The Sasaki–Ricci Flow on Sasakian 3-Spheres

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Abstract We show that on a Sasakian 3-sphere the Sasaki–Ricci flow initiating from a Sasakian metric of positive transverse scalar curvature converges to a gradient Sasaki–Ricci soliton. We also show the existence and uniqueness of gradient Sasaki–Ricci soliton on each Sasakian 3-sphere.

Keywords Sasaki–Ricci flow · Sasaki–Ricci soliton · Weighted Sasakian structure

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1 Introduction

Recently Sasaki–Einstein geometry, as an odd-dimensional cousin of Kähler–Einstein geometry, has played an important role in the Ads/CFT correspondence. The important problem in Sasaki–Einstein geometry is certainly to find Sasaki–Einstein metrics. Boyer, Galicki and their collaborators found many new Sasaki–Einstein metrics on quasi-regular Sasakian manifolds [4, 6, 7]. The first class of irregular Sasaki–Einstein metrics was found by Gauntlett, Martelli, Sparks and Waldram in [14, 15]. Another class of irregular Sasaki–Einstein metrics was found in [12] by studying the Sasaki–Ricci solitons.

In order to systematically study the existence of the Sasaki–Einstein metrics, we introduced in [21] a Sasaki–Ricci flow, motivated by the work of Lovrić,

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Min-Oo and Ruh [19] in transverse Riemannian geometry. The Sasaki–Ricci flow exploits the transverse structure of Sasakian manifolds. We have showed in [21] the well-posedness of the Sasaki–Ricci flow and global existence of the flow, together with a Cao [9] type result, i.e., the convergence in the case of negative and null basic first Chern class. For the more precise definitions see Sect. 2.

In the paper we want to consider the Sasaki–Ricci flow in the 3-dimensional case. 3-dimensional Sasakian manifolds are of the lowest dimension for Sasakian geometry. Nevertheless, the Sasakian structures on 3-dimensional manifolds are quite rich and have been well studied [1, 2, 13, 16]. A compact 3-manifold admits a Sasakian structure if and only if it is diffeomorphic to one of the following standard models: (i) S^3/Γ , (ii) Nil^3/Γ , (iii) $\widetilde{SL(2, \mathbb{R})}/\Gamma$, where Γ is a discrete subgroup of the isometry group with respect to the standard Sasakian metric in each case. Nil^3 is for the 3 nilpotent real matrices (i.e., the Heisenberg group), and $\widetilde{SL(2, \mathbb{R})}$ is the universal cover of $SL(2, \mathbb{R})$. Note that cases (i), (ii), and (iii) correspond to Sasakian manifolds with positive, null, and negative basic first Chern classes, respectively.

As mentioned above that on the Sasakian manifolds with null or negative basic first Chern classes the Sasaki–Ricci flow converges. Therefore, we focus in this paper on 3-dimensional Sasakian manifolds, i.e., Sasakian 3-spheres.

Let

$$S^3 = \{z \in \mathbb{C}^2, |z_1|^2 + |z_2|^2 = 1\}$$

and

$$\eta_0 = \sum_{i=1}^2 (x^i dy^i - y^i dx^i).$$

This η_0 together with the standard almost complex structure gives the standard Sasakian structure and the corresponding Sasakian manifold is the standard sphere. A weighted Sasakian structure on S^3 is given by

$$\eta_a = (a_1|z_1|^2 + a_2|z_2|^2)^{-1} \eta_0,$$

where a_1 and a_2 are any positive numbers. The Reeb vector field of the weighted Sasakian structure is

$$\xi_a = \sum_{i=1}^2 a_i \left(x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \right).$$

When a_1/a_2 is a rational number, the weighted Sasakian structure is quasi-regular; otherwise it is irregular. We know that for any Sasakian structure (S^3, η) on S^3 there is a weighted Sasakian structure such that

$$[d\eta]_B = [d\eta_a]_B,$$

where $[\cdot]_B$ is the basic homology class; for the proof see [2, Proposition 6]. See Sect. 2. The Sasakian structure (S^3, η) also has Reeb vector field ξ_a , and we say that

it is homologous to η_a . The Sasaki–Ricci flow deforms Sasakian structures within a fixed class $[\cdot]_B$.

Our main result is as follows.

Theorem 1.1 *For any initial Sasakian structure on S^3 of positive transverse scalar curvature, the Sasaki–Ricci flow converges exponentially to a gradient Sasaki–Ricci soliton.*

Moreover, we prove the existence and uniqueness of gradient Sasaki–Ricci soliton on any weighted Sasakian structure.

Theorem 1.2 *For any given homologous class of Sasakian structures on S^3 , there exists a unique gradient Sasaki–Ricci soliton.*

The proof follows closely Hamilton’s idea [18] on the Ricci flow on surfaces and relies on the Li–Yau–Harnack inequality and the entropy formula. Actually, the Sasaki–Ricci flow on Sasakian 3-spheres shares lots of properties of the Ricci flow on a 2-sphere and two-dimensional orbifolds, which was studied by Hamilton [18] and Wu [22], see also the work of Chow [10, 11]. In the expression of weighted Sasakian structures, if $a_1/a_2 = 1$, the characteristic foliation is regular and the leaf space is a 2-dimensional sphere. In this case the Sasaki–Ricci flow is equivalent to Hamilton’s Ricci flow on a 2-sphere. If $a_1/a_2 \neq 1$ is a rational number, the leaf space is a so-called bad orbifold with one or two orbifold points. Our Sasaki–Ricci flow in this case is equivalent to the Ricci flow on such orbifolds studied by Wu [22]. The key step in her proof is to establish some injectivity radius estimates and volume estimates. Note that Wu’s injectivity radius estimates are just for the orbifold points and the points in a region apart from the orbifold points at a distance. One main reason makes the injectivity radius estimate near orbifold points fail is that near a p -fold point, a shortest geodesic 1-gon may be generated by each of p copies of short geodesic segments on the universal covering. For the quasi-regular Sasakian structures, the Sasaki–Ricci flow can be reduced to the Ricci flow on bad orbifolds. Then one can apply Wu’s estimates to get the convergence of the Sasaki–Ricci flow to a soliton solution. The Ricci flow on bad orbifolds with negative curvature somewhere was also studied by Chow–Wu [11].

The remaining case of an irrational ratio a_1/a_2 is not covered by the work of Wu [22] and Chow–Wu [11]. In this case the leaf space has no manifold structure and it could be very wild. We have to work directly on the 3-dimensional manifold S^3 . Nevertheless, we manage to show the crucial volume estimate by using the Weyl tube formula.

The rest of paper is organized as follows. In Sect. 2 we recall the definitions of Sasakian manifolds, Sasaki–Ricci flow and Sasaki–Ricci soliton. In Sect. 3 we prove the convergence of the Sasaki–Ricci flow to a gradient Sasaki–Ricci soliton, by using the Harnack inequality and the entropy formula and leave the crucial volume estimates in Sect. 4. In Sect. 5 we study the gradient Sasaki–Ricci soliton explicitly.

2 Sasakian Manifolds and Sasaki–Ricci Flow

2.1 Sasakian Manifolds

For convenience of the reader, we recall briefly the definitions of Sasakian manifolds, its basic concepts and our Sasaki–Ricci flow. For more details we refer to [5, 21].

Let (M, g^M) be a Riemannian manifold, ∇^M the Levi-Civita connection of the Riemannian metric g^M , and let $R^M(X, Y)$ denote the Riemann curvature tensor of ∇^M . By a contact manifold we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is easy to check that there is a canonical vector field ξ defined by

$$\eta(\xi) = 1 \quad \text{and} \quad d\eta(\xi, X) = 0, \quad \text{for any vector field } X.$$

The vector field ξ is called the characteristic vector field or Reeb vector field. Let

$$\mathcal{D}_p = \ker \eta_p.$$

There is a decomposition of the tangential bundle TM

$$TM = \mathcal{D} \oplus L_\xi,$$

where L_ξ is the trivial bundle generated by the Reeb field ξ . A contact manifold with a Riemannian metric g^M and a tensor field Φ of type $(1, 1)$ satisfying

$$\Phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad g^M(\Phi X, \Phi Y) = g^M(X, Y) - \eta(X)\eta(Y)$$

is called an almost metric contact manifold. Such an almost metric contact manifold is called Sasakian if one of the following equivalent conditions holds:

- (1) There exists a Killing vector field ξ of unit length on M so that the Riemann curvature satisfies the condition

$$R^M(X, \xi)Y = g^M(\xi, Y)X - g^M(X, Y)\xi,$$

for any pair of vector fields X and Y on M .

- (2) The metric cone $(C(M), \bar{g}) = (\mathbb{R}_+ \times M, dr^2 + r^2 g^M)$ is Kähler.

For other equivalent definitions and the proof of the equivalence, see for instance [3]. By (2), a Sasakian manifold can be viewed as an odd-dimensional counterpart of a Kähler manifold.

A Sasakian manifold $(M, \xi, \eta, \Phi, g^M)$ is a Sasaki–Einstein manifold if g^M is an Einstein metric, i.e.,

$$\text{Ric}_{g^M} = c g^M,$$

for some constant c . Due to property (1) of the Sasaki–Einstein manifold, it is easy to see that $c = 2n > 0$. A generalized Sasaki–Einstein metric, η -Einstein manifold, is defined by

$$\text{Ric}_{g^M} = \lambda g^M + \nu \eta \otimes \eta, \tag{2.1}$$

for some constant λ and ν . It is easy to see that $\lambda + \nu = 2n$. For a recent study of η -Einstein manifolds and Sasaki–Einstein metrics, see [8, 12, 20].

2.2 Transverse Kähler Structures

In order to study the analytic aspect of Sasaki–Einstein metrics or η -Einstein manifolds, we need to consider the transverse structure of Sasakian manifolds. In this paper, we always assume that M is a Sasakian manifold with Sasakian Structure (ξ, η, g^M, Φ) . Let $\mathcal{F}_\xi$ be the characteristic foliation generated by ξ . On $\mathcal{D}$, it is naturally endowed with both a complex structure $\Phi|_{\mathcal{D}}$ and a symplectic structure $d\eta$. $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta, g^T)$ gives M a transverse Kähler structure with Kähler form $d\eta$ and transverse metric g^T defined by

$$g^T(X, Y) = d\eta(X, \Phi Y).$$

The metric g^T is clearly related to the Sasakian metric g^M by

$$g^M = g^T + \eta \otimes \eta.$$

There is a canonical quotient bundle of the foliation $\mathcal{F}_\xi$, $\nu(\mathcal{F}_\xi) = TM/L_\xi$ and an isomorphism between $\nu(\mathcal{F}_\xi)$ and $\mathcal{D}$. Let $p : TM \rightarrow \nu(\mathcal{F}_\xi)$ be the projection. g^T gives a bundle map $\sigma : \nu(\mathcal{F}_\xi) \rightarrow \mathcal{D}$ which splits the exact sequence

$$0 \rightarrow L_\xi \rightarrow TM \rightarrow \nu(\mathcal{F}_\xi) \rightarrow 0,$$

i.e., $p \circ \sigma = \text{id}$.

From the transverse metric g^T , one can define a transverse Levi-Civita connection on $\nu(\mathcal{F}_\xi)$ by

$$\nabla_X^T V = \begin{cases} (\nabla_X^M \sigma(V))^p, & \text{if } X \text{ is a section of } \mathcal{D}, \\ [\xi, \sigma(V)]^p, & \text{if } X = \xi, \end{cases} \quad (2.2)$$

where V is a section of $\nu(\mathcal{F}_\xi)$ and $X^p = p(X)$ the projection of X onto $\nu(\mathcal{F}_\xi)$ and ∇^M is the Levi-Civita connection associated to the Riemannian metric g^M on M . The transverse curvature operator is defined by

$$R^T(X, Y) = \nabla_X^T \nabla_Y^T - \nabla_Y^T \nabla_X^T - \nabla_{[X, Y]}^T$$

and transverse Ricci curvature by

$$\text{Ric}^T(X, Y) = g^M(R^T(X, e_i)e_i, Y),$$

where e_i is an orthonormal basis of $\mathcal{D}$. We remark that here we have used the identification between $\mathcal{D}$ and $\nu(\mathcal{F}_\xi)$. More precisely the transverse Ricci tensor is defined by

$$\text{Ric}^T(X, Y) = g^M(R^T(X, e_i)\sigma^{-1}(e_i), Y),$$

for $X, Y \in TM$. One can check that

$$\text{Ric}^T(X, Y) = \text{Ric}^M(X, Y) + 2g^T(X, Y). \quad (2.3)$$

A transverse Einstein metric g^T is a transverse metric satisfying

$$\text{Ric}^T = cg^T, \quad (2.4)$$

for certain constant c . It is clear that a Sasakian metric is a transverse Einstein metric if and only if it is an η -Einstein metric.

In order to introduce the Sasaki–Ricci flow, we first consider deformations of Sasakian structures which preserve the Reeb field ξ , and hence the characteristic foliation $\mathcal{F}_\xi$.

A p -form α on M is called basic if it satisfies $i(\xi)\alpha = 0$ and $\mathcal{L}_\xi\alpha = 0$. A function f is basic if and only if $\xi(f) = 0$. One can check that the exterior differential d preserves basic forms. Hence one can define the basic cohomology in a usual way. See [5]. Moreover, we consider the complexified bundle $\mathcal{D}^\mathbb{C} = \mathcal{D} \otimes \mathbb{C}$. Using the structure Φ we decompose $\mathcal{D}^\mathbb{C}$ into two subbundles $\mathcal{D}^{1,0}$ and $\mathcal{D}^{0,1}$, where $\mathcal{D}^{1,0} = \{X \in \mathcal{D}^\mathbb{C} \mid \Phi X = \sqrt{-1}X\}$ and $\mathcal{D}^{0,1} = \{X \in \mathcal{D}^\mathbb{C} \mid \Phi X = -\sqrt{-1}X\}$. Similarly, we decompose the complexified space $\Lambda_B^r \otimes \mathbb{C} = \oplus_{p+q=r} \Lambda_B^{p,q}$, where $\Lambda_B^{p,q}$ denotes the sheaf of germs of basic forms of type (p, q) . Define ∂_B and $\bar{\partial}_B$ by

$$\partial_B : \Lambda_B^{p,q} \rightarrow \Lambda_B^{p+1,q}, \quad \bar{\partial}_B : \Lambda_B^{p,q} \rightarrow \Lambda_B^{p,q+1},$$

which is the decomposition of d . Let $d_B = d|_{\Omega_B^p}$. We have $d_B = \partial_B + \bar{\partial}_B$. Let $d_B^c = \frac{1}{2}\sqrt{-1}(\bar{\partial}_B - \partial_B)$. Let $d_B^* : \Omega_B^{p+1} \rightarrow \Omega_B^p$ be the adjoint operator of $d_B : \Omega_B^p \rightarrow \Omega_B^{p+1}$. The basic Laplacian Δ_B is defined

$$\Delta_B = d_B^* d_B + d_B d_B^*.$$

Suppose that (ξ, η, Φ, g^M) defines a Sasakian structure on M . Let φ be a basic function. Put

$$\tilde{\eta} = \eta + d_B^c \varphi.$$

It is clear that

$$d\tilde{\eta} = d\eta + d_B d_B^c \varphi = d\eta + \sqrt{-1} \partial_B \bar{\partial}_B \varphi.$$

For small φ , $d\tilde{\eta}$ is non-degenerate in the sense that $\tilde{\eta} \wedge (d\tilde{\eta})^n \neq 0$. Set

$$\tilde{\Phi} = \Phi - \xi \otimes (d_B^c \varphi) \circ \Phi, \quad \tilde{g}^M = d\tilde{\eta} \circ (\text{Id} \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta}.$$

$(M, \xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g}^M)$ is also a Sasakian structure with $[d\tilde{\eta}]_B = [d\eta]_B$. It is this class of deformations we used in the definition of our Sasaki–Ricci flow.

There are other kinds of deformation. For instance, the so-called $\mathcal{D}$ -homothetic deformation is defined

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\Phi} = \Phi, \quad \bar{g}^M = ag^M + a(a-1)\eta \otimes \eta$$

for a positive constant a . Note that from an η -Einstein metric with $\lambda > -2$, one can use the $\mathcal{D}$ -homothetic deformation to get a Sasaki–Einstein metric. It was called also 0-type deformation.

A first type deformation of this Sasakian structure is a new Sasakian structure $(M, \eta', \mathcal{D}, \Phi', \xi')$, where $\eta' = f\eta$, for a positive function $f \neq \text{constant}$, and ξ' is the corresponding Reeb vector field, where $\Phi'|_{\mathcal{D}} = \Phi|_{\mathcal{D}}$. See for example, [1, 2, 13]. In this terminology our deformation is called the second type deformations. Here we would like to call them canonical deformations. A second type deformation η' of η is also called homologous to η (see [6]).

2.3 Sasaki–Ricci Flow

Let $\rho^T = \text{Ric}^T(\Phi \cdot, \cdot)$ and $\rho^M = \text{Ric}^M(\Phi \cdot, \cdot)$. ρ^T is called the transverse Ricci form. One can check that in view of (2.3) we have

$$\rho^T = \rho^M + 2d\eta. \quad (2.5)$$

ρ^T is a closed basic form and its basic cohomology class $[\rho^T]_B = c_B^1$ is the basic first Chern class. c_B^1 is called positive (negative, null resp.) if it contains a positive (negative, null resp.) representation. The transverse Einstein equation (2.4) can be written as

$$\rho^T = c d\eta, \quad (2.6)$$

for some constant c . A necessary condition for the existence of (2.6) is

$$c_B^1 = c[d\eta]_B.$$

By a $\mathcal{D}$ -homothetic deformation, it is natural to consider

$$c_B^1 = \kappa[d\eta]_B, \quad (2.7)$$

where $\kappa = 1, -1, 0$ corresponds to positive, negative and null c_B^1 .

Now we consider the following flow $(\xi, \eta(t), \Phi(t), g^M(t))$ with initial data $(\xi, \eta(0), \Phi(0), g^M(0)) = (\xi, \eta, \Phi, g^M)$:

$$\frac{d}{dt} g^T(t) = -(\text{Ric}_{g^M(t)}^T - \kappa g^T(t)), \quad (2.8)$$

or equivalently

$$\frac{d}{dt} d\eta(t) = -(\rho_{g^M(t)}^T - \kappa d\eta(t)). \quad (2.9)$$

We call (2.8) Sasaki–Ricci flow. In local coordinates the Sasaki–Ricci flow has the following form:

$$\frac{d}{dt} \varphi = \log \det(g_{i\bar{j}}^T + \varphi_{i\bar{j}}) - \log \det(g_{i\bar{j}}^T) + \kappa \varphi - F, \quad (2.10)$$

where the function F is a basic function such that

$$\rho_{g^M}^T - \kappa d\eta = d_B d_B^c F. \quad (2.11)$$

We showed in [21] that the well-posedness of the Sasaki–Ricci flow and a Cao type result, namely, if $\kappa = -1$ or 0 , then the Sasaki–Ricci flow converges to an η -Einstein metric. Unfortunately, in this case, there is no Sasaki–Einstein metric. Remark that one can obtain in the case $\kappa = -1$ a Sasaki–Einstein Lorentzian metric. The case $\kappa = 1$ is difficult. In general one can only expect to obtain a soliton type solution, namely a Sasaki–Ricci soliton. A Sasakian structure $(M, \xi, \eta, g^M, \Phi)$ is called a Sasaki–Ricci soliton if there is an Hamiltonian holomorphic vector field X with

$$\rho^T - d\eta = \mathcal{L}_X(d\eta),$$

where $\mathcal{L}_X$ is the Lie derivative. For the definition of Hamiltonian holomorphic vector field and the study of Sasaki–Ricci solitons on toric Sasakian manifolds we refer to [12].

A Sasakian manifold $(M, \xi, \eta, g^M, \Phi)$ is called quasi-regular if there is a positive integer k such that each point has a foliated coordinate chart $(U; x)$ such that each leaf of $\mathcal{F}_\xi$ passes through U at most k times, otherwise irregular. If $k = 1$ then the Sasakian manifold is called regular. Let $\mathcal{B}$ the leave space of $\mathcal{F}_\xi$. Then if M is regular if and only if $\mathcal{B}$ is a Kähler manifold and M is quasi-regular if and only if $\mathcal{B}$ is a Kähler orbifold. In these both cases, the Sasaki–Ricci flow on M is equivalent to the Kähler–Ricci flow on $\mathcal{B}$. But when the Sasakian manifold is irregular, $\mathcal{B}$ has no manifold structure.

3 Sasaki–Ricci Flow on Sasakian 3-Manifolds

Now let $(M, \xi, \eta, g^T, \Phi)$ be a closed 3-dimensional Sasakian manifold. For simplicity if there is no confusion we remove superscript T , since all quantities we are considering are transverse. Note that in 3-dimensional case $\Lambda_B^{1,1}$ is a line bundle. Since both the transverse Ricci form ρ_η and the transverse Kähler form $d\eta$ are real sections in $\Lambda_B^{1,1}$, the basic first Chern class must be positive, negative, or null. Actually, let

$$\int_M \rho_\eta \wedge \eta = \kappa \int_M d\eta \wedge \eta,$$

we must have

$$[\rho_\eta]_B = \kappa [d\eta]_B.$$

Let $(x, z = x^1 + ix^2)$ be the CR coordinates and

$$g_{ij} = d\eta \left(\frac{\partial}{\partial x^i}, \Phi \frac{\partial}{\partial x^j} \right).$$

Note that $R_{ij} = \frac{1}{2} R g_{ij}$, so we can rewrite the Sasaki–Ricci flow as

$$\frac{d}{dt} g_{ij} = (r - R) g_{ij}, \quad (3.1)$$

where r is the average of the transverse scalar curvature. For the reason mentioned in the Introduction, we now focus on the Sasaki–Ricci flow (3.1) defined on a Sasakian 3-sphere. Moreover we assume the flow (3.1) initiating from a metric of positive transverse scalar curvature. Note that the transverse scalar curvature R for a Sasakian manifold is a basic function.

Proposition 3.1 *Along the flow (3.1), we have*

$$\begin{aligned} \frac{d}{dt} \int_M d\eta \wedge \eta &= \int_M (r - R) d\eta \wedge \eta = 0; \\ \frac{d}{dt} R &= \Delta_B R + R(R - r); \\ \frac{d}{dt} \int_M R d\eta \wedge \eta &= 0. \end{aligned} \quad (3.2)$$

We now assume $R(0) > 0$. Note that on Sasakian 3-spheres, the scalar curvature $R^M = R - 2$. It follows from Proposition 3.1 that along the flow (3.1), the average transverse scalar curvature r stays as the same constant. If necessary we can make a 0-type deformation of the initial Sasakian structure, and always assume that

$$R_{\max} := \max_{x \in S^3} R(x) \geq r \geq 8. \quad (3.3)$$

It follows from the maximum principle that $R(t) > 0$. We follow Hamilton's approach [18] to prove the Harnack inequality and the entropy formula for the flow (3.1).

Theorem 3.2 *Suppose the flow (3.1) have a solution for $t < T^* (\leq +\infty)$ with $R(0) > 0$. Then for any two space-time points (x, τ) and (y, T) with $0 < \tau < T < T^*$, we have*

$$(e^{r\tau} - 1)R(x, \tau) \leq e^{\frac{1}{4}D} (e^{rT} - 1)R(y, T), \quad (3.4)$$

where

$$D = D((x, \tau), (y, T)) := \inf_{\gamma} \int_{\tau}^T |\dot{\gamma}(t)|_{g_t}^2 dt. \quad (3.5)$$

Here the infimum is taking over all piece-wisely smooth curves $\gamma(t)$, $t \in [\tau, T]$, with $\gamma(\tau) = x$ and $\gamma(T) = y$.

Proof Let $\gamma(t)$ be any piece-wisely smooth path joining x and y , and

$$L = \log R.$$

By the fact that R is a basic function, we have

$$g_t^M(\nabla^M L, \dot{\gamma}) = g_t^M(\nabla L, \dot{\gamma}) = g_t(\nabla L, \dot{\gamma}).$$

Hence

$$\frac{d}{dt}L(t, \gamma(t)) = \frac{\partial L}{\partial t} + g_t(\nabla L, \dot{\gamma}) \geq \frac{\partial L}{\partial t} - |\nabla L|_{g_t}^2 - \frac{1}{4}|\dot{\gamma}|_{g_t}^2.$$

It follows from (3.2) that

$$\frac{\partial L}{\partial t} = \Delta_B L + |\nabla L|_{g_t}^2 + (R - r).$$

Denote

$$Q = \frac{\partial L}{\partial t} - |\nabla L|_{g_t}^2 = \Delta_B L + (R - r).$$

One can compute that

$$\frac{\partial Q}{\partial t} = \Delta_B Q + 2g_t(\nabla L, \nabla Q) + 2\left|\nabla^2 L + \frac{1}{2}(R - r)g\right|_{g_t}^2 + rQ.$$

So we have

$$\frac{\partial Q}{\partial t} \geq \Delta_B Q + 2g_t(\nabla L, \nabla Q) + Q^2 + rQ,$$

which implies that

$$Q \geq \frac{-re^{rt}}{e^{rt} - 1}.$$

Therefore

$$\frac{d}{dt}L(t, \gamma(t)) \geq \frac{-re^{rt}}{e^{rt} - 1} - \frac{1}{4}|\dot{\gamma}|_{g_t}^2.$$

Taking $\gamma(t)$ to be a path achieving the minima D , we get

$$L(T, y) - L(\tau, x) \geq \int_{\tau}^T \left(\frac{-re^{rt}}{e^{rt} - 1} - \frac{1}{4}|\dot{\gamma}|_{g_t}^2 \right) dt = -\log \frac{e^{rT} - 1}{e^{r\tau} - 1} - \frac{1}{4}D. \quad \square$$

The following entropy formula for the flow (3.1) is an analog of Hamilton's entropy formula [18]. For the explicit computation we refer to [23].

Theorem 3.3 *Along the flow (3.1) with $R(0) > 0$, the integral $\int_M R \log R d\mu$ is non-increasing, where $d\mu = d\eta \wedge \eta$.*

Proof Let f be the basic function defined by

$$\Delta_B f = R - r,$$

and

$$M_{ij} = \nabla_i \nabla_j f - \frac{1}{2}(R - r)g_{ij}.$$

Then we have

$$\begin{aligned} \frac{d}{dt} \int_M R \log R \, d\mu &= \int_M (R - r)^2 \, d\mu - \int_M \frac{|\nabla R|_{g_t}^2}{R} \, d\mu \\ &= - \int_M \frac{|\nabla R + R \nabla f|_{g_t}^2}{R} \, d\mu - 2 \int_M |M_{ij}|_{g_t}^2 \, d\mu. \quad \square \end{aligned}$$

In order to combine the Harnack inequality (3.4) and the entropy formula to obtain uniform upper and lower bounds of R , we shall need a crucial volume estimate, see Lemma 3.4 below. To make the statement of Lemma 3.4, we first introduce some notations.

For any two points $x, y \in S^3$ and any transverse metric g , we define the transverse distance by

$$d_g(x, y) = \inf_{\gamma} \int_{\gamma} \left| \frac{d}{ds} \gamma(s) \right|_g \, ds, \quad (3.6)$$

here $\gamma(s)$ is any piece-wisely smooth curve joining x to y . For any $p \in S^3$, we denote

$$V_p(d, g) = \{q \in S^3 \mid d_g(p, q) \leq d\}.$$

Lemma 3.4 *Let (S^3, η, g) be a Sasakian 3-sphere with transverse metric g and positive transverse scalar curvature. Then there exists a positive constant C_0 , independent of second deformations of (η, g) , such that for any point $p \in S^3$, we have*

$$\text{Vol}_{g^M} \left(V_p \left(\frac{\pi}{2\sqrt{R_{\max}}}, g \right) \right) \geq \frac{C_0}{R_{\max}}. \quad (3.7)$$

Proof We will prove this Lemma in next section by using Weyl’s tube formula. We treat quasi-regular and irregular Sasakian 3-spheres separately. For quasi-regular Sasakian 3-spheres, we use essentially Wu’s injectivity radius estimate [22], see Proposition 4.3. For the case of irregular Sasakian 3-spheres, see Proposition 4.5. $\square$

Theorem 3.5 *Along the flow (3.1) with $R(0) > 0$, there exist constants $c_1 > 0$ and $c_2 < \infty$ such that*

$$c_1 \leq R(t) \leq c_2.$$

Proof Without loss of generality, let

$$\tau \geq 1, \quad T = \tau + \frac{1}{2R_{\max}(\tau)}.$$

It follows from (3.2) that $R_{\max}(T) \leq 2R_{\max}(\tau)$. Then by the definition of the flow (3.1), $g_t, t \in [\tau, T]$, are equivalent, i.e., there exist uniform positive constants δ_1, δ_2 such that

$$\delta_1 g(T) \leq g(t) \leq \delta_2 g(T).$$

Let $\gamma(t)$ be the shortest curve, joining x and y , with respect to $g(T)$. By the definition (3.5) of D , we have

$$D((x, \tau), (y, T)) \leq \delta_2 \int_{\tau}^T |\dot{\gamma}|_{g(T)}^2 dt = \delta_2 \frac{d_{g(T)}(x, y)^2}{T - \tau}. \quad (3.8)$$

Taking $x = y = p$ be a point where $R_{\max}(\tau)$ is achieved, it follow from the Harnack inequality (3.4) that

$$R_{\max}(T) \geq \frac{e^{r\tau} - 1}{e^{rT} - 1} R_{\max}(\tau). \quad (3.9)$$

We shall apply the Lemma 3.4 with

$$g = g(T), \quad d = \frac{\pi}{2\sqrt{R_{\max}(T)}}.$$

It follows from (3.8) and (3.9) that there exists a positive constant $C(r)$ such that

$$D((p, \tau), (q, T)) \leq C(r)\delta_2, \quad \text{for } q \in V_p\left(\frac{\pi}{2\sqrt{R_{\max}(T)}}, g(T)\right).$$

By the Harnack inequality (3.4) again, there exists a positive constant $C(r, \delta_2)$ such that

$$R(q, T) \geq C(r, \delta_2) R_{\max}(\tau), \quad \text{for } q \in V_p\left(\frac{\pi}{2\sqrt{R_{\max}(T)}}, g(T)\right).$$

Note that

$$R \log R \geq -\frac{1}{e}.$$

Integrating over S^3 at time T , we get

$$\begin{aligned} \int_{S^3} \left(R \log R + \frac{1}{e} \right) d\mu_{g^M(T)} &\geq \int_{V_p\left(\frac{\pi}{2\sqrt{R_{\max}(T)}}, g(T)\right)} R \log R d\mu_{g^M(T)} \\ &\geq C(r, \delta_2) R_{\max}(\tau) \log(C(r, \delta_2) R_{\max}(\tau)) \frac{C_0}{R_{\max}(T)} \\ &\geq \frac{1}{2} C_0 C(r, \delta_2) \log(C(r, \delta_2) R_{\max}(\tau)). \end{aligned}$$

Now by the entropy formula, we see that there exists a positive constant C_1 such that

$$R_{\max}(\tau) \leq C_1.$$

One can use the upper bound of R to conclude a lower bound of R . Since the volume is constant and R has a upper bound, by Lemma 3.4 we see that the transverse diameter must be bounded from above. Otherwise, we would have too much volume. Let x_0 be a point with $R(x_0, \tau) \geq r$. For $\tau \geq 1$, $g_t, t \in [\tau, \tau + 1]$, are equivalent. Hence

it follows from the upper bound of the transverse diameter that there exists a positive constant C such that

$$D((\tau, x_0), (\tau + 1, x)) \leq C, \quad \forall x.$$

It follows from the Harnack inequality that $R(\tau + 1)$ has a uniform lower bound. $\square$

We now show that the flow (3.1) converges to a gradient Sasaki–Ricci soliton. For a Sasakian 3-sphere (S^3, η) , let f be the basic function defined by

$$\Delta_B f = R - r,$$

and

$$M_{ij} = \nabla_i \nabla_j f - \frac{1}{2} \Delta_B f g_{ij}.$$

Then (S^3, η) is a gradient Sasaki–Ricci soliton if and only if

$$M_{ij} = 0.$$

We will show that the Sasaki–Ricci flow (3.1) converges to a solution satisfying $M_{ij} = 0$, which means the Sasaki–Ricci soliton solution is generated by diffeomorphisms related to $X = -\frac{1}{2} \nabla f$.

Theorem 3.6 *Along the Sasaki–Ricci flow (3.1) with $R(0) > 0$, we have*

$$|M_{ij}|^2 \leq C e^{-ct}.$$

Hence M_{ij} converges to zero exponentially.

Proof It is a direct consequence of the evolution equation of $|M_{ij}|^2$:

$$\frac{\partial}{\partial t} |M_{ij}|^2 = \Delta_B |M_{ij}|^2 - 2 |\nabla_k M_{ij}|^2 - 2R |M_{ij}|^2. \quad \square$$

So we have proved the following.

Theorem 3.7 *The Sasaki–Ricci flow (3.1) on a Sasakian 3-sphere with $R(0) > 0$ converges exponentially to a gradient Sasaki–Ricci soliton.*

4 Volume Estimates

In this section, we show Lemma 3.4. We use a Weyl type tube formula to prove the volume estimate (3.7). We treat quasi-regular Sasakian 3-spheres and irregular Sasakian structures separately. We first recall Weyl’s tube formula.

Let P^q be a q -dimensional embedded closed submanifold in M^n . A tube $T(P, r)$ of radius $r \geq 0$ about P is the set

$$T(P, r) = \{x \in M \mid \text{dist}_{g^M}(x, P) \leq r\}.$$

A hypersurface of the form

$$P_t = \{x \in T(P, r) \mid \text{dist}_{g^M}(x, P) = t\}$$

is called the tubular hypersurface at a distance t from P . Let $A_P^M(t)$ denote the $(n-1)$ -dimensional area of P_r , and $V_P^M(r)$ denote the n -dimensional volume of $T(P, r)$.

Let ν denote the normal bundle of P , and $\exp_\nu$ be the exponential map. Then we define $\text{minfoc}(P)$ to be the supremum of r such that

$$\exp_\nu : \{(p, v) \in \nu, |v| \leq r\} \rightarrow T(P, r)$$

is a diffeomorphism. For the exponential map $\exp_\nu$, we would like to note the following well-known fact: any geodesic $\gamma(t)$ in a Sasakian (S^3, g^M) with $\dot{\gamma}(0) \in \mathcal{D}_{\gamma(0)}$ must be horizontal.

Proposition 4.1 *Let $\gamma(t)$ be a geodesic in (S^3, g^M) with*

$$\gamma(0) = p, \quad \dot{\gamma}(0) \in \mathcal{D}_p.$$

Then we have

$$\dot{\gamma} \in \mathcal{D}_{\gamma(t)}.$$

Proof Let p be a given point in S^3 and $\gamma(t)$ be a geodesic of arc-lengthly parameterized through p in (S^3, g^M) . Set

$$\dot{\gamma} = H + V,$$

where H is the horizontal part and V is the vertical part. Then we have

$$\frac{d}{dt} g^M(\dot{\gamma}, \xi) = g^M(\dot{\gamma}, \nabla_{\dot{\gamma}}^M \xi) = g^M(\dot{\gamma}, \Phi H) = g^M(H, \Phi H) = 0.$$

□

Let $S^{n-q-1}(v_p)$ denote the unit sphere in v_p and $d\mu_\nu$ be the standard volume n -form on the normal bundle ν . One can introduce the following function on the normal bundle:

$$\Theta_u(p, t) = \frac{\exp_\nu^*[d\mu_M(\exp_\nu(p, tu))]}{d\mu_\nu(p, tu)}$$

for

$$0 \leq t < \text{minfoc}(P), \quad u \in S^{n-q-1}(v_p).$$

For $0 < t, r < \text{minfoc}(P)$, we have

$$A_P^M(t) = t^{n-q-1} \int_P \int_{S^{n-q-1}(v_p)} \Theta_u(p, t) du dp, \quad (4.1)$$

and

$$V_P^M(r) = \int_0^r A_P^M(t) dt. \quad (4.2)$$

Let $\sigma(x)$ be the distance function from P to x and

$$N = \nabla^M \sigma$$

be the formal outward unit normal vector field of $P_{\sigma(x)}$. On the tubular hypersurfaces P_t , let

$$Su = \nabla_u^M N,$$

here $u \in TP_t$, be the shape operator. The shape operator S satisfies the Riccati differential equation

$$S'(t) = -S(t)^2 + R_N^M, \quad (4.3)$$

here

$$S' = \nabla_N^M S, \quad R_N^M u = R^M(N, u)N, \quad u \in TP_t.$$

We also have

$$\Theta'_u(t) = \left(\text{tr} S(t) - \frac{n-q-1}{t} \right) \Theta_u(t), \quad (4.4)$$

here we omit the parameter $p \in P$. Now we can introduce a Weyl type tube formula which we will use in this section, see for instance [17].

Theorem 4.2 *Suppose that $P \subset M^n$ is a q -dimensional closed submanifold and the sectional curvature of M satisfies*

$$K^M \leq \lambda.$$

Then for $0 < r < \text{minfoc}(P)$, we have

$$\begin{aligned} A_P^M(r) \geq & \frac{2\pi^{\frac{n-q}{2}}}{\Gamma(\frac{n-q}{2})} \sum_{c=0}^{\lfloor \frac{q}{2} \rfloor} \frac{k_{2c}(R^P - R^M)}{(n-q)(n-q+2) \cdots (n-q+2c-2)} \\ & \cdot (\cos(r\sqrt{\lambda}))^{q-2c} \left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}} \right)^{n-q+2c-1}, \end{aligned} \quad (4.5)$$

here

$$k_0(R^P - R^M) = \int_P d\mu_P.$$

Corollary 1 *Let P be a closed embedding curve P in (S^3, g^M) and*

$$K^M \leq \lambda.$$

Then for $0 < r < \min \text{foc}(P)$, we have

$$A_P^M(r) \geq \pi L(P) \frac{\sin(2r\sqrt{\lambda})}{\sqrt{\lambda}}. \quad (4.6)$$

In the sequel we let

$$\lambda = K_{\max}^M.$$

Note that

$$R_{\max} = R_{\max}^M + 2 = 2K_{\max}^M(\mathcal{D}) + 6.$$

Then the assumption (3.3) is equivalent to

$$K_{\max}^M(\mathcal{D}) \geq 1.$$

The proof of Lemma 3.4 relies on a classification result [2, 13] of all Sasakian structures on S^3 . Let

$$S^3 = \{z \in \mathbb{C}^2, |z_1|^2 + |z_2|^2 = 1\}$$

and

$$\eta_0 = \sum_{i=1}^2 (x^i dy^i - y^i dx^i).$$

This η_0 together with the standard almost complex structure give the standard Sasakian structure and the corresponding Sasakian manifold is the standard sphere. A weighted Sasakian structure on S^3 is given by

$$\eta_a = (a_1|z_1|^2 + a_2|z_2|^2)^{-1} \eta_0,$$

here a_1 and a_2 are any positive numbers. The Reeb vector field of the weighted Sasakian structure is

$$\xi_a = \sum_{i=1}^2 a_i \left(x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \right),$$

which is generated by the action

$$e^{it} \cdot (z_1, z_2) = (e^{ia_1 t} z_1, e^{ia_2 t} z_2), \quad t \in [0, 2\pi].$$

When a_1/a_2 is a rational number, the weighted Sasakian structure is quasi-regular; otherwise it is irregular. Belgun's classification (see [2, Proposition 6]) tells that any Sasakian structure on S^3 is a second deformation of some weighted Sasakian structure on S^3 .

We will use formulas (4.2) and (4.6) to prove the volume estimate (3.7). We first deal with the case of quasi-regular Sasakian structures. The proof relies essentially on Wu's injectivity radius estimates [22]. In fact in the case of quasi-regular Sasakian structures, the volume estimates can be deduced directly from Wu's volume estimate.

Proposition 4.3 *Let (S^3, η, g) be a quasi-regular Sasakian 3-sphere with transverse metric g and positive transverse scalar curvature R . Then there exists a positive constant C_0 , which depends only on the first deformation class of η , such that for any orbit l_p passing through $p \in S^3$ we have*

$$V_{l_p}^M \left(\frac{\pi}{2\sqrt{R_{\max}}} \right) \geq \frac{C_0}{R_{\max}}. \quad (4.7)$$

Proof Without loss of generality let η be a second deformation of

$$\eta_a = (a_1|z_1|^2 + a_2|z_2|^2)^{-1} \eta_0, \quad a_1, a_2 \in \mathbb{N}, \quad a_1 < a_2, \quad \gcd(a_1, a_2) = 1.$$

Case I, $\mathbb{Z}_{a_2}$ -teardrop base space: If $a_1 = 1$, the only singular orbit is

$$l_{z_2} = \{(z_1, z_2) | z_1 = 0, |z_2| = 1\}.$$

Note that

$$L(l_{z_2}) = \frac{2\pi}{a_2},$$

and the generic orbits have length 2π . The base space $\mathcal{B}$ of this foliation, given by

$$\mathcal{B} = S^3 / l_{\xi_a}, \quad \pi : S^3 \rightarrow \mathcal{B},$$

is an orbifold with an orbifold point

$$Q = \pi(l_{z_2}).$$

The isotropy group at Q is $\mathbb{Z}_{a_2}$.

Let l be an orbit in S^3 ,

$$l_s = \{x \in S^3 | d_{g^M}(x, l) = s\},$$

and

$$T(l, r) = \{x \in S^3 | d_{g^M}(x, l) \leq r\}.$$

The injectivity radius of the orbifold point Q in the universal cover of $\mathcal{B} \setminus Q$ is greater than $\pi/\sqrt{R_{\max}}$, see [22]. Note that along any geodesic starting from l_{z_2} orthogonally there exists no focal point within distance $\pi/\sqrt{R_{\max}}$. It follows that

$$\text{minfoc}(l_{z_2}) \geq \frac{\pi}{\sqrt{R_{\max}}}.$$

Actually if not, there must be a shortest geodesic 1-gon at Q of length less than $2\pi/\sqrt{R_{\max}}$ on the universal cover of $\mathcal{B} \setminus Q$, which contradicts with Wu's injectivity radius estimate mentioned above.

Recall that

$$\lambda = K_{\max}^M (\geq 1).$$

By the formula (4.6), for $s < \text{minfoc}(l_p)$ we have

$$A_l^M(s) \geq \pi L(l) \frac{\sin(2s\sqrt{\lambda})}{\sqrt{\lambda}}.$$

Taking $r = \pi/4\sqrt{R_{\max}}$, we get

$$\begin{aligned} V_{l_{z_2}}^M\left(\frac{\pi}{4\sqrt{R_{\max}}}\right) &= \int_0^r A_{l_{z_2}}^M(s) ds \\ &\geq \frac{2\pi^2}{a_2} \int_0^{\frac{\pi}{4\sqrt{R_{\max}}}} \frac{\sin(2s\sqrt{\lambda})}{\sqrt{\lambda}} ds. \end{aligned} \quad (4.8)$$

Note that from (3.3), we have

$$R_{\max} = 2\lambda + 6 > \lambda,$$

so we get

$$V_{l_{z_2}}^M\left(\frac{\pi}{4\sqrt{R_{\max}}}\right) \geq \frac{2\pi^2}{a_2} \int_0^{\frac{\pi}{4\sqrt{R_{\max}}}} \frac{4s}{\pi} ds = \frac{\pi^3}{4a_2} \frac{1}{R_{\max}}. \quad (4.9)$$

For any orbit l such that

$$\text{dist}_{g^B}(\pi(l), Q) \leq \frac{\pi}{4\sqrt{R_{\max}}},$$

note that

$$T\left(l_{z_2}, \frac{\pi}{4\sqrt{R_{\max}}}\right) \subset T\left(l, \frac{\pi}{2\sqrt{R_{\max}}}\right),$$

by the estimate (4.9) we get

$$V_l^M\left(\frac{\pi}{2\sqrt{R_{\max}}}\right) \geq \frac{\pi^3}{4a_2} \frac{1}{R_{\max}}, \quad \forall l : \text{dist}_{g^B}(\pi(l), Q) \leq \frac{\pi}{4\sqrt{R_{\max}}}. \quad (4.10)$$

For any orbit l such that

$$\text{dist}_{g^B}(\pi(l), Q) \geq \frac{\pi}{4\sqrt{R_{\max}}},$$

it follows from Wu's injectivity radius estimate [22] that there exists a positive constant $C \leq \frac{1}{2}$ such that

$$\text{minfoc}(l) \geq \frac{C}{a_2\sqrt{R_{\max}}}.$$

Hence it follows in a similar way as (4.8) and (4.9) that

$$V_l^M\left(\frac{\pi}{4\sqrt{R_{\max}}}\right) \geq \frac{4\pi C^2}{a_2^2} \frac{1}{R_{\max}}, \quad \forall l : \text{dist}_{g^B}(\pi(l), Q) \geq \frac{\pi}{4\sqrt{R_{\max}}}. \quad (4.11)$$

By (4.10) and (4.11), we see that there exists a positive constant C such that

$$V_l^M \left(\frac{\pi}{2\sqrt{R_{\max}}} \right) \geq \frac{C}{a_2^2} \frac{1}{R_{\max}}, \quad \forall l. \quad (4.12)$$

Case II, $\mathbb{Z}_{a_1, a_2}$ -football base space: In case that $a_1 \geq 2$, there are just two singular orbits

$$l_{z_2} = \{z_1 = 0\}, \quad l_{z_1} = \{z_2 = 0\}.$$

Therefore, the base space of this foliation is an orbifold $\mathcal{B}$ with two orbifold points

$$Q_2 = \pi(l_{z_2}), \quad Q_1 = \pi(l_{z_1}),$$

here Q_2 is of $\mathbb{Z}_{a_2}$ isotropy group and Q_1 is of $\mathbb{Z}_{a_1}$ isotropy group. Note that

$$L(l_{z_2}) = \frac{2\pi}{a_2}, \quad L(l_{z_1}) = \frac{2\pi}{a_1},$$

and the generic orbits have length 2π .

The volume estimate (4.7) follows in this case similarly to the case I, using Wu's injectivity radius estimates on the base space and the formula (4.6). We just outline the main steps.

The injectivity radius of the orbifold point Q_2 (respectively Q_1) on the universal cover of $\mathcal{B} \setminus Q_1$ (respectively $\mathcal{B} \setminus Q_2$) is greater than $\pi/\sqrt{R_{\max}}$, which implies that

$$\text{minfoc}(l_{z_i}) \geq \frac{\pi}{\sqrt{R_{\max}}}, \quad i = 1, 2.$$

Hence we have similar volume estimates as (4.10):

$$V_l^M \left(\frac{\pi}{2\sqrt{R_{\max}}} \right) \geq \frac{\pi^3}{4a_i} \frac{1}{R_{\max}}, \quad \forall l : \text{dist}_{g^{\mathcal{B}}}(\pi(l), Q_i) \leq \frac{\pi}{4\sqrt{R_{\max}}}. \quad (4.13)$$

For any orbit l such that

$$\text{dist}_{g^{\mathcal{B}}}(\pi(l), Q_i) \geq \frac{\pi}{4\sqrt{R_{\max}}}, \quad i = 1, 2,$$

it follow from Wu's injectivity radius estimate [22] that there exists a positive constant $C \leq \frac{1}{2}$ such that

$$\text{minfoc}(l) \geq \frac{C}{a_2\sqrt{R_{\max}}}.$$

Hence we have

$$V_l^M \left(\frac{\pi}{4\sqrt{R_{\max}}} \right) \geq \frac{4\pi C^2}{a_2^2} \frac{1}{R_{\max}}, \quad (4.14)$$

for any l satisfies

$$\text{dist}_{g^{\mathcal{B}}}(\pi(l), Q_i) \geq \frac{\pi}{4\sqrt{R_{\max}}}, \quad i = 1, 2.$$

By (4.13) and (4.14), we see that there exists a positive constant C such that

$$V_l^M \left(\frac{\pi}{2\sqrt{R_{\max}}} \right) \geq \frac{C}{a_2^2} \frac{1}{R_{\max}}, \quad \forall l. \quad (4.15)$$

□

We now handle the irregular case. In this case, the base space is not even an orbifold. We will use Weyl's tube formula (4.5) to prove the volume estimate (3.4). However, we shall consider not only a tube about a closed curve but also a tube about a torus. Here the torus is the closure of some orbit. We first study the geometry of orbit closure.

Let (S^3, η, g^M) be an irregular Sasakian 3-sphere and the contact form η is of the first deformation class

$$\eta_a = (a_1|z_1|^2 + a_2|z_2|^2)^{-1} \eta_0, \quad 1 \leq a_1 < a_2, \quad (4.16)$$

here a_1/a_2 is an irrational number and η_0 is the canonical Sasakian structure given by

$$\eta_0 = \sum_{i=1}^2 (x^i dy^i - y^i dx^i).$$

The Reeb vector field determined by η is then

$$\xi_a = \sum_{i=1}^2 a_i \left(x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \right),$$

which is generated by the action

$$e^{it} \cdot (z_1, z_2) = (e^{ia_1 t} z_1, e^{ia_2 t} z_2), \quad t \in [0, +\infty).$$

Let T_{c_1} be the torus in S^3 given by

$$T_{c_1} = \{(z_1, z_2) \in S^3 : |z_1|^2 = c_1^2, |z_2|^2 = 1 - c_1^2\}.$$

For $c_1 = 0, 1$, the torus T_{c_1} degenerates, respectively, to closed orbits

$$l_{z_2} = \{(z_1, z_2) : |z_2| = 1\}, \quad l_{z_1} = \{(z_1, z_2) : |z_1| = 1\}.$$

Note that

$$L(l_{z_i}) = \frac{2\pi}{a_i}, \quad i = 1, 2.$$

Each orbit is along the torus containing it. In particular for $c_1 \in (0, 1)$ each orbit is dense in the torus. Hence the closure of each orbit is a torus and any two points contained in the same torus have transverse distance 0. The transverse distance between any two points contained in two different tori equals the (transverse) distance between these two tori.

For $c_1 \in (0, 1)$, let $P = T_{c_1}$ and $X \in \Gamma(TP)$ denote the unit vector field orthogonal to ξ_a and $N = \Phi X$. Both vector fields are determined up to choice of directions. For example we can choose

$$N = \frac{\nabla^M |z_1|}{|\nabla^M |z_1||}.$$

The vector field N is the unit normal vector field of the torus, and N integrates to geodesics in (S^3, g^M) . Note that

$$\nabla_{\xi_a}^M \xi_a = 0, \quad \nabla_X^M \xi_a = \Phi X = N,$$

so the mean curvature and the second fundamental form of P satisfy the relation

$$|H|^2 - |A|^2 = -2.$$

To prove the volume estimate (3.4) for irregular Sasakian 3-spheres, we encounter similar situation as for quasi-regular ones. We consider those points close to the ends l_{z_2} and l_{z_1} of the family of tori and those points in the middle part separately. In view of the formula (4.2), we still need to estimate the area of the torus in the middle part. For this aim, we shall apply the following Lemma.

Lemma 4.4 *Let $(S^3, \eta, \xi_a, \Phi, g^M)$ be a (quasi-regular or irregular) Sasakian 3-sphere of positive transverse sectional curvature. Fixing any torus T_{c_1} , let*

$$(T_{c_1})_t = \{\exp_{v(T_{c_1})} tN\}, \quad A(t) = \text{Area}[(T_{c_1})_t].$$

Then we have

$$A''(t) \leq 0. \quad (4.17)$$

Proof Note

$$A(t) = \int_{T_{c_1}} \Theta_N(p, t) d\mu_P, \quad p \in P = T_{c_1}.$$

Taking N as the formal outward unit normal vector field along the torus $(T_{c_1})_t$ and define the shape operator $S(t)$ by

$$\langle Su, v \rangle = \langle \nabla_u^M N, v \rangle, \quad u, v \in T(T_{c_1})_t.$$

Recall the Riccati equation (4.3) and Eq. (4.4), i.e.,

$$S'(t) = -S(t)^2 + R_N^M, \quad S' = \nabla_N^M S, \quad R_N^M u = R^M(N, u)N,$$

and

$$\Theta'_N(t) = \text{tr} S(t) \Theta_N(t).$$

Note that

$$(\text{tr} S(t))' = N \langle Se_i, e_i \rangle$$

$$\begin{aligned}
&= \langle \nabla_N^M (Se_i), e_i \rangle + \langle Se_i, \nabla_N^M e_i \rangle \\
&= \operatorname{tr} S' + 2 \langle \nabla_N^M e_i, Se_i \rangle.
\end{aligned}$$

Taking

$$e_1 = \xi_a, \quad e_2 = X = -\Phi N,$$

we get

$$\begin{aligned}
\langle \nabla_N^M e_i, Se_i \rangle &= \langle \nabla_N^M \xi_a, \nabla_{\xi_a}^M N \rangle + \langle \nabla_N^M X, \nabla_X^M N \rangle \\
&= -\langle X, \nabla_{\xi_a}^M N \rangle + \langle \nabla_N^M X, \xi_a \rangle \langle \nabla_X^M N, \xi_a \rangle \\
&= \langle \nabla_{\xi_a}^M X, N \rangle + \langle X, \nabla_N^M \xi_a \rangle \langle N, \nabla_X^M \xi_a \rangle \\
&= \langle \Phi X, N \rangle + \langle X, \Phi N \rangle \langle N, \Phi X \rangle \\
&= 0.
\end{aligned}$$

Therefore, we have

$$(\operatorname{tr} S(t))' = \operatorname{tr}(S').$$

Hence we have

$$\begin{aligned}
\Theta_N''(t) &= [(\operatorname{tr} S(t))^2 + \operatorname{tr} S'(t)] \Theta_N(t) \\
&= [(\operatorname{tr} S(t))^2 - \operatorname{tr} S^2(t) - \operatorname{Ric}^M(N, N)] \Theta_N(t) \\
&= [|H|^2 - |A|^2 - \operatorname{Ric}^M(N, N)] \Theta_N(t) \\
&= [-3 - K^M(X, N)] \Theta_N(t) \\
&= -K(X, N) \Theta_N(t) \\
&\leq 0.
\end{aligned}$$

□

Proposition 4.5 *Let (S^3, η, g) be an irregular Sasakian 3-sphere with transverse metric g and positive transverse scalar curvature R . Then there exists a positive constant C_0 , which depends only on the first deformation class of η , such that for any torus T_{c_1} we have*

$$V_{T_{c_1}}^M \left(\frac{\pi}{2\sqrt{R_{\max}}} \right) \geq \frac{C_0}{R_{\max}}. \quad (4.18)$$

Proof Let η be a second deformation of η_a given by (4.16). First, note that we may assume that

$$\operatorname{dist}(l_{z_1}, l_{z_2}) > \frac{\pi}{2\sqrt{R_{\max}}}. \quad (4.19)$$

Otherwise, we have the total volume of S^3 and the estimate (4.18) follow automatically.

By the formula (4.6), we have

$$A_{l_{z_2}}^M(t) \geq \pi L(l_{z_2}) \frac{\sin(2t\sqrt{\lambda})}{\sqrt{\lambda}}. \quad (4.20)$$

Taking $r = \frac{\pi}{4\sqrt{R_{\max}}}$, we get

$$\begin{aligned} V_{l_{z_2}}^M\left(\frac{\pi}{4\sqrt{R_{\max}}}\right) &= \int_0^r A_{l_{z_2}}^M(t) dt \\ &\geq \frac{2\pi^2}{a_2} \int_0^{\frac{\pi}{4\sqrt{R_{\max}}}} \frac{\sin(2t\sqrt{\lambda})}{\sqrt{\lambda}} dt \\ &\geq \frac{\pi^3}{4a_2} \frac{1}{R_{\max}}. \end{aligned}$$

In the same way, we have

$$V_{l_{z_1}}^M\left(\frac{\pi}{4\sqrt{R_{\max}}}\right) \geq \frac{\pi^3}{4a_1} \frac{1}{R_{\max}}.$$

For some i and a torus T_{c_1} such that

$$\text{dist}(T_{c_1}, l_{z_i}) \leq \frac{\pi}{4\sqrt{R_{\max}}},$$

note that

$$T\left(l_{z_i}, \frac{\pi}{4\sqrt{R_{\max}}}\right) \subset T\left(T_{c_1}, \frac{\pi}{2\sqrt{R_{\max}}}\right),$$

so we get

$$V_{T_{c_1}}^M\left(\frac{\pi}{2\sqrt{R_{\max}}}\right) \geq \frac{\pi^3}{4a_i} \frac{1}{R_{\max}}, \quad \forall i, T_{c_1} : \text{dist}(T_{c_1}, l_{z_i}) \leq \frac{\pi}{4\sqrt{R_{\max}}}. \quad (4.21)$$

Now for the torus T_{c_1} such that

$$\text{dist}(T_{c_1}, l_{z_i}) \geq \frac{\pi}{8\sqrt{R_{\max}}}, \quad i = 1, 2,$$

applying Lemma 4.4 and formula (4.20), we get

$$\text{Area}(T_{c_1}) \geq \min_i A_{l_{z_i}}^M\left(\frac{\pi}{8\sqrt{R_{\max}}}\right) \geq \frac{\pi^2}{a_2} \frac{1}{\sqrt{R_{\max}}}. \quad (4.22)$$

For any torus T_{c_1} such that

$$\text{dist}(T_{c_1}, l_{z_i}) \geq \frac{\pi}{4\sqrt{R_{\max}}}, \quad i = 1, 2,$$

by the assumption (4.19) and the estimate (4.22), we get

$$V_{T_{c_1}}^M \left(\frac{\pi}{2\sqrt{R_{\max}}} \right) > V_{T_{c_1}}^M \left(\frac{\pi}{8\sqrt{R_{\max}}} \right) \geq \frac{\pi}{4\sqrt{R_{\max}}} \frac{\pi^2}{a_2} \frac{1}{\sqrt{R_{\max}}} = \frac{\pi^3}{4a_2} \frac{1}{R_{\max}}. \quad (4.23)$$

The proof then follows from (4.21) and (4.23). $\square$

5 Sasaki–Ricci Solitons on S^3

In this section we consider existence and uniqueness of gradient Sasaki–Ricci soliton on S^3 . In particular we shall prove Theorem 1.2. A gradient Sasaki–Ricci soliton $(\eta, -\frac{1}{2}\nabla f)$ on a Sasakian S^3 satisfies the equation

$$\nabla^2 f - \frac{1}{2}\Delta_B f g_\eta = 0, \quad (5.1)$$

here f is the basic function defined by $R - r = \Delta_B f$.

By Belgun's work [2], we know that, up to second type deformations, weighted Sasakian structures are essentially all the Sasakian structures on S^3 . Along the Sasaki–Ricci flow, the volume and the total transverse scalar curvature are fixed. So we can first compute the average of the transverse scalar curvature, and then consider what second type deformation of a given weighted Sasakian structure satisfies the gradient Sasaki–Ricci soliton equation.

Let S^3 be the unit sphere in $\mathbb{C}^2$. The contact form of the canonical Sasakian structure on S^3 is given by

$$\eta_0 = \sum_{i=1}^2 (x^i dy^i - y^i dx^i).$$

The almost complex structure on the canonical Sasakian manifold (S^3, η_0) is induced by the complex structure of $\mathbb{C}^2$. Moreover, the contact distribution $\mathcal{D}$ is spanned by

$$\begin{aligned} X_1 &= -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial y^1} - y^1 \frac{\partial}{\partial y^2}, \\ X_2 &= \Phi X_1 = -x^2 \frac{\partial}{\partial y^1} + x^1 \frac{\partial}{\partial y^2} - y^2 \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial x^2}. \end{aligned}$$

Let

$$\sigma = a_1 |z_1|^2 + a_2 |z_2|^2, \quad a_1, a_2 > 0.$$

The contact form and Reeb vector field of a weighted Sasakian structure are, respectively, given by

$$\eta_a = \sigma^{-1} \eta_0, \quad \xi_a = \sum_{i=1}^2 a_i \left(x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \right). \quad (5.2)$$

Now let

$$Z_1 = |z_2|^2 \left(y^1 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial y^1} \right) - |z_1|^2 \left(y^2 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial y^2} \right),$$

$$Z_2 = \Phi Z_1 = |z_2|^2 \left(x^1 \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial y^1} \right) - |z_1|^2 \left(x^2 \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial y^2} \right)$$

and

$$Z = \sigma^{-1}(Z_1 - iZ_2) = -2i\sigma^{-1}|z_1|^2|z_2|^2 \left(\frac{1}{\bar{z}_1} \frac{\partial}{\partial z_1} - \frac{1}{\bar{z}_2} \frac{\partial}{\partial z_2} \right).$$

Here

$$Z_i \in \ker \eta_a, \quad [\xi_a, Z_i] = 0, \quad i = 1, 2,$$

and

$$\Phi Z = iZ, \quad [\xi_a, Z] = 0.$$

In particular on (S^3, η_a) , Z_1 is along the torus $T_{c_1} := \{|z_1|^2 = c_1^2, |z_2|^2 = 1 - c_1^2\}$, and Z_2 is perpendicular to Z_1 . Let g_a be the transverse metric associated to η_a and $\tilde{g}_a = g_a(Z, \bar{Z})$. Then we have

$$\tilde{g}_a = g_a(Z, \bar{Z}) = 2\sigma^{-3}|z_1|^2|z_2|^2, \quad [Z, \bar{Z}] = -2i\tilde{g}_a\xi_a.$$

So on (S^3, η_a) , we have $\nabla_Z \bar{Z} = \nabla_{\bar{Z}} Z = 0$.

Proposition 5.1 *The weighted Sasakian manifold (S^3, η_a) has*

$$R(g_a) = -24(a_1 - a_2)^2 \sigma^{-1}|z_1|^2|z_2|^2 - 16(a_1 - a_2)(|z_1|^2 - |z_2|^2) + 8\sigma,$$

$$r = 4(a_1 + a_2).$$

Proof The transverse scalar curvature can be computed by

$$R(g_a) = -2\tilde{g}_a^{-1} Z \bar{Z} \log(\tilde{g}_a).$$

Then we have

$$\begin{aligned} \int_{S^3} R\eta_a \wedge d\eta_a &= \int_{|z_1|^2=t} R\sigma^{-2}(2\pi\sqrt{t})(2\pi\sqrt{1-t}) \frac{dt}{|\nabla t|_{g_{\eta_0}^M}} \\ &= \int_0^1 R\sigma^{-2}2\pi^2 dt \\ &= 8\pi^2 \frac{a_1 + a_2}{a_1 a_2}, \end{aligned}$$

and

$$\int_{S^3} \eta_a \wedge d\eta_a = \int_0^1 \sigma^{-2}2\pi^2 dt = \frac{2\pi^2}{a_1 a_2}.$$

□

Theorem 5.2 *On each weighted Sasakian manifold (S^3, η_a) , there exists a unique Sasakian structure η homologous to η_a so that it is a gradient Sasaki–Ricci soliton.*

Proof The existence and uniqueness of Ricci soliton on bad orbifolds were obtained by Wu [22]. When (S^3, η_a) is a quasi-regular Sasakian structure, the problem is reduced to Wu’s result. So we focus on the irregular case.

Assume

$$\eta = \eta_a + i(\bar{\partial}_B - \partial_B)\varphi$$

is the contact form of a gradient Sasaki–Ricci soliton on (S^3, η_a) and g_η its transverse metric. Here φ is a basic function. So we have

$$\tilde{g} := d\eta(Z, \Phi \bar{Z}) = 2\sigma^{-3}|z_1|^2|z_2|^2 + \varphi_Z \bar{Z}, \quad R(g_\eta) = -2\tilde{g}^{-1}Z\bar{Z} \log \tilde{g}.$$

Let f be the basic function defined by $R(g_\eta) - r(g_\eta) = \Delta_B f$. By the definition of a gradient Sasaki–Ricci soliton, we have

$$\nabla^2 f - \frac{1}{2}\Delta_B f g_\eta = 0. \quad (5.3)$$

It’s equivalent to the system:

$$ZZf - Z(\log \tilde{g})Zf = 0, \quad (5.4)$$

and

$$Z\bar{Z}f = \frac{1}{2}(R(g_\eta) - r)\tilde{g}. \quad (5.5)$$

For an irregular Sasakian structure, all generic orbits of the characteristic foliation are dense in the torus T_{c_1} containing it. Hence basic function take constant value on each torus. Therefore, basic functions f and $\tilde{g}$ have vanishing derivatives in the direction of Z_1 . Note also that

$$[Z_1, Z_2] = -2\sigma^{-1}|z_1|^2|z_2|^2(\xi_a + Z_1).$$

By Eq. (5.4), we have

$$-\sigma^{-1}Z_2(\sigma^{-1}Z_2f) + \sigma^{-1}Z_2(\log \tilde{g})(\sigma^{-1}Z_2f) = 0.$$

So we get

$$Z_2f = -c\sigma\tilde{g}. \quad (5.6)$$

Let π_η be the projection $\pi_\eta : TS^3 \rightarrow \ker \eta$ and $\tilde{Z}_i = \pi_\eta Z_i$. The vector fields $\tilde{Z}_i$ satisfy

$$g_\eta(\tilde{Z}_i, \tilde{Z}_j) = d\eta(Z_1, Z_2)\delta_{ij} = \frac{1}{2}\sigma^2\tilde{g}\delta_{ij}.$$

On the other hand,

$$g_\eta(\nabla f, \tilde{Z}_2) = Z_2f = -c\sigma\tilde{g},$$

so we see that

$$\nabla f = -2c\sigma^{-1}\tilde{Z}_2. \quad (5.7)$$

Let $X = \sigma^{-1}Z_2$, then by (5.6) and the expression of $R(g_\eta)$, Eq. (5.5) can be written as

$$X^2 \log \tilde{g} - cX(\tilde{g}) + 2(a_1 + a_2)\tilde{g} = 0. \quad (5.8)$$

Without loss of generality, we assume $a_1 < a_2$. Let $\kappa = 2(a_1 + a_2)$ and

$$s(\sigma) = -\frac{a_1}{2} \log(\sigma - a_1) + \frac{a_2}{2} \log(a_2 - \sigma) \in (-\infty, +\infty).$$

Then we have $X(s) = 1$. Write $\tilde{g} = \tilde{g}(s)$, $\tilde{g}' = \frac{d\tilde{g}}{ds}$ and $\tilde{g}'' = \frac{d^2\tilde{g}}{ds^2}$. Therefore

$$\left(\frac{\tilde{g}'}{\tilde{g}}\right)' - c\tilde{g}' + \kappa\tilde{g} = 0. \quad (5.9)$$

That's the same equation appeared in [18]. Let $\tilde{g} = v'$. The function v is determined up to a constant. We include the constant in v . Then Eq. (5.9) integrates to

$$v'' - cv'^2 + \kappa vv' = 0. \quad (5.10)$$

Integrating the last equation again, we get

$$v' = \frac{\kappa}{c}v + \frac{\kappa}{c^2}(1 - ke^{cv}), \quad (5.11)$$

here k is a constant. Let $y = cv + 1$, $u = \kappa s/c$. We get

$$\frac{dy}{du} = y - ke^{y-1}. \quad (5.12)$$

Then

$$u = \int \frac{dy}{y - ke^{y-1}}. \quad (5.13)$$

So we have

$$\tilde{g} = \frac{y'}{c}, \quad \int \frac{dy}{y - ke^{y-1}} = \kappa s/c. \quad (5.14)$$

The denominator in (5.13) must have two zeroes, which happens precisely as $0 < k < 1$. Now let $0 < k < 1$, $y_1 = 1 - p < 1$ and $y_2 = 1 + q > 1$ be two solutions to $y = ke^{y-1}$. Moreover as $y \in (y_1, y_2)$ goes to y_1 and y_2 , u tends to minus infinity and positive infinity, respectively. Note that s goes to negative infinity and positive infinity as a point goes to the points $z_1 = 0$ and $z_2 = 0$, respectively.

We now consider the asymptotic behavior near the points $z_1 = 0$ and $z_2 = 0$. Suppose as $s \rightarrow -\infty$, we have an expansion

$$\tilde{g}(s) = b_1 e^{\lambda s} + b_2 e^{2\lambda s} + \dots$$

in powers of $e^{\lambda s}$. Similarly, as $s \rightarrow +\infty$, suppose we have an expansion

$$\tilde{g}(s) = d_1 e^{-\mu s} + d_2 e^{-2\mu s} + \dots$$

Near y_1 write $y = y_1 + h$. Then we have

$$y - ke^{y-1} = ph - \frac{1}{2}(1-p)h^2 + \dots$$

So we have $u = \frac{1}{p} \log h + \dots$, which implies that

$$dy = dh = pe^{pu} du + \dots$$

Hence

$$\tilde{g}(s) = v' = \frac{dv}{ds} = \frac{\kappa}{c^2} \frac{dy}{du} = \frac{\kappa p}{c^2} e^{pu} + \dots$$

So we have $\lambda = \frac{\kappa p}{c}$ if $c > 0$, or $\mu = -\frac{\kappa p}{c}$ if $c < 0$. In a similar way, we have $\mu = \frac{\kappa q}{c}$ if $c > 0$, or $\lambda = -\frac{\kappa q}{c}$ if $c < 0$.

Write φ as a function of $\sigma \in [a_1, a_2]$, then, by the expression of

$$\tilde{g} = 2\sigma^{-3}|z_1|^2|z_2|^2 + \varphi_Z \bar{Z},$$

we see that near the points with $z_2 = 0$ either $\tilde{g}$ is of order of $|z_2|^2$ or of order of $|z_2|^4$. However, by the nondegeneracy of g_η , it can only be of order of $|z_2|^2$. On the other hand

$$(a_2 - a_1)|z_2|^2 = e^{-\frac{2}{a_1}s} [(a_2 - a_1)|z_1|^2]^{\frac{a_2}{a_1}},$$

so we have

$$\mu = \frac{2}{a_1}.$$

Similarly near the points with $z_1 = 0$, $\tilde{g}$ is of order of $|z_1|^2$, and in the same way we have

$$\lambda = \frac{2}{a_2}.$$

We always have $p < q$, see [18]. Hence $c > 0$ and $\frac{\lambda}{\mu} = \frac{a_1}{a_2} = \frac{p}{q}$.

Therefore we see that to get a gradient Sasaki–Ricci soliton solution on an irregular Sasakian 3-sphere (S^3, η_a) is to find a constant $k \in (0, 1)$ such that the equation $y = ke^{y-1}$ has two solutions

$$y_1 = 1 - p < 1, \quad y_2 = 1 + q > 1,$$

and

$$0 < \frac{a_1}{a_2} = \frac{p}{q} < 1.$$

The existence and uniqueness then follow exactly as [22]. $\square$

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