Tangent nonlinear equation in context of fractal fractional operators with nonsingular kernel

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Abstract

In this manuscript, we investigate the approximate solutions to the tangent nonlinear packaging equation in the context of fractional calculus. It is an important equation because shock and vibrations are unavoidable circumstances for the packaged goods during transport from production plants to the consumer. We consider the fractal fractional Caputo operator and Atangana–Baleanu fractal fractional operator with nonsingular kernel to obtain the numerical consequences. Both fractal fractional techniques are equally good, but the Atangana–Baleanu Caputo method has an edge over Caputo method. For illustrations and clarity of our main results, we provided the numerical simulations of the approximate solutions and their physical interpretations. This paper contributes to the new applications of fractional calculus in packaging systems.

Keywords Fractal fractional caputo operator · Atangana–Baleanu fractal fractional operator · Tangent nonlinear packaging equation

Introduction

The packaging system model is an old problem in real-life problems and used in all life activities. Many products in the world, in commerce, in pharmaceutical products, the consumers buy goods and products which are carefully packaged. This problem is nowadays more important due to the

development of the industry, technology, and all products are obligatorily packaged. The problem of Packaging is first introduced in the literature by Newton in 1968. Newton is the first to consider the concept of damage boundary concept [1]. The importance of the packaging is to protect the products and goods to their interaction with the external environment phenomena. In general, shock and vibration are two phenomena that are not evitable during the displacement of the goods between two different points [2]. The vibrations are in general generated by the transports materials as the automobiles, the bus, the flights in the landings, the trains during the transits. It is essential to predict the behaviors of the packaging materials used for specific products or goods when external forces can be applied to it, as the vibrations or the shocks. That explains the investigations done in this direction in the packagings systems. Nowadays, there exist many researchers who focus on packagings systems, which constitute a field that receives some studies. In [3], Wang et al. propose a dynamic model coupled with linear and nonlinear stiffness for nonlinear cushioning packaging systems based on critical components. [4] proposed the approximate solution of the constructive equations proposed for packaging systems. They mainly develop He Chengtian's inequalities. In [5], Wang et al. introduce a novel concept of dropping the damage boundary surface. Under the advancement in the methods to get the analytical solutions for the

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fractional differential equations and differential equations in general, Kuang et al. in [2] propose the homotopy perturbation method to approach as well the exact solution of the constructive equation used to describe the Packaging systems. For History, the first packaged product appear in England at 1746. Note that the types of packaging materials used for the products or the goods depend on the shocks or the vibrations received by these goods or products during the transits. In general, the packaging systems are constituted by the plastics, the cartons, and the jettable papers. The main and urgent question is what is the impact of the packaging systems in an economy for a country. This question is important because, during the exportation or importation of the products or the goods, when the packagings are not well done, the firms can lose many products or goods during the transits. It impacts the utility functions for the considered firms. Note that the packaging systems have a part in the economic growth rate. As we know, in general, the growth is defined as an increase in the capacity of a country's economy to produce goods and services, using a comparison between two defined periods relatively short. The manufactured products need to be packaged, that is the way the packaging systems impact the economy of a country. Many researchers investigated the tangent nonlinear packaging system using variational iteration approach, homotopy perturbation approach and homotopy analysis approach, which are semi-analytical approaches.

Fractional calculus receives many interests these last decades. Many new fractional operators [6] and Mittag–Leffler [7, 8] kernel as Atangana–Baleanu (AB) derivative in Caputo sense [9] appear to enrich the existing fractional derivatives (FVs) as the Caputo derivative (CV) [10, 11] and the Riemann–Liouville derivative [10, 11] and their existing generalizations [12, 13]. Many applications related to the FVs operators (FDOs) have already been addressed in the literature, in physics [14, 15], in mechanical fluids [16], in finance models [17–19], in diffusion equations [20] and others. It is important to mention that there exist many FVs, which are not cited above, derived from the CV and the Riemann–Liouville (RL) derivatives.

In this paper, we consider the tangent nonlinear packaging equation with FV with Mittag–Leffler kernel. The first objective is to introduce the tangent nonlinear packaging equation in the context of fractional calculus. And the second objective is to propose a numerical scheme to get the approximate solutions of the proposed model for the packaging systems. We mainly use the Caputo and the Atangana–Baleanu–Caputo (ABC) fractal FV schemes recently introduced in fractional calculus in [21, 22]. The obtained solutions are represented, and the physical interpretations are made as well. The authors of [23] have used the conformable fractional-order technique to study the analytical and numerical solutions of HIV-1 infection with CD4+ T

cells. In [24], the author has given a review on Newland wavelets using fractional calculus. Benney–Lin equation is solved by using two fractional-order techniques [25], whereas the authors of [27] have used the q-HATM to find the numerical solution of phi-four equation. Rabotnov fractional exponential kernel is applied to model the anomalous heat transfer [26], whereas fractional homotopy transform method is used to solve the non-homogeneous heat conduction equation in fractal domain [28]. The authors of [29] have used the truncated M-fractional derivative with applications to fractional-order differential equation including classical properties.

In Zeliha et al. [30], have used the Atangana–Baleanu derivative with Mittag–Leffler kernel to solve the fractional Burger differential equation, whereas the authors of [31] have used the Adams Bashforth Moulton scheme with variable-order Atangana–Baleanu Caputo derivative to solve the chaotic system. The comparison of three different techniques are given in [32] for adsorption process of dye removal nonlinear equation. The writers of [33] have studied disease model with white noise. The local fractional q-homotopy transform method is used in [34] to study the fractional-order Tricomi equation. To study the Wick fractional stochastic PDEs, the authors of [35] have used the time conformable derivative. The authors of [36] have used the homotopy analysis method for the solution of linear fractional partial differential equations, whereas Mehdi Deghan et al. have used the same technique to study the fractional-order partial differential equations [39]. The mitigating internet bottleneck with quadratic–cubic nonlinearity containing the q-derivative has been considered that describes the control of internet traffic [37]. In [38], the researchers have used to study the dynamics of the fractional generalized CBS-BK equation.

The authors of [40] have used the fractal fractional-order techniques to find the dynamics of Corona Virus in Pakistan. Three different techniques with fractal fractional order are used in [41] and then compare the results, whereas the same authors of [42] have used the fractal fractional Caputo derivative to study diarrhea transmission dynamics under the use of real data. The authors of [43] have used the fractal fractional Caputo derivative to study the dynamics of rubella epidemic model. In [44], Ali et al. have used the fractal fractional derivative to study the dynamical behavior of HIV/AIDS epidemic model. Motivated by the above discussion on the fractional and fractal fractional-order techniques, we have used the fractal fractional-order techniques on the tangent nonlinear packaging system.

The paper is structured as follows: In section “Fractional calculus tools”, we recall the basics definitions and tools necessary for the present studies. In section “Constructive equations”, we recall the constructive equations related to the tangent nonlinear packaging system.

In section “Procedure solutions”, we describe the procedure of the solutions for the presented model. In section “Graphical representations and interpretations”, the graphical representations and interpretations have been performed. In section “Conclusion”, we finish this paper by concluding remarks.

Fractional calculus tools

In this paper, the fractal fractional calculus is defined and the related details are given in [22], which are used to derive significant results.

Definition 2.1 Let $z(\tau)$ be continuous and the fractal be differentiable on (c, d) with the order β . Then, we can use the following equation in the RL sense with a power-law-type kernel to calculate the fractal FV of $z(\tau)$ with the order μ :

$${}^{\text{FFP}}D_{0,\tau}^{\mu,\beta} z(\tau) = \frac{1}{\Gamma(m-\mu)} \frac{d}{d\tau^\beta} \int_0^\tau \frac{z(\xi)}{(\tau-\xi)^{\mu+1-m}} d\xi, \quad (1)$$

where $m-1 < \mu, \beta \leq m \in \mathbb{N}$ and $\frac{dz(\xi)}{d\xi^\beta} = \lim_{\tau \rightarrow \xi} \frac{z(\tau) - z(\xi)}{\tau^\beta - \xi^\beta}$.

Definition 2.2 Let $z(\tau)$ be continuous and the fractal be differentiable on (c, d) with the order β . Then, we can use the following equation in the RL sense with an exponentially decaying-type kernel to calculate the fractal FV of $z(\tau)$ with the order μ :

$${}^{\text{FFE}}D_{0,\tau}^{\mu,\beta} z(\tau) = \frac{M(\mu)}{(1-\mu)} \frac{d}{d\tau^\beta} \int_0^\tau \exp\left(\frac{-\mu(\tau-\xi)}{(1-\mu)}\right) z(\xi) d\xi, \quad (2)$$

where $\mu > 0, \beta \leq m \in \mathbb{N}$ and $M(0) = M(1) = 1$.

Definition 2.3 Let $z(\tau)$ be continuous and the fractal be differentiable on (c, d) with the order β . Then, we can use the following equation in the RL sense with a generalized Mittag-Leffler-type kernel to calculate the fractal FV of $z(\tau)$ with the order μ :

$${}^{\text{FFM}}D_{0,\tau}^{\mu,\beta} z(\tau) = \frac{AA(\mu)}{(1-\mu)} \frac{d}{d\tau^\beta} \int_0^\tau E_\mu\left(\frac{-\mu(\tau-\xi)^\mu}{(1-\mu)}\right) z(\xi) d\xi, \quad (3)$$

where $0 < \mu, \beta \leq 1$ and $AA(\mu) = 1 - \mu + \frac{\mu}{\Gamma(\mu)}$.

Definition 2.4 Let $z(\tau)$ be continuous on an open interval (c, d) . Then, the following equation explains how to calculate the fractal fractional integral of $z(\tau)$ with the order μ and a power-law-type kernel:

$${}^{\text{FFP}}J_{0,\tau}^\mu z(\tau) = \frac{\beta}{\Gamma(\mu)} \int_0^\tau (\tau-\xi)^{\mu-1} \xi^{\mu-1} z(\xi) d\xi. \quad (4)$$

Definition 2.5 Let $z(\tau)$ be continuous on an open interval (c, d) . Then, the following equation explains how to calculate the fractal fractional integral of $z(\tau)$ with the order μ and an exponentially decaying-type kernel:

$${}^{\text{FFE}}J_{0,\tau}^{\mu,\beta} z(\tau) = \frac{\mu\beta}{M(\mu)} \int_0^\tau \xi^{\mu-1} z(\xi) d\xi + \frac{\beta(1-\mu)\tau^{\beta-1} z(\tau)}{M(\mu)}. \quad (5)$$

Definition 2.6 Let $z(\tau)$ be continuous on an open interval (c, d) . Then, the following equation explains how to calculate the fractal fractional integral of $z(\tau)$ with the order μ and a generalized Mittag-Leffler-type kernel:

$$\begin{aligned} {}^{\text{FFM}}J_{0,\tau}^{\mu,\beta} z(\tau) &= \frac{\mu\beta}{AA(\mu)} \int_0^\tau \xi^{\mu-1} z(\xi) (\tau-\xi)^{\mu-1} d\xi \\ &\quad + \frac{\beta(1-\mu)\tau^{\beta-1} z(\tau)}{AA(\mu)}. \end{aligned} \quad (6)$$

Constructive equations

This section explains how to apply the fractal fractional calculus to the tangent nonlinear packaging system [2], which is usually available in packaging engineering. Also, it describes how to express the governing equations. In doing so, the Caputo and ABC fractal FDOs are used with fractal dimension β and fractional order μ .

$$m\ddot{x} + 2k_0 \frac{d_b}{\pi} \tan \frac{\pi x}{d_b} = 0, \quad (7)$$

$$x(0) = 0, \dot{x} = \sqrt{2gh}. \quad (8)$$

Where the coefficient m is the mass of the packaged product, the coefficient x is the displacement of the product, k_0 is the coupling stiffness of the cushioning material, d_b is the compression limit of the cushioning material, and finally, h is the dropping height.

Equation(7) should be converted into a first-order differential equation since the equation (7) is of second order. Then, the fractal FV demarcations should be applied to this system of equations. Thus, equation (7) can be given as:

$$\begin{aligned} {}^{**}D_{0,\tau}^{\mu,\beta} x(\tau) &= y(\tau), \\ {}^{**}D_{0,\tau}^{\mu,\beta} y(\tau) &= -2 \frac{k_0 d_b}{m\pi} \tan \frac{\pi x(\tau)}{d_b}, \end{aligned} \quad (9)$$

where the initial conditions $x(0) = 0$ and $y(0) = \sqrt{2gh}$, k_0 , d_b , g , h , and m are some real parameters.

It should be mentioned that the notation $**$ in the above equation (9) is the fractal FV under either Caputo (${}^{\text{FFP}}D_{0,\tau}^{\mu,\beta}$) or ABC (${}^{\text{FFM}}D_{0,\tau}^{\mu,\beta}$) operator. Also, Table 1 shows some values arbitrarily selected for the parameters involved in our systems for the simulation purpose.

Procedure solutions

At this point, under the Caputo (${}^{\text{FFP}}D_{0,\tau}^{\mu,\beta}$) and ABC (${}^{\text{FFM}}D_{0,\tau}^{\mu,\beta}$) operator, the structure of the two numerical methods is set. Also, under the fractal FDO of the Caputo type, the chaotic strange attractors (9) are expressed as follows:

$$\begin{aligned} {}^{\text{FFP}}D_{0,\tau}^{\mu,\beta}x(\tau) &= z_1(\tau, x, y), \\ {}^{\text{FFP}}D_{0,\tau}^{\mu,\beta}y(\tau) &= z_2(\tau, x, y), \end{aligned} \quad (10)$$

where, $z_1(\tau, x, y)$ and $z_2(\tau, x, y)$ are on the right-hand side of the related system under observation. To capture self-similarities in the above given system, the fractal dimension is introduced here with two state variables, namely $(x(\tau))$ and $(y(\tau))$. Since the fractional integral is differentiable, it is possible to convert the above system, (10), to the Volterra case. Also, the following equation is the converted form of the fractal FV in the RL sense:

$$\frac{1}{\Gamma(1-\mu)} \frac{d}{d\tau} \int_0^\tau \frac{z(\xi)(\tau-\xi)^\mu}{\beta\tau^{\beta-1}} d\xi, \quad (11)$$

using which our system turns into the following equation:

$$\begin{aligned} {}^{RL}D_{0,\tau}^{\mu,\beta}x(\tau) &= \beta\tau^{\beta-1}z_1(\tau, x, y), \\ {}^{RL}D_{0,\tau}^{\mu,\beta}y(\tau) &= \beta\tau^{\beta-1}z_2(\tau, x, y). \end{aligned} \quad (12)$$

Now, the RL derivative should be replaced with the CV so that the integer-order initial conditions could be used. Then, the RL fractional integral should be applied on both sides, which yields the following equation:

$$\begin{aligned} x(\tau) &= x(0) + \frac{\beta}{\Gamma(\mu)} \int_0^\tau (\tau-\xi)^{\mu-1} \xi^{\beta-1} z_1(\xi, x, y) d\xi, \\ y(\tau) &= y(0) + \frac{\beta}{\Gamma(\mu)} \int_0^\tau (\tau-\xi)^{\mu-1} \xi^{\beta-1} z_2(\xi, x, y) d\xi. \end{aligned} \quad (13)$$

Using the approach at τ_{n+1} , the numerical method of the related system could be presented. Thus, the system (13) turns into the following equation:

$$\begin{aligned} x(\tau_{n+1}) &= x(0) \\ &+ \frac{\beta}{\Gamma(\mu)} \int_0^{\tau_{n+1}} (\tau_{n+1}-\xi)^{\mu-1} \xi^{\beta-1} z_1(\xi, x, y) d\xi, \\ y(\tau_{n+1}) &= y(0) \\ &+ \frac{\beta}{\Gamma(\mu)} \int_0^{\tau_{n+1}} (\tau_{n+1}-\xi)^{\mu-1} \xi^{\beta-1} z_2(\xi, x, y) d\xi. \end{aligned} \quad (14)$$

Then, the integral obtained above could be approximated to:

$$\begin{aligned} x_{n+1} &= x(0) \\ &+ \frac{\beta}{\Gamma(\mu)} \sum_{k=0}^n \int_{\tau_k}^{\tau_{k+1}} (\tau_{n+1}-\xi)^{\mu-1} \xi^{\beta-1} z_1(\xi, x, y) d\xi, \\ y_{n+1} &= y(0) \\ &+ \frac{\beta}{\Gamma(\mu)} \sum_{k=0}^n \int_{\tau_k}^{\tau_{k+1}} (\tau_{n+1}-\xi)^{\mu-1} \xi^{\beta-1} z_2(\xi, x, y) d\xi. \end{aligned} \quad (15)$$

The function $\xi^{\beta-1} z_1(\xi, x, y)$ is approximated within the finite interval $[\tau_k, \tau_{k+1}]$ using the Lagrangian piece-wise interpolation, such that:

$$\begin{aligned} Z_{1k}(\xi) &= \frac{(\xi-\tau_{k-1})\tau_k^{\beta-1}}{\tau_k-\tau_{k-1}} z_1(\tau_k, x_k, y_k) \\ &- \frac{(\xi-\tau_k)\tau_{k-1}^{\beta-1}}{\tau_k-\tau_{k-1}} z_1(\tau_{k-1}, x_{k-1}, y_{k-1}), \\ Z_{2k}(\xi) &= \frac{(\xi-\tau_{k-1})\tau_k^{\beta-1}}{\tau_k-\tau_{k-1}} z_2(\tau_k, x_k, y_k) \\ &- \frac{(\xi-\tau_k)\tau_{k-1}^{\beta-1}}{\tau_k-\tau_{k-1}} z_2(\tau_{k-1}, x_{k-1}, y_{k-1}). \end{aligned} \quad (16)$$

Therefore, the following equation can be obtained:

$$\begin{aligned} x_{n+1} &= x(0) \\ &+ \frac{\beta}{\Gamma(\mu)} \sum_{k=0}^n \int_{\tau_k}^{\tau_{k+1}} (\tau_{n+1}-\xi)^{\mu-1} Z_{1k}(\xi) d\xi, \\ y_{n+1} &= y(0) \\ &+ \frac{\beta}{\Gamma(\mu)} \sum_{k=0}^n \int_{\tau_k}^{\tau_{k+1}} (\tau_{n+1}-\xi)^{\mu-1} Z_{2k}(\xi) d\xi. \end{aligned} \quad (17)$$

Simplifying the integrals on the right-hand side of the above equation, the numerical method for the systems under the fractal fractional CV operator is obtained as follows:

$$\begin{aligned}
x_{n+1} &= x(0) \\
&+ \frac{\beta(\Delta\tau)^\mu}{\Gamma(\mu+2)} \sum_{k=0}^n \left[\tau_k^{\beta-1} z_1(\tau_k, x_k, y_k) ((n+1-k)^\mu \right. \\
&(n-k+2+\mu) \\
&- (n-k)^\mu (n-k-2+2\mu)) \\
&- \tau_{k-1}^{\beta-1} z_1(\tau_{k-1}, x_{k-1}, y_{k-1}) \\
&\left. ((n+1-k)^{\mu+1} - (n-k)^\mu (n-k+1+\mu)) \right], \\
y_{n+1} &= y(0)
\end{aligned} \tag{18}$$

$$\begin{aligned}
&+ \frac{\beta(\Delta\tau)^\mu}{\Gamma(\mu+2)} \sum_{k=0}^n \left[\tau_k^{\beta-1} z_2(\tau_k, x_k, y_k) ((n+1-k)^\mu \right. \\
&(n-k+2+\mu) \\
&- (n-k)^\mu (n-k-2+2\mu)) \\
&- \tau_{k-1}^{\beta-1} z_2(\tau_{k-1}, x_{k-1}, y_{k-1}) \\
&\left. ((n+1-k)^{\mu+1} - (n-k)^\mu (n-k+1+\mu)) \right],
\end{aligned}$$

where z_1 and z_2 are the right-hand sides of the related system (9) under observation.

Now, the fractal fractional differential operator of ABC type should be considered. In this regard, we can have the following equation:

$$\begin{aligned}
{}^{ABR}D_{0,\tau}^{\mu,\beta} x(\tau) &= \beta \tau^{\beta-1} z_1(\tau, x, y), \\
{}^{ABR}D_{0,\tau}^{\mu,\beta} y(\tau) &= \beta \tau^{\beta-1} z_2(\tau, x, y).
\end{aligned} \tag{19}$$

By continuing the above procedure to arrive at the AB in the Caputo sense and applying the AB integral, the following equation is obtained:

$$\begin{aligned}
x(\tau) &= x(0) + \frac{\beta \tau^{\beta-1} (1-\mu)}{AA(\mu)} z_1(\tau, x, y) \\
&+ \frac{\mu\beta}{AA(\mu)\Gamma(\mu)} \int_0^\tau (\tau - \xi)^{\mu-1} \xi^{\beta-1} z_1(\xi, x, y) d\xi, \\
y(\tau) &= y(0) + \frac{\beta \tau^{\beta-1} (1-\mu)}{AA(\mu)} z_2(\tau, x, y) \\
&+ \frac{\mu\beta}{AA(\mu)\Gamma(\mu)} \int_0^\tau (\tau - \xi)^{\mu-1} \xi^{\beta-1} z_2(\xi, x, y) d\xi.
\end{aligned} \tag{20}$$

At this point, the numerical method of this system is presented using the approach given at τ_{n+1} . Thus, the system, (16), turns into the following equation:

$$\begin{aligned}
x(\tau_{n+1}) &= x(0) + \frac{\beta \tau_n^{\beta-1} (1-\mu)}{AA(\mu)} z_1(\tau_n, x_n, y_n) \\
&+ \frac{\mu\beta}{AA(\mu)\Gamma(\mu)} \int_0^{\tau_{n+1}} (\tau_{n+1} - \xi)^{\mu-1} \xi^{\beta-1} z_1(\xi, x, y) d\xi, \\
y(\tau_{n+1}) &= y(0) + \frac{\beta \tau_n^{\beta-1} (1-\mu)}{AA(\mu)} z_2(\tau_n, x_n, y_n) \\
&+ \frac{\mu\beta}{AA(\mu)\Gamma(\mu)} \int_0^{\tau_{n+1}} (\tau_{n+1} - \xi)^{\mu-1} \xi^{\beta-1} z_2(\xi, x, y) d\xi.
\end{aligned} \tag{21}$$

Using the approximation of the integrals, the above system can be rewritten as

$$\begin{aligned}
x(\tau_{n+1}) &= x(0) + \frac{\beta \tau_n^{\beta-1} (1-\mu)}{AA(\mu)} z_1(\tau_n, x_n, y_n) \\
&+ \frac{\mu\beta}{AA(\mu)\Gamma(\mu)} \sum_{k=0}^n \int_{\tau_k}^{\tau_{k+1}} (\tau_{n+1} - \xi)^{\mu-1} \xi^{\beta-1} z_1(\xi, x, y) d\xi, \\
y(\tau_{n+1}) &= y(0) + \frac{\beta \tau_n^{\beta-1} (1-\mu)}{AA(\mu)} z_2(\tau_n, x_n, y_n) \\
&+ \frac{\mu\beta}{AA(\mu)\Gamma(\mu)} \sum_{k=0}^n \int_{\tau_k}^{\tau_{k+1}} (\tau_{n+1} - \xi)^{\mu-1} \xi^{\beta-1} z_2(\xi, x, y) d\xi.
\end{aligned} \tag{22}$$

Then, the following numerical method for the fractal FV could be obtained by approximating $\xi^{\beta-1} z_1(\xi, x, y)$ and $\xi^{\beta-1} z_2(\xi, x, y)$ within the finite interval $[\tau_k, \tau_{k+1}]$ using the Lagrangian piece-wise interpolation under the ABC type operator:

$$\begin{aligned}
x_{n+1} &= x(0) + \frac{\beta \tau_n^{\beta-1}(1-\mu)}{AA(\mu)} z_1(\tau_n, x_n, y_n) \\
&\quad + \frac{\beta(\Delta\tau)^\mu}{\Gamma(\mu+2)AA(\mu)} \\
&\quad \times \sum_{k=0}^n \left[\tau_k^{\beta-1} z_1(\tau_k, x_k, y_k) ((n+1-k)^\mu (n-k+2+\mu) \right. \\
&\quad \left. - (n-k)^\mu (n-k-2+2\mu)) \right. \\
&\quad \left. - \tau_{k-1}^{\beta-1} z_1(\tau_{k-1}, x_{k-1}, y_{k-1}) ((n+1-k)^{\mu+1} \right. \\
&\quad \left. - (n-k)^\mu (n-k+1+\mu)) \right], \\
y_{n+1} &= y(0) + \frac{\beta \tau_n^{\beta-1}(1-\mu)}{AA(\mu)} z_2(\tau_n, x_n, y_n) \\
&\quad + \frac{\beta(\Delta\tau)^\mu}{\Gamma(\mu+2)AA(\mu)} \\
&\quad \times \sum_{k=0}^n \left[\tau_k^{\beta-1} z_2(\tau_k, x_k, y_k) ((n+1-k)^\mu (n-k+2+\mu) \right. \\
&\quad \left. - (n-k)^\mu (n-k-2+2\mu)) \right. \\
&\quad \left. - \tau_{k-1}^{\beta-1} z_2(\tau_{k-1}, x_{k-1}, y_{k-1}) ((n+1-k)^{\mu+1} \right. \\
&\quad \left. - (n-k)^\mu (n-k+1+\mu)) \right],
\end{aligned} \tag{23}$$

where z_1, z_2 are the right hand sides of the respective system (9) under observation.

Hence, the numerical solution for system (9) using fractal FV under Caputo and the ABC type operators are, respectively, given by

$$\begin{aligned}
x_{n+1} &= x(0) + \frac{\beta(\Delta\tau)^\mu}{\Gamma(\mu+2)} \sum_{k=0}^n \left[\tau_k^{\beta-1} \left(y_k \right. \right. \\
&\quad \left. \left. ((n+1-k)^\mu (n-k+2+\mu) \right. \right. \\
&\quad \left. \left. - (n-k)^\mu (n-k-2+2\mu)) \right. \right. \\
&\quad \left. \left. - \tau_{k-1}^{\beta-1} \left(y_{k-1} \right) ((n+1-k)^{\mu+1} \right. \right. \\
&\quad \left. \left. - (n-k)^\mu (n-k+1+\mu)) \right] \right], \\
y_{n+1} &= y(0) + \frac{\beta(\Delta\tau)^\mu}{\Gamma(\mu+2)} \sum_{k=0}^n \left[\tau_k^{\beta-1} \left(-2 \frac{k_0 d_b}{m\pi} \tan \frac{\pi x_k}{d_b} \right) \right. \\
&\quad \left. ((n+1-k)^\mu (n-k+2+\mu) \right. \\
&\quad \left. - (n-k)^\mu (n-k-2+2\mu)) \right. \\
&\quad \left. - \tau_{k-1}^{\beta-1} \left(-2 \frac{k_0 d_b}{m\pi} \tan \frac{\pi x_{k-1}}{d_b} \right) \right. \\
&\quad \left. ((n+1-k)^{\mu+1} \right. \\
&\quad \left. - (n-k)^\mu (n-k+1+\mu)) \right];
\end{aligned} \tag{24}$$

$$\begin{aligned}
x_{n+1} &= x(0) + \frac{\beta \tau_n^{\beta-1}(1-\mu)}{AA(\mu)} \left(y_n \right) + \frac{\beta(\Delta\tau)^\mu}{\Gamma(\mu+2)AA(\mu)} \\
&\quad \times \sum_{k=0}^n \left[\tau_k^{\beta-1} \left(y_k \right) ((n+1-k)^\mu (n-k+2+\mu) \right. \\
&\quad \left. - (n-k)^\mu (n-k-2+2\mu)) \right. \\
&\quad \left. - \tau_{k-1}^{\beta-1} \left(y_{k-1} \right) ((n+1-k)^{\mu+1} \right. \\
&\quad \left. - (n-k)^\mu (n-k+1+\mu)) \right], \\
y_{n+1} &= y(0) + \frac{\beta \tau_n^{\beta-1}(1-\mu)}{AA(\mu)} \left(-2 \frac{k_0 d_b}{m\pi} \tan \frac{\pi x_n}{d_b} \right) \\
&\quad + \frac{\beta(\Delta\tau)^\mu}{\Gamma(\mu+2)AA(\mu)} \\
&\quad \times \sum_{k=0}^n \left[\tau_k^{\beta-1} \left(-2 \frac{k_0 d_b}{m\pi} \tan \frac{\pi x_k}{d_b} \right) ((n+1-k)^\mu \right. \\
&\quad \left. (n-k+2+\mu) - (n-k)^\mu (n-k-2+2\mu)) \right. \\
&\quad \left. - \tau_{k-1}^{\beta-1} \left(-2 \frac{k_0 d_b}{m\pi} \tan \frac{\pi x_{k-1}}{d_b} \right) ((n+1-k)^{\mu+1} \right. \\
&\quad \left. - (n-k)^\mu (n-k+1+\mu)) \right].
\end{aligned} \tag{25}$$

Graphical representations and interpretations

Section “Procedure solutions” discussed some numerical methods. This section uses these methods to analyze the dynamic behaviors of the two systems under observation using the fractal FDOs of Caputo and the ABC operators. In this section, the simulation time $\tau = 100$ seconds and the time step size $\Delta\tau = 0.01$ were used for all cases with varying values of the fractal dimension β and fractional order μ under different initial conditions. To perform the simulations, the parameter values shown in Table 1 were used.

System’s simulations

Left (Caputo) to right (AB)

Figures 1, 2 and 3 illustrate the dynamic behavior of the system (9) in terms of both techniques. An analysis of the figures showed that for the fractional order $\mu = 1$ and the fractal dimension $\beta = 1$, the system motion is repeated periodically about the steady states (0, 0). However, for other values of μ , namely 0.98, 0.96, 0.94, 0.92, 0.90, the nonlinearity of the system gradually disappears and the system converges

Table 1 Parameters and values

Symbol	Definition	Units	Value	Source
m	Mass of the packaged product	Kg	1–11	[2]
k_0	Coupling stiffness of the cushioning material	Ncm^{-1}	2–15	[2]
d_b	Compression limit of the cushioning material	–	0.8–1.2	[2]
h	Dropping Height	m	0.01–0.3	[2]
g	Gravity acceleration	m/sec^2	0.8–0.99	[2]
$\sqrt{2gh}$	Dropping shock velocity of the product	m/sec	–	[2]
r	Compression limit of the cushioning material	Ncm^{-3}	0.01–0.3	[2]
a	Compression limit of the cushioning material	Ncm^{-5}	0.0001–0.3	[2]

Fig. 1 The dynamic behavior of $x(\tau)$ using fractal FDO of Caputo and ABC types **Fig. 2** The dynamic behavior of $y(\tau)$ using fractal FDO of Caputo and ABC types

asymptotically to the steady states. Observing Fig. 1, we can easily see that the dynamic of variate x dies out earlier for $\mu = 0.98, 0.96, 0.94, 0.92, 0.9$ and $\beta = 1$ using ABC operator than the Caputo operator. Similar behaviour is true for variate y (Fig. 2). When observing 1D (Figs. 1, 2) or 2D (Fig. 3) phase plots, for $\mu = 1 = \beta$, the system remains chaotic and there are 11 periodic loops. But if we decrease the fractional

index $\mu = 2\%$ and $\beta = 1$, the systems chaotic goes down gradually. CPU time fractal fractional Caputo and fractal fractional ABC for Figs. 1, 2 and 3 are 1462.226164 seconds and 1501.7666166 seconds. An analysis of Figs. 4, 5 and 6 showed that the fractal FV in Caputo sense increases its periodic movement gradually for the fractional order $\mu = 1$ and the fractal dimension $\beta = 0.95$. However, in the ABC sense,

Fig. 3 Phase plots of the system using fractal FDO of Caputo and ABC types

Fig. 4 The dynamic behavior of $x(\tau)$ using fractal FDO of Caputo and ABC types

Fig. 5 The dynamic behavior of $y(\tau)$ using fractal FDO of Caputo and ABC types

Fig. 6 Phase plots of the system using fractal FDO of Caputo and ABC types

its periodic movement about the equilibrium point remains for fractal FV. When we decrease the μ values 2%, i.e., 0.98, 0.96, 0.94, 0.92, 0.90, the system gradually converges to the steady states. When observing Fig. 6 for $\mu = 1$, $\beta = 0.95$, the system remains chaotic for both the fractal techniques. But for Caputo method, the amplitude of the periodic loops is increasing with the passage of time, that is why you clusters of blue lines in 2D phase plots. But when we reduce the fractional order μ by 0.02 and $\beta = 0.95$, the chaoticity of the system decreases slowly. Besides, there are 12 periodic loops for Caputo method and 7 for ABC method for $t = 0 \rightarrow 100$. CPU time fractal fractional Caputo and ABC for Figs. 4, 5 and 6 are 1481.489937 seconds and 1559.761734 seconds.

An analysis of the simulations 7–9 showed that, as the fractional order is decreased by 2% and the fractal dimension $\beta = 0.80$ is taken, the system converges to the equilibrium point asymptotically. However, it does not apply for $\mu = 1$ and $\beta = 0.80$. The $x(\tau)$ and $y(\tau)$ values gradually increase their periodicity and continue their motion. CPU time fractal

fractional Caputo and fractal fractional ABC for Figs. 7, 8 and 9 are 1504.855900 seconds and 1664.698435 seconds.

Finally, the guiding principle is that for integer value of the fractional order $\mu = 1$ and fractal dimensions $\beta = 1, 0.95$ and 0.80, the system has periodic motions and is highly nonlinear. However, when the fractional order 2% decreases and fractal dimensions $\beta = 1, 0.95$ and 0.80, the system approaches the steady states asymptotically.

Conclusion

In the present paper, the tangent nonlinear packaging equation was discussed in the context of the Fractal fractional Caputo and ABC derivatives. These schemes are introduced here to obtain the numerical approximation of the solutions of the proposed model. For different values of fractal dimensions and fractal orders, the numerical solutions were presented graphically. It can be seen that the results

Fig. 7 The dynamic behavior of $x(\tau)$ using fractal FDO of Caputo and ABC types

Fig. 8 The dynamic behavior of $y(\tau)$ using fractal FDO of Caputo and ABC types

Fig. 9 Phase plots of the system using fractal FDO of Caputo and ABC types

obtained using the FV are in good agreement with the classical results early studied in the literature. The major finding of the present study is attributed to the performance of fractal fractional Atangana–Baleanu Caputo with fractal fractional Caputo method. Hence, for this particular nonlinear model, it is concluded that the fractal fractional Atangana–Baleanu Caputo is more capable enough to design real world problems. Since the tangent nonlinear equation is a complex physical phenomenon, it is, therefore believed that fractional derivative with Atangana bi-order is suitable to see the complexities of the under observed models. It would be interesting for the body of literature in this domain to compare the solutions presented in this work with the ones obtained using the Grunwald–Letnikov nonstandard finite difference method, which is a numerical scheme. This comparative study established the best ways to approach the exact solution of the packaging systems model. We believe that the future of mathematical modeling for every complex

real-world problem depends on these new fractal fractional differential operators.

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