$F(X)$ is empty for general X if $d > 2n - 3$ and is smooth with dimension $2n - 3 - d$ for general X if $d \leq 2n - 3$. If $d \leq 2n - 4$ (with the only exceptional case as quadric surface), then $F(X)$ is always connected, thus is irreducible for general X .

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Further in [5], Professor J. Kollár raised the following question.

Question 1.3 Let X be a hyper-surface of degree d . Is there a similar degree bound for the generic smoothness and connectedness of $C(X)$?

We noted that this type of question is related to a folklore conjecture in the topic of rationally connected varieties.

Conjecture 1.4 *Let X be a smooth projective rationally connected variety over $\mathbb{C}$. Is the 1-Griffiths group of 1-cycles homologous to 0 modulo algebraic equivalence trivial?*

We generalize Theorem 1.2 to the case of lines and conics in low degree complete intersections, and hence solving Question 1.3. We note here that Bonavero and Horing already obtained related results using cohomology computations on Grassmannians, see Section 2.3 and Proposition 3.1 in [2].

The main result of this paper is the following theorem.

Theorem 1.5 *Let X be a complete intersection of type $d_1, \dots, d_c$. Then,*

- *$F(X)$ is empty for general X if $d_1 + d_2 + \dots + d_c + c > 2n - 2$, smooth with dimension $2n - 2 - (d_1 + d_2 + \dots + d_c + c)$ for general X if $d_1 + d_2 + \dots + d_c + c \leq 2n - 2$. If $d_1 + d_2 + \dots + d_c + c \leq 2n - 3$ (with the only exceptional case as quadric surface), then $F(X)$ is always connected, and thus is irreducible for general X .*
- *$C(X)$ is empty for general X if $d_1 + d_2 + \dots + d_c + c/2 > (3n - 2)/2$, smooth with dimension $3n - 1 - 2(d_1 + d_2 + \dots + d_c) - c$ for general X if $d_1 + d_2 + \dots + d_c + c/2 \leq (3n - 2)/2$, and always connected if $d_1 + d_2 + \dots + d_c + c/2 \leq (3n - 3)/2$ (and the rest $n \leq 3$ case is trivial), in particular $\bar{M}_{0,0}(X, 2)$ is connected.*

A crucial part of the proof is some generic dimension count on conics, the rest is the same as the proof in [5].

We have an easy corollary:

Corollary 1.6 *On a complete intersection X with type $(d_1, \dots, d_c)$ and $d_1 + d_2 + \dots + d_c + c/2 \leq (3n - 2)/2$, all degree 2 rational curves on X can be degenerated to union of two lines, or double of one line.*

Another application of Theorem 1.5, as suggested by Professor C. Voisin, is the solution to Conjecture 1.4 for Fano complete intersection with index at least 2.

Theorem 1.7 *For a Fano complete intersection X with type $(d_1, \dots, d_c)$ with index ≥ 2 , the 1-Griffiths group generated by algebraic 1-cycles homologous to 0 modulo algebraic equivalence is trivial.*

The readers are encouraged to find more applications of Theorem 1.5. By the same method, it should not be hard to get the (general) sharp bound for hyper-quadrics in complete intersections:

Question 1.8 Generalize Theorem 1.5 to the case of hyper-quadrics in complete intersections.

2 Preliminaries

2.1 Connectedness Lemma

First, we need a connectedness lemma for later use.

Lemma 2.1 *Let X, Y be noetherian schemes, with proper morphism $f : X \rightarrow Y$, assume Y is smooth and $\pi^1(Y) = \{1\}$. Let $S(f) \subset X$ be the locus where f is not smooth. If $\text{codim}(S(f), X) \geq 2$, then f has connected fibers.*

Proof Consider the Stein factorization of f :

$$X \xrightarrow{g} X' \xrightarrow{h} Y,$$

where g is proper with connected fibers and h is finite. By the assumption, we know that h is generically étale (when the characteristic is p , smoothness of the map at general point implies that $K(X')/K(Y)$ is a separable extension). Let $D \subset Y$ be its branch divisor, if $D \neq 0$, f will not be smooth in $f^{-1}\{D\} \subset X$ which is of codimension 1, contradicting the assumption. So $D = 0$, and h will be étale outside a codimension ≥ 2 locus E , and hence

$$h|_{X' - h^{-1}\{E\}} : X' - h^{-1}\{E\} \rightarrow Y - E$$

is a étale cover of $Y - E$. Now by [3, Corollary X.3.3]

$$\pi^1(Y - E) = \pi^1(Y) = \{1\}$$

which means $\deg(h) = 1$, so $f = g$ has connected fibers. $\square$

2.2 Smoothness of $C(\mathbb{P}^n)$

The following is an easy exercise, which we will need later:

Exercise 2.2 $\text{Hilb}_{2t+1}(\mathbb{P}^n)$ is isomorphic to $\text{Proj}(E)$ of a rank-6 vector bundle E over $\text{Grass}(3, n+1)$, and in particular smooth and irreducible.

2.3 Generic Dimension Count on Conics

2.3.1 Smooth Conic Case

Lemma 2.3 *Let m be the multiplication map*

$$H^0(\mathbb{P}^1, \mathcal{O}(1)) \times \left(\bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(d_i - 1)) \right) \rightarrow \bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(d_i)).$$

For codimension 1 linear subspace

$$V \subset \bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(d_i)),$$

let

$$m^{-1}(V) := \{f \in \bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(d_i - 1)), f \cdot H^0(\mathbb{P}^1, \mathcal{O}(1))\} \subset V\}.$$

Let $(f_{i,1}, \dots, f_{i,c}), i = 1, \dots, \dim m^{-1}(V)$ be a basis of $m^{-1}(V)$.

Then, either of the followings is true:

- There is a point $p \in \mathbb{P}^1$ such that

$$\text{rank}\{f_{i,j}(p)\} < c.$$

- $m^{-1}(V)$ is of codimension 2 in $\bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(d_i - 1))$.

Proof First, we note that $H^0(\mathbb{P}^1, \mathcal{O}(d))$ can be identified with all polynomials spanned by $\{1, x, x^2, \dots, x^d\}$. Let $x_1, \dots, x_c$ be generators for each $H^0(\mathbb{P}^1, \mathcal{O}(d_i))$, and $H^0(\mathbb{P}^1, \mathcal{O}(1))$ spanned by $\{1, x\}$, and then m is simply defined by

$$m(x, x_1^{i_1}, \dots, x_c^{i_c}) = (x_1^{i_1+1}, \dots, x_c^{i_c+1}).$$

Denote the coordinate of x_i^j to be $x_{i,j}$, and let V be defined by equation

$$\sum_{1 \leq i \leq d_1} a_{1,i} x_{1,i} + \sum_{1 \leq i \leq d_2} a_{2,i} x_{2,i} \cdots + \sum_{1 \leq i \leq d_c} a_{c,i} x_{c,i} = 0,$$

then $m^{-1}(V)$ is defined by

$$\sum_{1 \leq i \leq d_1-1} a_{1,i} x_{1,i} + \sum_{1 \leq i \leq d_2-1} a_{2,i} x_{2,i} \cdots + \sum_{1 \leq i \leq d_c-1} a_{c,i} x_{c,i} = 0$$

and

$$\sum_{1 \leq i \leq d_1-1} a_{1,i} x_{1,i+1} + \sum_{1 \leq i \leq d_2-1} a_{2,i} x_{2,i+1} \cdots + \sum_{1 \leq i \leq d_c-1} a_{c,i} x_{c,i+1} = 0.$$

So it will be of codimension 2 unless there is $p = [s, t] \in \mathbb{P}^1$ such that

$$s \cdot a_{i,j} = t \cdot a_{i,j+1}.$$

We may assume that $p = [0, 1]$, then $m^{-1}(V)$ is defined by

$$a_{1,d_1} x_{1,d_1-1} + \dots a_{c,d_c} x_{c,d_c-1} = 0.$$

The matrix $\{f_{i,j}(p)\}$ will has elements as

$$f_{i,j}(p) = x_{j,d_1-1}(f_i)$$

the defining equation of $m^{-1}(V)$ simply implies that this $\dim m^{-1}(V) \times c$ matrix satisfies a non-trivial linear relation in its j -index, so its rank will be strictly less than c . $\square$

Lemma 2.4 *Let m_2 (resp. m_4) be the multiplication map*

$$H^0(\mathbb{P}^1, \mathcal{O}(2)) \times \left(\bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(2d_i - 2)) \right) \rightarrow \bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(2d_i))$$

resp.

$$H^0(\mathbb{P}^1, \mathcal{O}(4)) \times \left(\bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(2d_i - 4)) \right) \rightarrow \bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(2d_i)).$$

For codimension 1 linear subspace

$$V \subset \bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(2d_i)),$$

let

$$m_2^{-1}(V) := \{f \in \bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(2d_i - 2)), f \cdot H^0(\mathbb{P}^1, \mathcal{O}(2)) \subset V\}$$

resp.

$$m_4^{-1}(V) := \{f \in \bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(2d_i - 4)), f \cdot H^0(\mathbb{P}^1, \mathcal{O}(4)) \subset V\}.$$

Let $(f_{i,1}, \dots, f_{i,c}), i = 1, \dots, \dim m_2^{-1}(V)$ (resp. $\dim m_4^{-1}(V)$) be a basis of $m_2^{-1}(V)$ (resp. $m_4^{-1}(V)$).

Then, either of the followings is true:

- There are 2 (resp. 4) points $p_1, p_2 \in \mathbb{P}^1$ (resp. $p_1, p_2, p_3, p_4 \in \mathbb{P}^1$) such that

$$\text{rank}\{f_{i,j}(p_i)\} < c$$

(p_i 's might coincide with each other, resulting in even lower codimension of $m_*^{-1}(V)$).

- $m_2^{-1}(V)$ (resp. $m_4^{-1}(V)$) is of codimension 3 (resp. 5) in $\bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(2d_i - 2))$ (resp. $\bigoplus_{i=1, \dots, c} H^0(\mathbb{P}^1, \mathcal{O}(2d_i - 4))$).

Proof We only prove the m_2 part here: $H^0(\mathbb{P}^1, \mathcal{O}(2d))$ can be identified with all polynomials spanned by $\{1, x, x^2, \dots, x^{2d}\}$. Let $x_1, \dots, x_c$ be generators for each $H^0(\mathbb{P}^1, \mathcal{O}(2d_i))$, and $H^0(\mathbb{P}^1, \mathcal{O}(2))$ spanned by $\{1, x, x^2\}$.

Then, m_2 is simply defined by

$$m_2(x^i, x_1^{i_1}, \dots, x_c^{i_c}) = (x_1^{i_1+i}, \dots, x_c^{i_c+i}).$$

Denote the coordinate of x_i^j as $x_{i,j}$, and let V be defined by equation:

$$\sum_{1 \leq i \leq 2d_1} a_{1,i} x_{1,i} + \sum_{1 \leq i \leq 2d_2} a_{2,i} x_{2,i} \cdots + \sum_{1 \leq i \leq 2d_c} a_{c,i} x_{c,i} = 0.$$

Then $m_2^{-1}(V)$ is defined by

$$\begin{aligned} \sum_{1 \leq i \leq 2d_1-2} a_{1,i} x_{1,i} + \sum_{1 \leq i \leq 2d_2-2} a_{2,i} x_{2,i} \cdots + \sum_{1 \leq i \leq 2d_c-2} a_{c,i} x_{c,i} &= 0, \\ \sum_{1 \leq i \leq 2d_1-2} a_{1,i} x_{1,i+1} + \sum_{1 \leq i \leq 2d_2-2} a_{2,i} x_{2,i+1} \cdots + \sum_{1 \leq i \leq 2d_c-2} a_{c,i} x_{c,i+1} &= 0 \end{aligned}$$

and

$$\sum_{1 \leq i \leq 2d_1-2} a_{1,i} x_{1,i+2} + \sum_{1 \leq i \leq 2d_2-2} a_{2,i} x_{2,i+2} \cdots + \sum_{1 \leq i \leq 2d_c-2} a_{c,i} x_{c,i+2} = 0.$$

So it will be of codimension 3 unless there is $p_1 = [s_1, t_1], p_2 = [s_2, t_2] \in \mathbb{P}^1$ such that

$$a_{i,j+2} = (s_1/t_1 + s_2/t_2) \cdot a_{i,j+1} + s_1/t_1 \cdot s_2/t_2 \cdot a_{i,j};$$

then, everything would be clear after solving this recurrence relation. $\square$

2.3.2 Union of Two Lines Case

Lemma 2.5 Let D be union of two $\mathbb{P}^1$'s glued at a point with nodal singularity, $\mathcal{O}(d)$ be the line bundle gluing from $\mathcal{O}(d)$'s of two pieces.

Let m'_1 (resp. m'_2) be the multiplication map

$$H^0(D, \mathcal{O}(1)) \times \left(\bigoplus_{i=1, \dots, c} H^0(D, \mathcal{O}(d_i - 1)) \right) \rightarrow \bigoplus_{i=1, \dots, c} H^0(D, \mathcal{O}(d_i))$$

resp.

$$H^0(D, \mathcal{O}(2)) \times \left(\bigoplus_{i=1, \dots, c} H^0(D, \mathcal{O}(d_i - 2)) \right) \rightarrow \bigoplus_{i=1, \dots, c} H^0(D, \mathcal{O}(d_i)).$$

For codimension 1 linear subspace

$$V \subset \bigoplus_{i=1, \dots, c} H^0(D, \mathcal{O}(d_i)),$$

let

$$m_1'^{-1}(V) := \{f \in \bigoplus_{i=1, \dots, c} H^0(D, \mathcal{O}(d_i - 1)), f \cdot H^0(D, \mathcal{O}(1)) \subset V\}$$

resp.

$$m_2'^{-1}(V) := \{f \in \bigoplus_{i=1, \dots, c} H^0(D, \mathcal{O}(d_i - 2)), f \cdot H^0(D, \mathcal{O}(2)) \subset V\}.$$

Let $(f_{i,1}, \dots, f_{i,c}), i = 1, \dots, \dim m_1'^{-1}(V)$ (resp. $\dim m_2'^{-1}(V)$) be a basis of $m_1'^{-1}(V)$ (resp. $m_2'^{-1}(V)$).

Then, either of the followings is true:

- There are 2 (resp. 4) points $p_1, p_2 \in D$ (resp. $p_1, p_2, p_3, p_4 \in D$) such that

$$\text{rank}\{f_{i,j}(p_i)\} < c$$

(p_i 's might coincide with each other, resulting in even lower codimension of $m_*'^{-1}(V)$).

- $m_1'^{-1}(V)$ (resp. $m_2'^{-1}(V)$) is of codimension 3 (resp. 5) in $\bigoplus_{i=1, \dots, c} H^0(D, \mathcal{O}(d_i - 1))$ (resp. $\bigoplus_{i=1, \dots, c} H^0(D, \mathcal{O}(d_i - 2))$).

Proof The sections here come from glueing each section from two lines, and the argument in Lemma 2.3 applies. $\square$

2.3.3 Double of a Line Case

Lemma 2.6 Let L be double $\mathbb{P}^1$ embedded in $\mathbb{P}^2$ as double of a line, $\mathcal{O}(d)$ be the line bundle restricting from $\mathcal{O}(d)$ of $\mathbb{P}^2$.

Let m_1'' (resp. m_2'') be the multiplication map

$$H^0(L, \mathcal{O}(1)) \times \left(\bigoplus_{i=1, \dots, c} H^0(L, \mathcal{O}(d_i - 1)) \right) \rightarrow \bigoplus_{i=1, \dots, c} H^0(L, \mathcal{O}(d_i))$$

resp.

$$H^0(L, \mathcal{O}(2)) \times \left(\bigoplus_{i=1, \dots, c} H^0(L, \mathcal{O}(d_i - 2)) \right) \rightarrow \bigoplus_{i=1, \dots, c} H^0(L, \mathcal{O}(d_i)).$$

For codimension 1 linear subspace

$$V \subset \bigoplus_{i=1, \dots, c} H^0(L, \mathcal{O}(d_i)),$$

let

$$m''^{-1}_1(V) := \{f \in \bigoplus_{i=1, \dots, c} H^0(L, \mathcal{O}(d_i - 1)), f \cdot H^0(L, \mathcal{O}(1))\} \subset V$$

resp.

$$m''^{-1}_2(V) := \{f \in \bigoplus_{i=1, \dots, c} H^0(L, \mathcal{O}(d_i - 2)), f \cdot H^0(L, \mathcal{O}(2))\} \subset V.$$

Let $(f_{i,1}, \dots, f_{i,c}), i = 1, \dots, \dim m''^{-1}_1(V)$ (resp. $\dim m''^{-1}_2(V)$) be a basis of $m''^{-1}_1(V)$ (resp. $m''^{-1}_2(V)$).

Then, either of the followings is true:

- There are 1 (resp. 2) point $p_1 \in \mathbb{P}^1$ (resp. $p_1, p_2 \in \mathbb{P}^1$) such that

$$\text{rank}\{f_{i,j}(p_i)\} < c$$

(p_i 's might coincide with each other, resulting in even lower codimension of $m''^{-1}_*(V)$).

- $m''^{-1}_1(V)$ (resp. $m''^{-1}_2(V)$) is of codimension 3 (resp. 5) in $\bigoplus_{i=1, \dots, c} H^0(L, \mathcal{O}(d_i - 1))$ (resp. $\bigoplus_{i=1, \dots, c} H^0(L, \mathcal{O}(d_i - 2))$).

Proof In this case, we have $H^0(L, \mathcal{O}(d)) \cong H^0(\mathbb{P}^1, \mathcal{O}(d)) \oplus t \cdot H^0(\mathbb{P}^1, \mathcal{O}(d - 1))$, where t is a formal variable with $t^2 = 0$; now the multiplication map $H^0(L, \mathcal{O}(d_1)) \times H^0(L, \mathcal{O}(d_2)) \rightarrow H^0(L, \mathcal{O}(d_1 + d_2))$ is simply sending $(s_1 + ts_2, s'_1 + ts'_2)$ to $s_1s'_1 + t(s_2s'_1 + s_1s'_2)$ and the rest follows from the argument in Lemma 2.3. $\square$

3 Proof

Theorem 3.1 Let X be a complete intersection of type $d_1, \dots, d_c$. Then,

- $F(X)$ is empty for general X if $d_1 + d_2 + \dots + d_c + c > 2n - 2$, smooth with dimension $2n - 2 - (d_1 + d_2 + \dots + d_c + c)$ for general X if $d_1 + d_2 + \dots + d_c + c \leq 2n - 2$. If $d_1 + d_2 + \dots + d_c + c \leq 2n - 3$ (with the only exceptional case as quadric surface), then $F(X)$ is always connected, thus is irreducible for general X .
- $C(X)$ is empty for general X if $d_1 + d_2 + \dots + d_c + c/2 > (3n - 2)/2$, smooth with dimension $3n - 1 - 2(d_1 + d_2 + \dots + d_c) - c$ for general X if $d_1 + d_2 + \dots + d_c + c/2 \leq (3n - 2)/2$, always connected if $d_1 + d_2 + \dots + d_c + c/2 \leq (3n - 3)/2$ ($n \leq 3$ case is trivial), in particular $\bar{M}_{0,0}(X, 2)$ is connected.

Proof For line case, the proof is almost word to word translation of the arguments in Section V.4 of [5], with the [5, Lemma 4.3.11] used there replaced by Lemma 2.3 above.

For conic case, consider the incidence correspondence

$$\mathcal{I} \subset \text{Hilb}_{2t+1}(\mathbb{P}^n) \times H := H_{n,d_1} \times H_{n,d_2} \times \cdots \times H_{n,d_c},$$

where H parametrizes all complete intersections of degree $(d_1, \dots, d_c)$ and $\mathcal{I}$ is defined by all hyper-surfaces defining the complete intersections that vanish at the conic $[C] \in \text{Hilb}_{2t+1}$.

We have two projections

$$\mathcal{I} \xrightarrow{p_1} \text{Hilb}_{2t+1}(\mathbb{P}^n), \mathcal{I} \xrightarrow{p_2} H.$$

One can see that p_1 has smooth fibers as product of linear spaces in H_{n,d_i} 's. So by smoothness of $\text{Hilb}_{2t+1}(\mathbb{P}^n)$, we have that $\mathcal{I}$ is smooth.

Now one has to show that p_2 has connected fibers: For any $C \subset X$ in $\mathcal{I}$, let C be defined by

$$x_0 = \cdots = x_{n-3} = L_2(x_{n-2}, x_{n-1}, x_n) = 0,$$

where L_2 is a possibly reducible degree 2 polynomial in the plane $x_0 = \cdots = x_{n-3} = 0$ and let X be defined by $f_1 = \cdots = f_c = 0$. Then, $C \subset X$ implies that

$$f_i = f_{i,0}x_0 + f_{i,1}x_1 + \cdots + f_{i,n-3}x_{n-3} + f_{i,n-2}L_2.$$

Let $Z(\mathcal{I}) \subset \mathcal{I}$ be the set where $\text{rank}\{f_{i,j}\}(p) < c$ for some point $p \in C$, where $j = 0, 1, \dots, n-3$. It is easy to see that these two conditions are independent of the $x_0, \dots, x_{n-3}, L_2$ we choose, and

$$\text{codim}(Z(\mathcal{I}), \mathcal{I}) \geq 2$$

if $n \geq 4$ and $d_1 + d_2 + \cdots + d_c + c/2 \leq (3n-1)/2$. ($n \leq 3$ case is trivial).

Let $S(\mathcal{I}) \subset \mathcal{I}$ be the locus where p_2 is not smooth along C .

In the locus $\mathcal{I} - Z(\mathcal{I})$ where X is smooth at all points of C , consider the normal bundle sequence:

$$0 \rightarrow N_{C|X} \rightarrow \mathcal{O}_C(1)^{n-2} \oplus \mathcal{O}_C(2) \xrightarrow{\{f_{i,j}\}} \bigoplus_{i=1,\dots,c} \mathcal{O}_C(d_i) \rightarrow 0.$$

Then, $Z(\mathcal{I}) - S(\mathcal{I})$ is contained in the locus where

$$H^1(C, N_{C|X}) = 0,$$

which is equivalent to

$$H^0(C, \mathcal{O}(d_i)) = \sum_j f_{i,j} H^0(C, \mathcal{O}(1)) + f_{i,n-2} H^0(C, \mathcal{O}(2)).$$

So $S(\mathcal{I})$ is cut out by

$$\cup_V \{C \subset X \mid \sum_{i,j} f_{i,j} H^0(C, \mathcal{O}(1)) + \sum_i f_{i,n-2} H^0(C, \mathcal{O}(2)) \subset V\},$$

where V varies through all possible codimension 1 linear subspaces of

$$\bigoplus_{i=1,\dots,c} H^0(C, \mathcal{O}(d_i)).$$

Now Lemmas 2.4–2.6 show that

$$\text{codim}(S(\mathcal{I}), X - Z(\mathcal{I})) \geq 3n - 1 - 2(d_1 + \dots + d_c) - c,$$

and the general fiber of p_2 has dimension $3n - 2 - 2(d_1 + \dots + d_c) - c$, so the general smoothness follows.

When

$$3n - 1 - 2(d_1 + \dots + d_c) - c \geq 2,$$

namely $d_1 + \dots + d_c + c \leq (3n - 3)/2$, we have

$$\text{codim}(S(\mathcal{I}), \mathcal{I} - Z(\mathcal{I})) \geq 3n - 1 - 2(d_1 + \dots + d_c) - c \geq 2$$

and

$$\text{codim}(Z(\mathcal{I}), \mathcal{I}) \geq 2,$$

which implies the connectedness of fibers of p_2 by Lemma 2.1. $\square$

4 Application to Conjecture 1.4

We begin with a proposition which is inspired by Professor Jason Starr, which is also applied in [6].

Theorem 4.1 (By Professor Jason Starr). *Let X be a (possibly singular) complete intersection of type $(d_1, \dots, d_c)$ with $d_1 + \dots + d_c \leq n - 1$ in $\mathbb{P}(V) \cong \mathbb{P}^n$. Then, every rational curve on X is algebraically equivalent to an effective sum of lines.*

We will need the following connectedness result of Hartshorne.

Lemma 4.2 (Hartshorne, [4]). *Let X be a subscheme in $\mathbb{P}^N$ defined by M homogeneous polynomials, and let Y be a closed subset of X with dimension less than $N - M - 1$. Then, $X - Y$ is connected.*

Proof of Theorem 4.1 by Professor Jason Starr We use induction on the degree of the considered rational curve. The degree 1 case is obvious.

Assume the statement is true for all rational curves whose degree is less than $e(\geq 2)$. Fix a copy C of $\mathbb{P}^1$, and consider the parameter space

$$\mathbb{P} := \mathbb{P}(\text{Hom}(V^*, H^0(C, \mathcal{O}(e)))).$$

If we choose homogeneous coordinates $x_0, \dots, x_n$ on $\mathbb{P}(V)$, then $\mathbb{P}$ parametrizes $(n+1)$ -tuples $[u_0, \dots, u_n]$ of homogeneous degree e polynomials on C .

Let X be a complete intersection of codimension c of type $(d_1, \dots, d_c)$ defined by nonzero degree d_i homogeneous polynomials F_i ($1 \leq i \leq c$) on $\mathbb{P}(V)$. Define $\mathbb{P}_X$ to be the closed subset of $\mathbb{P}$ parameterizing $[u_0, \dots, u_n]$ such that $F_i(u_0, \dots, u_n)$ equals 0 for all $1 \leq i \leq c$. The subvariety $\mathbb{P}_X$ is defined by homogeneous polynomials of degree

$$\underbrace{d_1, \dots, d_1}_{ed_1+1}, \dots, \underbrace{d_c, \dots, d_c}_{ed_c+1}$$

on $\mathbb{P}$.

Let $B \subset \mathbb{P}$ be the closed subvariety parameterizing tuples $[u_0, \dots, u_n]$ where $\text{span}\{u_0, \dots, u_n\}$ in $H^0(C, \mathcal{O}(e))$ is 1-dimensional, i.e., every pair u_i, u_j satisfies a scalar linear relation. The codimension of B in $\mathbb{P}$ is ne .

Let $D \subset \mathbb{P}$ be the closed subvariety parameterizing $(n+1)$ -tuples $[u_0, \dots, u_n]$ that have a common zero in C . Clearly, D contains B . The point outside D parametrizes a degree e map from C to $\mathbb{P}^n$.

Now, we see that $\mathbb{P}_X \cap B$ parameterizes a degree e polynomial in two variables (up to scaling) and a point in X . So its dimension is $n - c + e$. Then, in the situation of Lemma 4.2, take Y to be $\mathbb{P}_X \cap B$ and $N = ne + n + e$, $M = e(d_1 + \dots + d_c) + c$. The condition $n - c + e < N - M - 1$ is simply

$$e(n - d_1 - d_2 - \dots - d_c) > 1.$$

Therefore, $\mathbb{P}_X - B$ is connected for all $e \geq 2$.

Now, let $u = [u_0, \dots, u_n]$ be any element of $\mathbb{P}_X - D$, e.g., a parameterized, degree e morphism from C to X . Let u' be a degree e multiple cover of a line in X . Then, u' is algebraically equivalent to a union of lines. Since $\mathbb{P}_X - B$ is connected, there exists a connected curve mapping to $\mathbb{P}_X - B$ whose image connects these two points. We may assume that it is a chain of irreducible components, with each consecutive pair meeting in a single node.

If none of the nodes is contained in D , then after deleting finitely many points from the curve, none of which are the nodes, we may assume that the entire connected curve is disjoint from D . Thus, this connected curve parameterizes a family of stable maps. Then, u is algebraically equivalent to u' , and hence algebraically equivalent to a union of lines.

So it remains to consider the case when some of the nodes lie in D but not in B . Consider the first such node, starting from the component containing u . Approaching

the node from the component closer to u , we have a family of elements in the complement of D which approaches a point of D , i.e., we have a family of stable maps which approaches a point of D . But this is a point of D which is not in B . So the corresponding rational map from C to X is non-constant of degree strictly less than e . It follows that the limiting stable map has at least one component of positive degree which is strictly less than e , i.e., this is a point in the boundary. But then, each component of this boundary stable map has degree strictly less than e , hence by induction each component is algebraically equivalent to a union of lines. Thus, u is also algebraically equivalent to a union of lines. $\square$

Now we can prove the following theorem.

Theorem 4.3 (Suggested by Professor Claire Voisin). *For a Fano complete intersection X with type $(d_1, \dots, d_c)$ with index ≥ 2 , the 1-Griffiths group generated by algebraic 1-cycles homologous to 0 modulo algebraic equivalence is trivial.*

Proof Assume that $\mathcal{C}$ is an algebraic 1-cycles which is homologous to zero. By the main theorem of [7] and [6], $\mathcal{C}$ is algebraically equivalent to a $\mathbb{Z}$ -linear combination of rational curves on X , and by Proposition 4.1, it is further algebraically equivalent to

$$\sum_i n_i \mathcal{C}_i$$

where n_i values are integers and $\mathcal{C}_i$ values are conics on X , or algebraically equivalent to

$$\sum_i m_i \mathcal{L}_i$$

where m_i values are integers and $\mathcal{L}_i$ values are lines on X . By Lefschetz hyperplane theorem, the $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ and is determined by the degree map, so that $\mathcal{C}$ is homologous to zero is equivalent to

$$\sum_i n_i = \sum_i m_i = 0$$

and finally by Theorem 1.5, we can assume that $\mathcal{C}_i$ values are all equal to $\mathcal{C}_1$ and $\mathcal{L}_i$ values are all equal to $\mathcal{L}_1$, so finally $\mathcal{C}$ is algebraically equivalent to $\sum_i n_i \mathcal{C}_1 = \sum_i m_i \mathcal{L}_1 = 0$. $\square$

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