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ORIGINAL RESEARCH

Existence of the solution of nonlinear fractional differential equations via new best proximity point results

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Abstract

In this paper, we obtain some best proximity point results by introducing the concepts of proximal p-contractions of the first type and proximal p -contractions of the second type on partial metric spaces. Thus, some famous results in the literature such as the main result of Altun et al. (Acta Math Hung 162:393–402, 2020) and Basha (J Approx Theory 163(11):1772–1781, 2011) have been extended. Also, we provide some examples where our results are applicable and the results in Haghi et al. (Topol Appl 160:450–454, 2013) are not. Hence, our results are a real generalization of some results in metric spaces and partial metric spaces. Finally, we obtain sufficient conditions for the existence of the solution of nonlinear fractional differential equations via our results.

Keywords Best proximity point · Partial metric space · Nonlinear fractional differential equations

Introduction

In 1922, Banach [7] proved a fundamental theorem, known as the Banach contraction principle, which is considered the beginning of the fixed point theory on metric spaces. Because of its applicability, many authors have studied to generalize this principle [15, 21]. One of the interesting and nice generalizations has been obtained by Popescu [19]. He presented a contractive condition, named p-contraction, which expands Banach contraction as follows:

Theorem 1 *Let $\phi : (\Xi, \theta) \to (\Xi, \theta)$ be a p-contraction mapping, that is, there exists q in $[0, 1)$ satisfying*

$$\theta(\phi\varpi, \phi\varsigma) \le q[\theta(\varpi, \varsigma) + |\theta(\varpi, \phi\varpi) - \theta(\varsigma, \phi\varsigma)|],$$

for all $\varpi, \varsigma \in \Xi$ where (Ξ, θ) is a complete metric space. Hence, there is a point ϖ in Ξ satisfying $\varpi = \phi\varpi$.

Recently, many authors extended the fixed point theory in different way by considering nonself mappings. Let Λ, Π be nonempty subsets of a metric space (Ξ, θ) and $\phi : \Lambda \to \Pi$ be a mapping. If $\Lambda \cap \Pi$ is empty, then there is no a point ϖ in Λ such that $\phi\varpi = \varpi$. Therefore, it is reasonable to investigate whether there is a point $\varpi^* \in \Lambda$ satisfying $\theta(\varpi^*, \phi\varpi^*)$ is minimum. In this sense, Basha and Veeramani [10] gave the notion of best proximity point. We say that the point $\varpi^* \in \Lambda$ is a best proximity point of ϕ if it satisfies $\theta(\varpi^*, \phi\varpi^*) = \theta(\Lambda, \Pi)$. Best proximity point theorems are logical expansions of fixed point results because every best proximity point turns into a fixed point when $\Lambda = \Pi = \Xi$. Thus, several authors have researched this subject [4, 5, 9, 20, 23–25].

We now review several properties linked to the best proximity point theory. We regard the following sets:

$$\Lambda_0 = \{\varpi \in \Lambda : \theta(\varpi, \varsigma) = \theta(\Lambda, \Pi) \text{ for some } \varsigma \in \Pi\},$$

and

$$\Pi_0 = \{\varsigma \in \Pi : \theta(\varpi, \varsigma) = \theta(\Lambda, \Pi) \text{ for some } \varpi \in \Lambda\},$$

where $\theta(\Lambda, \Pi) = \inf\{\theta(\varpi, \varsigma) : \varpi \in \Lambda \text{ and } \varsigma \in \Pi\}$.

We give the definitions of proximal contractions of the first and second kind [8].

Definition 1 Let (Ξ, θ) be a metric space. A mapping $\phi : \Lambda \to \Pi$ is called a proximal contraction of first kind if there exists q in $[0, 1)$ such that, for all $\xi_1, \xi_2, \varpi_1, \varpi_2 \in \Lambda$,

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$$\left.\begin{array}{l}\theta(\xi_1, \phi\varpi_1) = \theta(\Lambda, \Pi)\\ \theta(\xi_2, \phi\varpi_2) = \theta(\Lambda, \Pi)\end{array}\right\} \Rightarrow \theta(\xi_1, \xi_2) \le q\theta(\varpi_1, \varpi_2).$$

Definition 2 Let (Ξ, θ) be a metric space. A mapping $\phi : \Lambda \to \Pi$ is called a proximal contraction of second kind if there exists q in $[0, 1)$ such that, for all $\xi_1, \xi_2, \varpi_1, \varpi_2 \in \Lambda$,

$$\left.\begin{array}{l}\theta(\xi_1, \phi\varpi_1) = \theta(\Lambda, \Pi)\\ \theta(\xi_2, \phi\varpi_2) = \theta(\Lambda, \Pi)\end{array}\right\} \Rightarrow \theta(\phi\xi_1, \phi\xi_2) \le q\theta(\phi\varpi_1, \phi\varpi_2).$$

If every sequence $\{\varpi_r\}$ in Λ satisfying $\theta(\varsigma, \varpi_r) \to \theta(\varsigma, \Lambda)$ for some $\varsigma \in \Pi$ has a subsequence $\{\varpi_{r_k}\}$ such that $\varpi_{r_k} \to \varpi$ for some $\varpi \in \Lambda$, then Λ is said to be approximately compact with respect to Π.

For proximal contractions of the first and second kind, the following theorems are the primary results.

Theorem 2 ([8]) *Let (Ξ, θ) be a complete metric space, $\emptyset \neq \Lambda, \Pi \subseteq \Xi$ with $\Lambda_0 \neq \emptyset$ where Π is approximatively compact with respect to Λ and Λ is closed. Suppose that $\phi : \Lambda \to \Pi$ is a continuous proximal contraction of the first kind and $\phi(\Lambda_0) \subseteq \Pi_0$. Then, there exists a unique element ϖ in Λ such that $\theta(\varpi, \phi\varpi) = \theta(\Lambda, \Pi)$.*

Theorem 3 ([8]) *Let (Ξ, θ) be a complete metric space and $\emptyset \neq \Lambda, \Pi \subseteq \Xi$with $\Lambda_0 \neq \emptyset$ where Λ is approximately compact with respect to Π and Λ is closed. Assume that $\phi : \Lambda \to \Pi$ is a continuous proximal contraction of the second kind and $\phi(\Lambda_0) \subseteq \Pi_0$. Then, there exists an element ϖ in Λ such that $\theta(\varpi, \phi\varpi) = \theta(\Lambda, \Pi)$. Moreover, if ϖ^* is another best proximity point of ϕ , then $\phi\varpi$ and $\phi\varpi^*$ are identical.*

On the other hand, Matthews [18] obtained another generalization of the Banach contraction principle by giving a famous notion called a partial metric. Then, numerous fixed point results in partial metric spaces have been obtained in various ways [1, 2, 6, 11, 22]. We now remind the concept of the partial metric space and its some properties.

Definition 3 ([18]) Let $\Xi \neq \emptyset$ and $\kappa : \Xi \times \Xi \to [0, \infty)$. If κ satisfies the following conditions, for all $\varpi, \varsigma, z \in \Xi$,

(p1) $\kappa(\varpi, \varpi) = \kappa(\varpi, \varsigma) = \kappa(\varsigma, \varsigma)$ if and only if $\varpi = \varsigma$,
(p2) $\kappa(\varpi, \varpi) \le \kappa(\varpi, \varsigma)$,
(p3) $\kappa(\varpi, \varsigma) = \kappa(\varsigma, \varpi)$,
(p4) $\kappa(\varpi, \varsigma) \le \kappa(\varpi, z) + \kappa(z, \varsigma) - \kappa(z, z)$,

then κ is called partial metric, and (Ξ, κ) is called partial metric space.

It can be easily seen that a metric space is a partial metric space, but the converse may not happen. In fact, let $\Xi = [0, \infty)$ and $\kappa : \Xi \times \Xi \to [0, \infty)$ be a function defined by $\kappa(\varpi, \varsigma) = \max\{\varpi, \varsigma\}$ for all $\varpi, \varsigma \in \Xi$. Although (Ξ, κ) is a partial metric space, it is not a metric space.

Let (Ξ, κ) be a partial metric space. The sets

$$B_\kappa(\varpi, \varepsilon) = \{\varsigma \in \Xi : \kappa(\varpi, \varsigma) < \kappa(\varpi, \varpi) + \varepsilon\},$$

for all $\varpi \in \Xi$ and $\varepsilon > 0$ is called open ball.

Let $\varpi \in \Xi$ and $\{\varpi_r\}$ be a sequence in Ξ. Hence, $\{\varpi_r\}$ converges to ϖ with respect to τ_κ if and only if

$$\lim_{r\to\infty} \kappa(\varpi_r, \varpi) = \kappa(\varpi, \varpi).$$

If $\lim_{r,m\to\infty} \kappa(\varpi_r, \varpi_m)$ exists and is finite, then the sequence $\{\varpi_r\}$ in Ξ is said to be a Cauchy sequence. Further, (Ξ, κ) is said to be complete if every Cauchy sequence $\{\varpi_r\}$ in Ξ converges to a point $\varpi \in \Xi$ with respect to τ_κ such that

$$\kappa(\varpi, \varpi) = \lim_{m,r\to\infty} \kappa(\varpi_r, \varpi_m).$$

Let $\kappa^s : \Xi \times \Xi \to [0, \infty)$ be a function defined by

$$\kappa^s(\varpi, \varsigma) = 2\kappa(\varpi, \varsigma) - \kappa(\varpi, \varpi) - \kappa(\varsigma, \varsigma).$$

Then, κ^s is a metric on Ξ.

We now give the following lemma which gives the relationship between κ^s and κ.

Lemma 1 ([18]) *Let (Ξ, κ) be a partial metric space.*

(i) *$\{\varpi_r\}$ is a Cauchy sequence in (Ξ, κ^s) if and only if $\{\varpi_r\}$ is a Cauchy sequence in (Ξ, κ).*
(ii) *(Ξ, κ^s) is a complete metric space if and only if (Ξ, κ) is a complete partial metric space.*
(iii) *Consider a sequence $\{\varpi_r\}$ in (Ξ, κ) and $\varpi \in \Xi$. Then, we get*

$$\lim_{r\to\infty} \kappa^s(\varpi_r, \varpi) = 0 \Longleftrightarrow \kappa(\varpi, \varpi) = \lim_{r\to\infty} \kappa(\varpi_r, \varpi) = \lim_{r,m\to\infty} \kappa(\varpi_r, \varpi_m).$$

The following lemma is important because it gives the characterization of the closed set in partial metric spaces.

Lemma 2 ([1]) *Let (Ξ, κ) be a partial metric space and $\emptyset \neq \Lambda \subseteq \Xi$. Define $\kappa(\varpi, \Lambda) = \inf\{\kappa(\varpi, a) : a \in \Lambda\}$. Then, we have*

$$\varpi \in \overline{\Lambda} \Longleftrightarrow \kappa(\varpi, \Lambda) = \kappa(\varpi, \varpi),$$

where $\overline{\Lambda}$ is closure of Λ with respect to τ_κ.

Definition 4 ([18]) Let (Ξ, κ) be a partial metric space and $\phi : \Xi \to \Xi$ be a mapping. We say that ϕ is a continuous mapping with respect to τ_κ if

$$\lim_{r,m\to\infty} \kappa(\varpi_r, \varpi_m) = \lim_{r\to\infty} \kappa(\varpi_r, \varpi^*) = \kappa(\varpi^*, \varpi^*),$$

then

$$\lim_{r,m\to\infty} \kappa(\phi\varpi_r, \phi\varpi_m) = \lim_{r\to\infty} \kappa(\phi\varpi_r, \phi\varpi^*) = \kappa(\phi\varpi^*, \phi\varpi^*).$$

In the present paper, we introduce the new concepts of proximal p-contraction of the first type and proximal p-contraction of the second type on partial metric spaces. Then, we obtain some best proximity point results for such mappings. Hence, we generalize and extend some famous results in the literature such as the main result of [3] and [8]. Recently, Haghi et al. [13] proved that some results on partial metric spaces are equivalent to the results in the context of a metric spaces. But, this case is not valid for our main results. Therefore, we provide some important examples where our results are applicable and their results are not. Hence, our results are a real generalization of some results in metric spaces and partial metric spaces. Finally, with the help of our results, we obtain sufficient conditions for the existence of the solution of nonlinear fractional differential equations.

Main results

Considering partial metric spaces, we begin this section by giving the definition of approximately compact.

Definition 5 Let (Ξ, κ) be a partial metric space, and $\emptyset \neq \Lambda, \Pi \subseteq \Xi$. If every sequence $\{\varpi_r\}$ in Λ satisfying $\kappa(\varsigma, \varpi_r) \to \kappa(\varsigma, \Lambda)$ for some $\varsigma \in \Pi$ has a subsequence $\{\varpi_{r_k}\}$ such that

$$\lim_{k,l\to\infty} \kappa(\varpi_{r_k}, \varpi_{r_l}) = \lim_{k\to\infty} \kappa(\varpi_{r_k}, \varpi) = \kappa(\varpi, \varpi) = 0,$$

for some $\varpi \in \Lambda$, then Λ is said to be κ -approximately compact with respect to Π.

After that, we introduce the following definition of proximal p -contraction of the first type.

Definition 6 Let (Ξ, κ) be a partial metric space and $\emptyset \neq \Lambda, \Pi \subseteq \Xi$. A mapping $\phi : \Lambda \to \Pi$ is said to be a proximal p-contraction of the first type if there exists q in [0, 1) such that, for all $\xi_1, \xi_2, \varpi_1, \varpi_2 \in \Lambda$ with $\xi_1 \neq \xi_2$,

$$\left.\begin{array}{l}\kappa(\xi_1, \phi\varpi_1) = \kappa(\Lambda, \Pi)\\ \kappa(\xi_2, \phi\varpi_2) = \kappa(\Lambda, \Pi)\end{array}\right\} \Rightarrow \kappa(\xi_1, \xi_2) \leq q\Big(\kappa(\varpi_1, \varpi_2) + \big|\kappa(\xi_1, \varpi_1) - \kappa(\xi_2, \varpi_2)\big|\Big). \tag{1}$$

Now, we give our first main result.

Theorem 4 *Let (Ξ, κ) be a complete partial metric space and $\emptyset \neq \Lambda, \Pi \subseteq \Xi$ where Λ is closed and Π is a κ-approximately compact with respect to Λ. Assume that the followings hold:*

(i) *$\Lambda_0 \neq \emptyset$ and $\phi(\Lambda_0) \subseteq \Pi_0$.*
(ii) *$\phi : \Lambda \to \Pi$ is a proximal p-contraction of the first type.*

Then, there exists ϖ^* in Λ such that $\kappa(\varpi^*, \phi\varpi^*) = \kappa(\Lambda, \Pi)$.

Proof Let ϖ_0 be an arbitrary point in Λ_0. Because of the fact that $\phi\varpi_0 \in \phi(\Lambda_0) \subseteq \Pi_0$, there exists $\varpi_1 \in \Lambda_0$ satisfying

$$\kappa(\varpi_1, \phi\varpi_0) = \kappa(\Lambda, \Pi).$$

Similarly, since $\phi\varpi_1 \in \phi(\Lambda_0) \subseteq \Pi_0$, there exists $\varpi_2 \in \Lambda_0$ satisfying

$$\kappa(\varpi_2, \phi\varpi_1) = \kappa(\Lambda, \Pi).$$

Performing this process again, one can create a sequence $\{\varpi_r\}$ in Λ satisfying

$$\kappa(\varpi_{r+1}, \phi\varpi_r) = \kappa(\Lambda, \Pi), \tag{2}$$

for all $r \geq 1$. Assume that $\varpi_{r_0} = \varpi_{r_0+1}$ for some $r_0 \geq 1$, then, from (2), ϖ_{r_0+1} is a best proximity point of ϕ. We now get $\varpi_r \neq \varpi_{r+1}$ for all $r \geq 1$. Because ϕ is a proximal p-contraction of the first type, we have, for all $r \geq 1$,

$$\kappa(\varpi_r, \varpi_{r+1}) \leq q\Big(\kappa(\varpi_{r-1}, \varpi_r) + \big|\kappa(\varpi_r, \varpi_{r+1}) - \kappa(\varpi_{r-1}, \varpi_r)\big|\Big).$$

If there exists $r_0 \geq 1$ such that $\kappa(\varpi_{r_0-1}, \varpi_{r_0}) \leq \kappa(\varpi_{r_0}, \varpi_{r_0+1})$, then we have

$$\begin{aligned}\kappa(\varpi_{r_0}, \varpi_{r_0+1}) \leq& q\Big(\kappa(\varpi_{r_{0-1}}, \varpi_{r_0})\\ &+\Big|\kappa(\varpi_{r_0}, \varpi_{r_0+1}) - \kappa(\varpi_{r_0-1}, \varpi_{r_0})\Big|\Big)\\ =& q\Big(\kappa(\varpi_{r_0-1}, \varpi_{r_0})\\ &+\kappa(\varpi_{r_0}, \varpi_{r_0+1}) - \kappa(\varpi_{r_0-1}, \varpi_{r_0})\Big)\\ =& q\kappa(\varpi_{r_0}, \varpi_{r_0+1})\\ <& \kappa(\varpi_{r_0}, \varpi_{r_0+1}),\end{aligned}$$

which is a contradiction. Therefore, we can conclude that $\kappa(\varpi_r, \varpi_{r+1}) < \kappa(\varpi_{r-1}, \varpi_r)$ for all $r \geq 1$. Hence, we get

$$\begin{aligned}\kappa(\varpi_r, \varpi_{r+1}) \leq& q\Big(\kappa(\varpi_{r-1}, \varpi_r) + \kappa(\varpi_{r-1}, \varpi_r) - \kappa(\varpi_r, \varpi_{r+1})\Big)\\ =& 2q\kappa(\varpi_{r-1}, \varpi_r) - q\kappa(\varpi_r, \varpi_{r+1}),\end{aligned}$$

and so, for all $r \geq 1$,

$$\kappa(\varpi_r, \varpi_{r+1}) \leq \left(\frac{2q}{q+1}\right)\kappa(\varpi_{r-1}, \varpi_r).$$

Using the last inequality, we have, for all $r \geq 1$,

$$\begin{aligned}\kappa(\varpi_r,\varpi_{r+1}) &\leq \left(\frac{2q}{q+1}\right)\kappa(\varpi_{r-1},\varpi_r)\\ &\leq \left(\frac{2q}{q+1}\right)^2\kappa(\varpi_{r-2},\varpi_{r-1})\\ &\ \ \vdots\\ &\leq \left(\frac{2q}{q+1}\right)^r\kappa(\varpi_0,\varpi_1).\end{aligned}$$

Hence, we can obtain, for all $m, r \geq 1$ with $m > r$,

$$\begin{aligned}\kappa(\varpi_r,\varpi_m) \leq&\kappa(\varpi_r,\varpi_{r+1})+\kappa(\varpi_{r+1},\varpi_{r+2})+\ldots+\kappa(\varpi_{m-1},\varpi_m)\\ \leq&\left(\frac{2q}{q+1}\right)^r\kappa(\varpi_0,\varpi_1)+\left(\frac{2q}{q+1}\right)^{r+1}\kappa(\varpi_0,\varpi_1)\\ &+\cdots+\left(\frac{2q}{q+1}\right)^{m-1}\kappa(\varpi_0,\varpi_1)\\ =&\left[1+\frac{2q}{q+1}+\cdots+\left(\frac{2q}{q+1}\right)^{m-r-1}\right]\left(\frac{2q}{q+1}\right)^r\kappa(\varpi_0,\varpi_1)\\ \leq&\frac{\left(\frac{2q}{q+1}\right)^r}{1-\frac{2q}{q+1}}\kappa(\varpi_0,\varpi_1).\end{aligned}$$

Hence, $\lim_{r,m\to\infty}\kappa(\varpi_r,\varpi_m)=0$, and so $\{\varpi_r\}$ is a Cauchy sequence in Λ. Since Λ is a closed subset of complete partial metric space (Ξ,κ), there is ϖ^* in Λ satisfying

$$\lim_{r,m\to\infty}\kappa(\varpi_r,\varpi_m)=\lim_{r\to\infty}\kappa(\varpi_r,\varpi^*)=\kappa(\varpi^*,\varpi^*)=0. \tag{3}$$

Moreover, from (2) and (3), it can be seen that

$$\begin{aligned}\kappa(\varpi^*,\Pi) \leq&\kappa(\varpi^*,\phi\varpi_r)\\ \leq&\kappa(\varpi^*,\varpi_{r+1})+\kappa(\varpi_{r+1},\phi\varpi_r)\\ =&\kappa(\varpi^*,\varpi_{r+1})+\kappa(\Lambda,\Pi)\\ \leq&\kappa(\varpi^*,\varpi_{r+1})+\kappa(\varpi^*,\Pi).\end{aligned}$$

Therefore, $\kappa(\varpi^*,\phi\varpi_r)\to\kappa(\varpi^*,\Pi)$ as $r\to\infty$. Since Π is a κ-approximately compact with respect to Λ, there exists a subsequence $\{\phi\varpi_{r_k}\}$ of $\{\phi\varpi_r\}$ such that

$$\lim_{k,l\to\infty}\kappa(\phi\varpi_{r_k},\phi\varpi_{r_l})=\lim_{k\to\infty}\kappa(\phi\varpi_{r_k},\varsigma^*)=\kappa(\varsigma^*,\varsigma^*)=0,$$

for some $\varsigma^*\in\Pi$. Therefore, from (2) and Lemma 1 (iii), we have

$$\begin{aligned}\kappa(\Lambda,\Pi) \leq&\kappa(\varpi^*,\varsigma^*)\\ \leq&\kappa(\varpi^*,\varpi_{r_k+1})+\kappa(\varpi_{r_k+1},\phi\varpi_{r_k})+\kappa(\phi\varpi_{r_k},\varsigma^*)\\ =&\kappa(\varpi^*,\varpi_{r_k+1})+\kappa(\Lambda,\Pi)+\kappa(\phi\varpi_{r_k},\varsigma^*),\end{aligned}$$

and so taking the limit as $k\to\infty$, we have

$$\kappa(\varpi^*,\varsigma^*)=\kappa(\Lambda,\Pi).$$

Hence, we get $\varpi^*\in\Lambda_0$. Also, due to the fact that $\phi\varpi^*\in\phi(\Lambda_0)\subseteq\Pi_0$, there exists $z\in\Lambda_0$ such that

$$\kappa(z,\phi\varpi^*)=\kappa(\Lambda,\Pi). \tag{4}$$

We can consider $\varpi_{r+1}\neq z$ for all $r\geq 1$. Otherwise, since $\varpi_r\neq\varpi_{r+1}$ for all $r\geq 1$, we can find subsequences satisfying $\varpi_{r_k+1}\neq z$ for all $k\geq 1$. Then, from (2) and (4), we have

$$\kappa(\varpi_{r+1},z)\leq q\Big(\kappa(\varpi_r,\varpi^*)+\big|\kappa(\varpi_r,\varpi_{r+1})-\kappa(\varpi^*,z)\big|\Big), \tag{5}$$

for all $r\geq 1$. Taking the limit as $r\to\infty$ in the last inequality, from (3) and (5), we get

$$\kappa(\varpi^*,z)\leq q\kappa(\varpi^*,z),$$

which implies $\varpi^*=z$. Hence, from (4), there exists ϖ^* in Λ such that $\kappa(\varpi^*,\phi\varpi^*)=\kappa(\Lambda,\Pi)$. $\square$

The following example is important as it shows that the approach of Haghi et al. [13] cannot be applied to Theorem 4.

Example 1 Let $\Xi=(\{0\}\cup[1,\infty))\times(\{0\}\cup[1,\infty))$ and $\kappa:\Xi\times\Xi\to[0,\infty)$ defined by

$$\kappa(\varpi,\varsigma)=\begin{cases}\frac{\varpi_1}{2}+\varpi_2 & ,\quad \varpi=\varsigma\\ \varpi_1+\varsigma_1+\max\{\varpi_2,\varsigma_2\} & ,\quad \text{otherwise}\end{cases},$$

where $\varpi=(\varpi_1,\varpi_2)$ and $\varsigma=(\varsigma_1,\varsigma_2)\in\Xi$. Then, (Ξ,κ) is a complete partial metric space. Also, for all $\varpi\in\Xi$, we have

$$B_\kappa\left(\varpi,\frac{\varpi_1}{3}\right)=\{\varpi\}\text{ for all }\varpi_1>0,$$

and

$$B_\kappa\big((0,\varpi_2),\varepsilon\big)=\{0\}\times\big\{\{0\}\cup[1,\varpi_2+\varepsilon)\big\}.$$

Consider the sets

$$\Lambda=\{(3,8),(1,11)\},$$

and

$$\Pi=\{(1,7),(0,10)\}.$$

Then, we have $\kappa(\Lambda,\Pi)=12$, $\Lambda_0=\Lambda$ and $\Pi_0=\Pi$. Further, it can be seen that Λ is closed. If we define a mapping $\phi:\Lambda\to\Pi$ by

$$\phi(\varpi_1,\varpi_2)=\begin{cases}(1,7) & ,\ \varpi_1=3\\ (0,10) & ,\ \varpi_1=1\end{cases},$$

then we get $\phi(\Lambda_0)\subseteq\Pi_0$ and ϕ is continuous mapping with respect to τ_κ. Also, the mapping ϕ satisfies (1) for $q=\frac{16}{17}$. Indeed, let ξ,ν,ϖ,ς be arbitrary points in Λ with $\xi\neq\nu$ satisfying

$\kappa(\xi, \phi\varpi) = \kappa(\Lambda, \Pi)$
$\kappa(v, \phi\varsigma) = \kappa(\Lambda, \Pi).$

Then, we get $\varpi = \xi \neq v = \varsigma \in \{(3,8),(1,11)\}$. In this case, we have

$$\begin{aligned}\kappa(\xi, v) =&15\\ \leq&\frac{16}{17}\left\{15+\left|\frac{3}{2}+8-\frac{1}{2}-11\right|\right\}\\ =&q\{\kappa(\varpi,\varsigma)+|\kappa(\xi,\varpi)-\kappa(v,\varsigma)|\}.\end{aligned}$$

Thus, all hypotheses of Theorem 4 are satisfied, and so ϕ has a best proximity point in Λ. Now, consider the standard metric θ defined by

$$\theta(\varpi,\varsigma)=\begin{cases}0 & ,\ \varpi=\varsigma\\ \kappa(\varpi,\varsigma) & ,\ \varpi\neq\varsigma\end{cases},$$

as in Proposition 2.1 of [13]. However, for $\xi = \varpi = (3,8)$ and $\varsigma = v = (1,11) \in \Lambda$, we have

$$\theta(\xi,v)=15>15q=q\{\theta(\varpi,\varsigma)+|\theta(\xi,\varpi)-\theta(v,\varsigma)|\},$$

for all $q \in [0,1)$. Therefore, ϕ does not satisfy (1) with respect to θ.

Taking $\Lambda = \Pi = \Xi$ in Theorem 4, we obtain the following fixed point result.

Corollary 1 *Let $\phi : (\Xi,\kappa) \to (\Xi,\kappa)$ be a mapping where (Ξ,κ) is a complete partial metric space. Then, there is a point ϖ^* in Ξ satisfying $\varpi^* = \phi\varpi^*$ if there is q in $[0, 1)$ satisfying, for all $\varpi \neq \varsigma$,*

$$\kappa(\phi\varpi,\phi\varsigma)\leq q(\kappa(\varpi,\varsigma)+|\kappa(\varpi,\phi\varpi)-\kappa(\varsigma,\phi\varsigma)|). \tag{6}$$

Now, we introduce the following definition of proximal p-contraction of the second type.

Definition 7 Let (Ξ,κ) be a partial metric space and $\emptyset \neq \Lambda, \Pi \subseteq \Xi$. A mapping $\phi : \Lambda \to \Pi$ is said to be proximal p-contraction of the second type if there exists q in [0, 1) such that, for all $\xi_1, \xi_2, \varpi_1, \varpi_2 \in \Lambda$ with $\phi\xi_1 \neq \phi\xi_2$,

$$\left.\begin{aligned}\kappa(\xi_1,\phi\varpi_1)&=\kappa(\Lambda,\Pi)\\ \kappa(\xi_2,\phi\varpi_2)&=\kappa(\Lambda,\Pi)\end{aligned}\right\}\Rightarrow\kappa(\phi\xi_1,\phi\xi_2)$$
$$\leq q\left(\begin{array}{c}\kappa(\phi\varpi_1,\phi\varpi_2)\\ +\left|\kappa(\phi\xi_1,\phi\varpi_1)-\kappa(\phi\xi_2,\phi\varpi_2)\right|\end{array}\right).$$

Now, we give our second main result.

Theorem 5 *Let (Ξ,κ) be a complete partial metric space and $\emptyset \neq \Lambda, \Pi \subseteq \Xi$ where Π is closed and Λ is a κ-approximately compact with respect to Π. Assume that the followings hold:*

(i) *$\Lambda_0 \neq \emptyset$ and $\phi(\Lambda_0) \subseteq \Pi_0$.*
(ii) *$\phi : \Lambda \to \Pi$ is a proximal p-contraction of the second type.*
(iii) *ϕ is a continuous mapping with respect to τ_κ.*

Then, there exists ϖ^* in Λ such that $\kappa(\varpi^*, \phi\varpi^*) = \kappa(\Lambda, \Pi)$.

Proof Let $\varpi_0 \in \Lambda_0$ be any point. Since $\phi\varpi_0 \in \phi(\Lambda_0) \subseteq \Pi_0$, there exists $\varpi_1 \in \Lambda_0$ satisfying

$$\kappa(\varpi_1,\phi\varpi_0)=\kappa(\Lambda,\Pi).$$

Similarly, since $\phi\varpi_1 \in \phi(\Lambda_0) \subseteq \Pi_0$, there exists $\varpi_2 \in \Lambda_0$ satisfying

$$\kappa(\varpi_2,\phi\varpi_1)=\kappa(\Lambda,\Pi).$$

Repeating this process, we can construct a sequence $\{\varpi_r\}$ in Λ such that, for all $r \geq 1$,

$$\kappa(\varpi_{r+1},\phi\varpi_r)=\kappa(\Lambda,\Pi). \tag{7}$$

Assume that $\phi\varpi_{r_0} = \phi\varpi_{r_0+1}$ for some $r_0 \geq 1$, then from (7), ϖ_{r_0+1} is a best proximity point of ϕ. We now get $\phi\varpi_r \neq \phi\varpi_{r+1}$ for all $r \geq 1$. Because ϕ is a proximal p-contraction of the second type, we have, for all $r \geq 1$,

$$\kappa(\phi\varpi_r,\phi\varpi_{r+1})\leq q\Big(\kappa(\phi\varpi_{r-1},\phi\varpi_r)$$
$$+\left|\kappa(\phi\varpi_r,\phi\varpi_{r+1})-\kappa(\phi\varpi_{r-1},\phi\varpi_r)\right|\Big).$$

Suppose that the inequality $\kappa(\phi\varpi_{r_0-1},\phi\varpi_{r_0}) \leq \kappa(\phi\varpi_{r_0},\phi\varpi_{r_0+1})$ holds for some $r_0 \geq 1$, then we have

$$\begin{aligned}\kappa(\phi\varpi_{r_0},\phi\varpi_{r_0+1})\leq&q\Big(\kappa(\phi\varpi_{r_0-1},\phi\varpi_{r_0})\\ &+\left|\kappa(\phi\varpi_{r_0},\phi\varpi_{r_0+1})-\kappa(\phi\varpi_{r_0-1},\phi\varpi_{r_0})\right|\Big)\\ =&q\Big(\kappa(\phi\varpi_{r_0-1},\phi\varpi_{r_0})+\kappa(\phi\varpi_r,\phi\varpi_{r+1})\\ &-\kappa(\phi\varpi_{r_0-1},\phi\varpi_{r_0})\Big)\\ =&q\kappa(\phi\varpi_{r_0},\phi\varpi_{r_0+1})\\ <&\kappa(\phi\varpi_{r_0},\phi\varpi_{r_0+1}).\end{aligned}$$

This is a contradiction. Thus, we can conclude that $\kappa(\phi\varpi_r,\phi\varpi_{r+1}) < \kappa(\phi\varpi_{r-1},\phi\varpi_r)$ for all $r \geq 1$. Therefore, we get

$$\begin{aligned}\kappa(\phi\varpi_r,\phi\varpi_{r+1})\leq&q\Big(\kappa(\phi\varpi_{r-1},\phi\varpi_r)\\ &+\kappa(\phi\varpi_r,\phi\varpi_{r-1})-\kappa(\phi\varpi_r,\phi\varpi_{r+1})\Big)\\ =&2q\kappa(\phi\varpi_{r-1},\phi\varpi_r)-q\kappa(\phi\varpi_r,\phi\varpi_{r+1}),\end{aligned}$$

and so, for all $r \geq 1$,

$$\kappa(\phi\varpi_r,\phi\varpi_{r+1})\leq\left(\frac{2q}{q+1}\right)\kappa(\phi\varpi_{r-1},\phi\varpi_r).$$

Using the last inequality, we have

$$\begin{aligned}\kappa(\phi\varpi_r,\phi\varpi_{r+1}) \leq& \left(\frac{2q}{q+1}\right)\kappa(\phi\varpi_{r-1},\phi\varpi_r)\\ \leq& \left(\frac{2q}{q+1}\right)^2\kappa(\phi\varpi_{r-2},\phi\varpi_{r-1})\\ &\vdots\\ \leq& \left(\frac{2q}{q+1}\right)^r\kappa(\phi\varpi_0,\phi\varpi_1),\end{aligned}$$

for all $r \geq 1$. Now, for $m, r \geq 1$ with $m > r$, we obtain

$$\begin{aligned}&\kappa(\phi\varpi_r,\phi\varpi_m) \leq \kappa(\phi\varpi_r,\phi\varpi_{r+1}) + \kappa(\phi\varpi_{r+1},\phi\varpi_{r+2})\\ &\quad + \cdots + \kappa(\phi\varpi_{m-1},\phi\varpi_m)\\ &\leq \left(\frac{2q}{q+1}\right)^r\kappa(\phi\varpi_0,\phi\varpi_1) + \left(\frac{2q}{q+1}\right)^{r+1}\kappa(\phi\varpi_0,\phi\varpi_1)\\ &\quad + \cdots + \left(\frac{2q}{q+1}\right)^{m-1}\kappa(\phi\varpi_0,\phi\varpi_1)\\ &= \left[1 + \frac{2q}{q+1} + \cdots + \left(\frac{2q}{q+1}\right)^{m-r-1}\right]\left(\frac{2q}{q+1}\right)^r\kappa(\phi\varpi_0,\phi\varpi_1)\\ &\leq \frac{\left(\frac{2q}{q+1}\right)^r}{1-\frac{2q}{q+1}}\kappa(\phi\varpi_0,\phi\varpi_1).\end{aligned}$$

Therefore, we have $\lim_{r,m\to\infty}\kappa(\phi\varpi_r,\phi\varpi_m) = 0$. Thus, $\{\phi\varpi_r\}$ is a Cauchy sequence in Π. Because of the fact that Π is a closed subset of complete partial metric space (Ξ,κ), there is $\varsigma^* \in \Pi$ satisfying

$$\lim_{r,m\to\infty}\kappa(\phi\varpi_r,\phi\varpi_m) = \lim_{r\to\infty}\kappa(\phi\varpi_r,\varsigma^*) = \kappa(\varsigma^*,\varsigma^*) = 0.$$

Moreover, from (7), it can be seen that

$$\begin{aligned}\kappa(\varsigma^*,\Lambda) \leq& \kappa(\varsigma^*,\varpi_{r+1})\\ \leq& \kappa(\varsigma^*,\phi\varpi_r) + \kappa(\varpi_{r+1},\phi\varpi_r)\\ =& \kappa(\varsigma^*,\phi\varpi_r) + \kappa(\Lambda,\Pi)\\ \leq& \kappa(\varsigma^*,\phi\varpi_r) + \kappa(\varsigma^*,\Lambda).\end{aligned}$$

Hence, we get $\kappa(\varsigma^*,\varpi_{r+1}) \to \kappa(\varsigma^*,\Lambda)$ as $r \to \infty$. Since Λ is a κ-approximately compact with respect to Π, there exists a subsequence $\{\varpi_{r_k}\}$ of $\{\varpi_r\}$ such that

$$\lim_{k,l\to\infty}\kappa(\varpi_{r_k},\varpi_{r_l}) = \lim_{k\to\infty}\kappa(\varpi_{r_k},\varpi^*) = \kappa(\varpi^*,\varpi^*) = 0, \tag{8}$$

for some $\varpi^* \in \Lambda$. Since ϕ is a continuous mapping with respect to κ, we get

$$\lim_{k,l\to\infty}\kappa(\phi\varpi_{r_k},\phi\varpi_{r_l}) = \lim_{r\to\infty}\kappa(\phi\varpi_{r_k},\phi\varpi^*) = \kappa(\phi\varpi^*,\phi\varpi^*) \tag{9}$$

Hence, we get

$$\begin{aligned}\kappa(\Lambda,\Pi) \leq& \kappa(\varpi^*,\phi\varpi^*)\\ \leq& \left\{\begin{array}{c}\kappa(\varpi^*,\varpi_{r_k+1}) + \kappa(\varpi_{r_k+1},\phi\varpi_{r_k}) + \kappa(\phi\varpi_{r_k},\phi\varpi^*)\\ -\kappa(\varpi_{r_k+1},\varpi_{r_k+1}) - \kappa(\phi\varpi_{r_k},\phi\varpi_{r_k})\end{array}\right\}\\ =& \left\{\begin{array}{c}\kappa(\varpi^*,\varpi_{r_k+1}) + \kappa(\Lambda,\Pi) + \kappa(\phi\varpi_{r_k},\phi\varpi^*)\\ -\kappa(\varpi_{r_k+1},\varpi_{r_k+1}) - \kappa(\phi\varpi_{r_k},\phi\varpi_{r_k})\end{array}\right\},\end{aligned}$$

and so taking the limit as $k \to \infty$, we have $\kappa(\varpi^*,\phi\varpi^*) = \kappa(\Lambda,\Pi)$. □

The following example is important as it shows that the approach of Haghi et al. [13] cannot be applied to Theorem 5.

Example 2 Let (Ξ,κ) be as in Example 1. Let $\Lambda = \{(4,9),(2,12)\}$ and $\Pi = \{(2,9),(1,11)\}$. Then, we have $\kappa(\Lambda,\Pi) = 15$, $\Lambda_0 = \Lambda$ and $\Pi_0 = \Pi$. Since the single point $\{(\varpi_1,\varpi_2)\}$ is closed whenever $\varpi_1 > 0$ for all $\varpi = (\varpi_1,\varpi_2) \in \Pi$, Π is closed. Also, it can be seen that Λ is κ-approximately compact with respect to Π. Define a mapping $\phi : \Lambda \to \Pi$ by

$$\phi\varpi = \left\{\begin{array}{ll}(2,9)\ , & \varpi = (4,9)\\ (1,11)\ , & \varpi = (2,12).\end{array}\right.,$$

Hence, ϕ is a continuous mapping with respect to τ_κ. Further, we have $\phi(\Lambda_0) \subseteq \Pi_0$. We will demonstrate that ϕ is a proximal p-contraction of the second type for $q = \frac{29}{31}$. Let $\xi, v, \varpi, \varsigma$ be arbitrary points in Λ with $\phi\xi \neq \phi v$ satisfying

$$\begin{array}{l}\kappa(\xi,\phi\varpi) = \kappa(\Lambda,\Pi)\\ \kappa(v,\phi\varsigma) = \kappa(\Lambda,\Pi)\end{array}.$$

Then, we get $\phi\varpi = \phi\xi \neq \phi v = \phi\varsigma \in \{(2,9),(1,11)\}$. In this case, we have

$$\begin{aligned}\kappa(\phi\xi,\phi v) =& 14\\ \leq& \frac{29}{31}\left\{14 + \left|\frac{1}{2} + 11 - 10\right|\right\}\\ =& q\{\kappa(\phi\varpi,\phi\varsigma) + |\kappa(\phi\xi,\phi\varpi) - \kappa(\phi v,\phi\varsigma)|\}.\end{aligned}$$

Thus, all hypotheses of Theorem 5 are satisfied, and so ϕ has a best proximity point in Λ. Now, consider the metric θ defined by

$$\theta(\varpi,\varsigma) = \left\{\begin{array}{ll}0, & \varpi = \varsigma\\ \kappa(\varpi,\varsigma), & \varpi \neq \varsigma\end{array}\right.,$$

as in Proposition 2.1 of [13]. However, for $\phi\varpi = \phi\xi = (2,9)$ and $\phi v = \phi\varsigma = (1,11)$, we have

$$\theta(\phi\xi,\phi v)=14>14q=q\{\theta(\phi\varpi,\phi\varsigma)+|\theta(\phi\xi,\phi\varpi)-\theta(\phi v,\phi\varsigma)|\},$$

for all $q\in[0,1)$. Therefore, ϕ does not satisfy (1) with respect to θ.

If we take $\Lambda=\Pi=\Xi$ in Theorem 5, then we present the following fixed point result.

Corollary 2 *Let $\phi:(\Xi,\kappa)\to(\Xi,\kappa)$ be a continuous mapping with respect to τ_κ where (Ξ,κ) is a complete partial metric space. Then, there is a point ϖ^* in Ξ satisfying $\varpi^*=\phi\varpi^*$ if there exists q in $[0,1)$ such that for all $\varpi,\varsigma\in\Xi$ with $\phi\varpi\neq\phi\varsigma$,*

$$\kappa(\phi^2\varpi,\phi^2\varsigma)\leq q\Big(\kappa(\phi\varpi,\phi\varsigma)+\Big|\kappa(\phi^2\varpi,\phi\varpi)-\kappa(\phi^2\varsigma,\phi\varsigma)\Big|\Big).$$

Nonlinear fractional differential equations

The existence and uniqueness of the solution to nonlinear fractional differential equations are examined in this section. First, let us recall some basic definitions of fractional calculus (see [12, 14, 16, 17, 26, 27]). The Caputo derivative of h of order $\alpha>0$ is defined by

$$^{C}D^{\alpha}(h(\beta))=\frac{1}{\Gamma(r-\alpha)}\int_0^\beta(\beta-\jmath)^{r-\alpha-1}h^{(r)}(\jmath)\mathrm{d}\jmath,$$

where $h:[0,\infty)\to\mathbb{R}$ is a continuous function and $r=[\alpha]+1$ with $[\alpha]$ denoting the integer part of a positive real number α, and Γ is the gamma function

The following nonlinear fractional differential equation of Caputo type

$$^{C}D^{\alpha}(h(\beta))=h(\beta,\varpi(\beta)),\tag{10}$$

with integral boundary conditions

$$\varpi(0)=0 \text{ and } \varpi(1)=\int_0^\delta\varpi(\jmath)\mathrm{d}\jmath,$$

where $h:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and $1<\alpha\leq2$, $0<\delta<1$, $\varpi\in C[0,1]$. Since h is a continuous, it is clear that Eq. (10) is equal to the integral equation.

$$\begin{aligned}\varpi(\beta)=&\frac{1}{\Gamma(\alpha)}\int_0^\beta(\beta-\jmath)^{\alpha-1}h(\jmath,\varpi(\jmath))\mathrm{d}\jmath\\&-\frac{2\beta}{(2-\delta^2)\Gamma(\alpha)}\int_0^1(1-\jmath)^{\alpha-1}h(\jmath,\varpi(\jmath))\mathrm{d}\jmath\\&+\frac{2\beta}{(2-\delta^2)\Gamma(\alpha)}\int_0^\delta\left(\int_0^\jmath(\jmath-m)^{\alpha-1}h(m,\varpi(m))\mathrm{d}m\right)\mathrm{d}\jmath.\end{aligned}\tag{11}$$

Theorem 6 *Assume the following conditions are true:*

(i) for all $\varpi\in C[0,1]$ and $\beta\in[0,1]$, the mapping $\phi:\Xi\to\Xi$

$$\begin{aligned}\phi\varpi(\beta)=&\frac{1}{\Gamma(\alpha)}\int_0^\beta(\beta-\jmath)^{\alpha-1}h(\jmath,\varpi(\jmath))\mathrm{d}\jmath\\&-\frac{2\beta}{(2-\delta^2)\Gamma(\alpha)}\int_0^1(1-\jmath)^{\alpha-1}h(\jmath,\varpi(\jmath))\mathrm{d}\jmath\\&+\frac{2\beta}{(2-\delta^2)\Gamma(\alpha)}\int_0^\delta\left(\int_0^\jmath(\jmath-m)^{\alpha-1}h(m,\varpi(m))\mathrm{d}m\right)\mathrm{d}\jmath,\end{aligned}$$

is a continuous mapping.

(ii) there is $q\in[0,1)$ such that

$$|h(\jmath,\varpi(\jmath))|+|h(\jmath,\varsigma(\jmath))|$$
$$\leq\frac{\Gamma(\alpha+1)}{5}q\left\{\begin{array}{c}|\varpi(\jmath)|+|\varsigma(\jmath)|\\+\left|\begin{array}{l}\sup_{\jmath\in[0,1]}|\varpi(\jmath)|+\sup_{\jmath\in[0,1]}|\phi\varpi(\jmath)|\\-\sup_{\jmath\in[0,1]}|\varsigma(\jmath)|-\sup_{\jmath\in[0,1]}|\phi\varsigma(\jmath)|\end{array}\right|\end{array}\right\}.$$

Then, the problem (10) has a solution.

Proof Let $\Xi=C[0,1]$ and $\kappa:\Xi\times\Xi\to[0,\infty)$ be a function defined by

$$\kappa(\xi,v)=\begin{cases}\sup_{\beta\in[0,1]}|\xi(\beta)|, & \xi=v\\\sup_{\beta\in[0,1]}|\xi(\beta)|+\sup_{\beta\in[0,1]}|v(\beta)|, & \xi\neq v\end{cases},$$

for all $\xi,v\in\Xi$ and $\beta\in[0,1]$. Hence, (Ξ,κ) is a complete partial metric space. Now, we will demonstrate that ϕ holds (6). For all $\varpi,\varsigma\in\Xi$ and $\beta\in[0,1]$, we have

$$
\begin{aligned}
|\phi\varpi(\beta)| + |\phi\varsigma(\beta)| = &\left| \frac{1}{\Gamma(\alpha)} \int_0^\beta (\beta-\jmath)^{\alpha-1} h(\jmath, \varpi(\jmath)) \mathrm{d}\jmath \right. \\
&- \frac{2\beta}{(2-\delta^2)\Gamma(\alpha)} \int_0^1 (1-\jmath)^{\alpha-1} h(\jmath, \varpi(\jmath)) \mathrm{d}\jmath \\
&\left. + \frac{2\beta}{(2-\delta^2)\Gamma(\alpha)} \int_0^\delta \left(\int_0^\jmath (\jmath-m)^{\alpha-1} h(m, \varpi(m)) \mathrm{d}m \right) \mathrm{d}\jmath \right| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_0^\beta (\beta-\jmath)^{\alpha-1} h(\jmath, \varsigma(\jmath)) \mathrm{d}\jmath \right. \\
&- \frac{2\beta}{(2-\delta^2)\Gamma(\alpha)} \int_0^1 (1-\jmath)^{\alpha-1} h(\jmath, \varsigma(\jmath)) \mathrm{d}\jmath \\
&\left. + \frac{2\beta}{(2-\delta^2)\Gamma(\alpha)} \int_0^\delta \left(\int_0^\jmath (\jmath-m)^{\alpha-1} h(m, \varsigma(m)) \mathrm{d}m \right) \mathrm{d}\jmath \right| \\
\le &\frac{1}{\Gamma(\alpha)} \left\{ \int_0^\beta |\beta-\jmath|^{\alpha-1} (|h(\jmath, \varpi(\jmath))| + |h(\jmath, \varsigma(\jmath))|) \mathrm{d}\jmath \right\} \\
&+ \frac{2\beta}{(2-\delta^2)\Gamma(\alpha)} \left\{ \int_0^1 (1-\jmath)^{\alpha-1} (|h(\jmath, \varpi(\jmath))| + |h(\jmath, \varsigma(\jmath))|) \mathrm{d}\jmath \right\} \\
&+ \frac{2\beta}{(2-\delta^2)\Gamma(\alpha)} \left\{ \int_0^\delta \left(\int_0^\jmath (\jmath-m)^{\alpha-1} \begin{pmatrix} |h(m, \varpi(m))| \\ +|h(m, \varsigma(m))| \end{pmatrix} \mathrm{d}m \right) \mathrm{d}\jmath \right\} \\
\le &\sup_{\beta\in[0,1]} \left\{ \int_0^\beta \left(\begin{matrix} \frac{|\beta-\jmath|^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{5} \\ \times q \left\{ \begin{matrix} |\varpi(\jmath)| + |\varsigma(\jmath)| \\ + \left| \begin{matrix} \sup_{\jmath\in[0,1]} |\varpi(\jmath)| \\ + \sup_{\jmath\in[0,1]} |\phi\varpi(\jmath)| \\ - \sup_{\jmath\in[0,1]} |\varsigma(\jmath)| \\ - \sup_{\jmath\in[0,1]} |\phi\varsigma(\jmath)| \end{matrix} \right| \end{matrix} \right\} \end{matrix} \right) \mathrm{d}\jmath \right\} \\
&+ \frac{2\beta}{(2-\delta^2)} \left\{ \int_0^1 \left(\begin{matrix} \frac{(1-\jmath)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{5} \\ \times q \left\{ \begin{matrix} |\varpi(\jmath)| + |\varsigma(\jmath)| \\ + \left| \begin{matrix} \sup_{\jmath\in[0,1]} |\varpi(\jmath)| \\ + \sup_{\jmath\in[0,1]} |\phi\varpi(\jmath)| \\ - \sup_{\jmath\in[0,1]} |\varsigma(\jmath)| \\ - \sup_{\jmath\in[0,1]} |\phi\varsigma(\jmath)| \end{matrix} \right| \end{matrix} \right\} \end{matrix} \right) \mathrm{d}\jmath \right\} \\
&+ \frac{2\beta}{(2-\delta^2)} \left\{ \int_0^\delta \left(\int_0^\jmath \left[\begin{matrix} \frac{(\jmath-m)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{5} \\ \times q \left\{ \begin{matrix} |\varpi(m)| + |\varsigma(m)| \\ + \left| \begin{matrix} \sup_{m\in[0,1]} |\varpi(m)| \\ + \sup_{m\in[0,1]} |\phi\varpi(m)| \\ - \sup_{m\in[0,1]} |\varsigma(m)| \\ - \sup_{m\in[0,1]} |\phi\varsigma(m)| \end{matrix} \right| \end{matrix} \right\} \end{matrix} \right] \mathrm{d}m \right) \mathrm{d}\jmath \right\} \\
\le &\frac{\Gamma(\alpha+1)}{5} q\{\kappa(\varpi,\varsigma) + |\kappa(\varpi,\phi\varpi) - \kappa(\varsigma,\phi\varsigma)|\} \\
&\times \sup_{\beta\in[0,1]} \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{2\beta}{(2-\delta^2)} \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \right) \right\} \\
\le &q\{\kappa(\varpi,\varsigma) + |\kappa(\varpi,\phi\varpi) - \kappa(\varsigma,\phi\varsigma)|\}
\end{aligned}
$$

which implies that

$$
\sup_{\beta\in[0,1]} |\phi\varpi(\beta)| + \sup_{\beta\in[0,1]} |\phi\varsigma(\beta)| \le q\{\kappa(\varpi,\varsigma) + |\kappa(\varpi,\phi\varpi) - \kappa(\varsigma,\phi\varsigma)|\}.
$$

Hence, we get

$$
\kappa(\phi\varpi, \phi\varsigma) \le q\{\kappa(\varpi,\varsigma) + |\kappa(\varpi,\phi\varpi) - \kappa(\varsigma,\phi\varsigma)|\}.
$$

As a result, all of Corollary 1's hypotheses are met, and thus ϕ has a fixed point. Consequently, there is a solution to nonlinear fractional differential equation of Caputo type (10). □

Conclusion

Using best proximity point theory on complete partial metric spaces, we aim to expand several famous conclusions that have already been in the literature. We first introduce some new concepts, named proximal p -contractions of the first type and proximal p-contractions of the second type, and then prove some best proximity point theorems for such mappings on partial metric spaces. We also get some notable examples where our results are valid and the results in [13] are not. Hence, our results are a real generalization of some of the results that exist in standard metric spaces and partial metric spaces. Finally, with the help of our results, we obtain sufficient conditions for the existence of the solution of nonlinear fractional differential equations.

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