Asymptotic expansions for certain exponential-type operators connected with $2x^{3/2}$

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Abstract

In the present paper, we consider the complete asymptotic expansion of certain exponential-type operators connected with $2x^{3/2}$. Also, a modification of such exponential-type operators is provided, which preserve the function e^{Ax} .

Keywords Exponential-type operators · Asymptotic expansion · Modified Bessel function · Degree of approximation

Mathematics Subject Classification 41A36 · 41A25

Introduction

Ismail and May [5, (3.16)] defined the operators

$$\begin{aligned} (T_n f)(x) &= \int_0^\infty k_n(x, t) f(t) dt \\ &= e^{-n\sqrt{x}} \left\{ n \int_0^\infty e^{-nt/\sqrt{x}} t^{-1/2} I_1(2n\sqrt{t}) f(t) dt + f(0) \right\}, \end{aligned} \quad (1)$$

where $x \in (0, \infty)$. The kernel is given by

$$k_n(x, t) = e^{-n\sqrt{x}} \left(n e^{-nt/\sqrt{x}} t^{-1/2} I_1(2n\sqrt{t}) + \delta(t) \right), \quad (2)$$

where $\delta(t)$ is the Dirac delta function and I_1 denotes the modified Bessel function of first kind defined by

$$I_1(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{1+2k}}{k! \Gamma(k+2)}.$$

The operators (1) are exponential-type operators as the kernels $k_n(x, t)$ of the operators T_n satisfy the partial differential equation

$$\frac{\partial}{\partial x} [k_n(x, t)] = \frac{n(t-x)}{p(x)} k_n(x, t), \quad (3)$$

with $p(x) = 2x^{3/2}$. The operators (1) reproduce constant and linear functions.

Note that the operators (1) can alternatively be written in the form

$$\begin{aligned} (T_n f)(x) &= \frac{n}{\sqrt{x}} \sum_{k=1}^{\infty} s_k(n\sqrt{x}) \int_0^\infty s_{k-1}\left(\frac{nt}{\sqrt{x}}\right) f(t) dt \\ &\quad + e^{-n\sqrt{x}} f(0), \end{aligned} \quad (4)$$

where $x \in (0, \infty)$ and

$$s_k(x) = e^{-x} \frac{x^k}{k!}.$$

We observe that the operators T_n are closely related to the well-known Phillips operators given by

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$$(P_n f)(x) = n \sum_{k=0}^{\infty} s_k(nx) \int_0^{\infty} s_{k-1}(nt) f(t) dt + e^{-nx} f(0). \quad (5)$$

Obviously, we have $(P_{n/\sqrt{x}} f)(x) = (T_n f)(x)$, but unlike the operators T_n , Phillips operators are not exponential type operators.

The operators T_n individually have not been studied by researchers much, due to its complicated behaviour. Very recently, Abel and Gupta [1] studied the rate of convergence of these operators on functions of bounded variation. Also, Gupta [2] and Gupta et al. [3] resp. gave some direct results in ordinary and simultaneous approximation. In the present article, we establish the complete asymptotic expansion of the operators T_n as n tends to infinity.

Asymptotic expansion

In the following, we use the falling factorial given by $z^0 = 1$, and $z^m = z(z-1)(z-2) \cdots (z-m+1)$, for $m \in \mathbb{N}$.

Firstly, we derive a concise representation of the r -th order moment $(T_n e_r)(x)$ where $e_r(t) = t^r$, $r \in \mathbb{N}_0$, denote the monomials. Obviously, we have $(T_n e_0)(x) = 1$.

Lemma 1 For $r \in \mathbb{N}$, the r -th order moment of the Ismail–May operators T_n has the representation

$$(T_n e_r)(x) = \sum_{k=0}^{r-1} \frac{1}{n^k} \binom{r-1}{k} r^k x^{r-\frac{k}{2}}.$$

Proof For $r \in \mathbb{N}$, we have

$$\begin{aligned} (T_n e_r)(x) &= e^{-n\sqrt{x}} \left\{ n \int_0^{\infty} e^{-nt/\sqrt{x}} t^{-1/2} I_1(2n\sqrt{t}) e_r(t) dt + e_r(0) \right\} \\ &= e^{-n\sqrt{x}} \left\{ n \sum_{v=0}^{\infty} \frac{n^{2v+1}}{v!(v+1)!} \int_0^{\infty} e^{-nt/\sqrt{x}} t^{v+r} dt + \delta_{r,0} \right\} \\ &= e^{-n\sqrt{x}} \sum_{v=0}^{\infty} \frac{n^{2v+2} \left(\frac{\sqrt{x}}{n}\right)^{v+r+1}}{v!} \frac{(v+r)!}{(v+1)!} \\ &= e^{-n\sqrt{x}} n^2 \left(\frac{\sqrt{x}}{n}\right)^{r+1} \left[\left(\frac{\partial}{\partial z}\right)^{r-1} z^r \exp\left(n^2 \frac{\sqrt{x}}{n} z\right) \right]_{z=1}. \end{aligned}$$

Application of the Leibniz rule for differentiation yields

$$\begin{aligned} (T_n e_r)(x) &= e^{-n\sqrt{x}} \frac{x^{(r+1)/2}}{n^{r-1}} \sum_{k=0}^{r-1} \binom{r-1}{k} r^k z^{r-k} \\ &\quad \cdot \left(n\sqrt{x} \right)^{r-1-k} e^{n\sqrt{x}z} \Big|_{z=1} \\ &= r! \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{x^{r-\frac{k}{2}}}{n^k (r-k)!} \end{aligned}$$

which is the desired formula. $\square$

Now we turn to the central moments $(T_n \psi_x^s)(x)$. For real x , we define $\psi_x(t) = t - x$.

Lemma 2 For $s = 1, 2, \dots$, the central moment of the Ismail–May operator T_n possesses the representation

$$(T_n \psi_x^s)(x) = \sum_{k=\lfloor (s+1)/2 \rfloor}^{s-1} n^{-k} \binom{s}{k} k^{s-k} (k-1)^{2k-s} x^{s-\frac{k}{2}}.$$

Proof For $s \in \mathbb{N}$, we have

$$\begin{aligned} (T_n \psi_x^s)(x) &= \sum_{r=0}^s \binom{s}{r} (-x)^{s-r} \sum_{k=0}^{r-1} \frac{1}{n^k} \binom{r-1}{k} r^k x^{r-\frac{k}{2}} \\ &= \sum_{k=0}^{s-1} n^{-k} \sum_{r=k+1}^s (-x)^{s-r} \binom{s}{r} \binom{r-1}{k} (r-1)^k x^{r-\frac{k}{2}}. \end{aligned}$$

Using the binomial identity $\binom{s}{r} \binom{r}{k} = \binom{s}{k} \binom{s-k}{r-k}$ we obtain

$$\begin{aligned} (T_n \psi_x^s)(x) &= \sum_{k=0}^{s-1} n^{-k} \binom{s}{k} x^{s-\frac{k}{2}} \sum_{r=k+1}^s (-1)^{s-r} \binom{s-k}{r-k} (r-1)^k \\ &= \sum_{k=0}^{s-1} n^{-k} \binom{s}{k} x^{s-\frac{k}{2}} \sum_{r=1}^{s-k-r} (-1)^{s-k} \binom{s-k}{r} \\ &\quad \left[\left(\frac{\partial}{\partial z}\right)^k (z^{r+k-1}) \right]_{z=1} \\ &= \sum_{k=0}^{s-1} n^{-k} \binom{s}{k} x^{s-\frac{k}{2}} \left[\left(\frac{\partial}{\partial z}\right)^k z^{k-1} (z-1)^{s-k} \right]_{z=1} \end{aligned}$$

and the desired formula follows by an application of the Leibniz rule for differentiation. $\square$

Lemma 3 The central moments of the operators T_n satisfy

$$(T_n \psi_x^s)(x) = O(n^{-\lfloor (s+1)/2 \rfloor}) \quad (n \rightarrow \infty).$$

Let $C_\gamma[0, \infty)$ be the class of continuous functions f on $[0, \infty)$ satisfying the exponential growth condition $f(t) = O(e^{\gamma t})$ as $t \rightarrow \infty$, for some $\gamma > 0$.

The next theorem is the main result of this section.

Theorem 1 Let $q \in \mathbb{N}$ and $x \in (0, \infty)$. For each function $f \in C_\gamma[0, \infty)$, which has a derivative of order $2q$ at the point x , the operators T_n possess the asymptotic expansion

$$(T_n f)(x) = f(x) + \sum_{k=1}^q \frac{1}{k!n^k} \sum_{s=1}^k \binom{k}{s} (k-1)^{k-s} f^{(k+s)}(x) x^{s+\frac{k}{2}} + o(n^{-q})$$

as $n \rightarrow \infty$.

Proof We apply a general result by Sikkema [7]. In view of the localization result (Proposition 1), we can assume that the function f has (at most) polynomial growth as the variable tends to infinity. Using Lemma 2, we obtain

$$\begin{aligned} (T_n f)(x) &= f(x) + \sum_{s=1}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{k=\lfloor (s+1)/2 \rfloor}^{s-1} n^{-k} \binom{s}{k} k^{s-k} \\ &\quad (k-1)^{2k-s} x^{s-\frac{k}{2}} + o(n^{-q}) \\ &= f(x) + \sum_{k=1}^q n^{-k} \sum_{s=k+1}^{2k} \frac{f^{(s)}(x)}{s!} \binom{s}{k} k^{s-k} \\ &\quad (k-1)^{2k-s} x^{s-\frac{k}{2}} + o(n^{-q}) \\ &= f(x) + \sum_{k=1}^q n^{-k} \sum_{s=1}^k \frac{f^{(k+s)}(x)}{k!s!} k^s \\ &\quad (k-1)^{k-s} x^{s+\frac{k}{2}} + o(n^{-q}) \\ &= f(x) + \sum_{k=1}^q \frac{1}{k!n^k} \sum_{s=1}^k \binom{k}{s} \\ &\quad (k-1)^{k-s} f^{(k+s)}(x) x^{s+\frac{k}{2}} + o(n^{-q}) \end{aligned}$$

as $n \rightarrow \infty$. $\square$

More explicitly, we have

$$\begin{aligned} (T_n f)(x) &= f(x) + \frac{x^{3/2}}{n} f^{(2)}(x) \\ &\quad + \frac{x^2}{n^2} \left(f^{(3)}(x) + \frac{x}{2} f^{(4)}(x) \right) \\ &\quad + \frac{x^{5/2}}{n^3} \left(f^{(4)}(x) + x f^{(5)}(x) + \frac{x^2}{6} f^{(6)}(x) \right) \\ &\quad + o(n^{-3}) \end{aligned}$$

as $n \rightarrow \infty$.

A direct corollary is the following Voronovskaja-type formula.

Corollary 1 Let $x \in \mathbb{R}$. For each function $f \in C_\gamma[0, \infty)$, which has a second derivative at the point x , we have the asymptotic relation

$$\lim_{n \rightarrow \infty} n \left((T_n f)(x) - f(x) \right) = x^{3/2} f''(x).$$

Now we prove a localization result which is of use in the proof of Theorem 1. It is of interest in its own. In what follows let $\exp$ denote the exponential function. For real γ , we write $\exp_\gamma(t) := e^{\gamma t}$.

Proposition 1 (Localisation theorem) Let $\gamma, \delta > 0$ and fix $x > 0$. If $f \in C_\gamma[0, \infty)$ vanishes on the interval $(x - \delta, x + \delta) \cap (0, \infty)$ then

$$(T_n f)(x) = O(n^{-m}) \quad (n \rightarrow \infty),$$

for arbitrary large $m > 0$.

Proof Let $m \in \mathbb{N}$. For a certain positive constant C , we have

$$\begin{aligned} |(T_n f)(x)| &\leq \int_{|t-x| \geq \delta} k_n(x, t) e^{\gamma t} dt \\ &\leq C \delta^{-m} \int_0^\infty k_n(x, t) |t-x|^m e^{\gamma t} dt. \end{aligned}$$

Application of the Schwarz inequality yields

$$|(T_n f)(x)| \leq C \delta^{-m} \sqrt{(T_n \exp_{2\gamma})(x)} \sqrt{(T_n \psi_x^{2m})(x)}.$$

The first root is finite, by [5, Prop. 2.5], if n is sufficiently large. The second root satisfies

$$\sqrt{(T_n \psi_x^{2m})(x)} = O(n^{-m}) \quad (n \rightarrow \infty),$$

by Lemma 3. This completes the proof. $\square$

Preservation of the exponential function e^{Ax}

King [6] considered a modification of the Bernstein polynomials so as to preserve the function e_2 . Also the moments of several operators have been discussed in [4]. We start with the following modified form of operators (1) with $(\tilde{T}_n f)(x) = (T_n f)(a_n(x))$, i.e.,

$$\begin{aligned} (\tilde{T}_n f)(x) &= \int_0^\infty k_n(a_n(x), t) f(t) dt \\ &= e^{-n\sqrt{a_n(x)}} \left\{ n \int_0^\infty e^{-nt/\sqrt{a_n(x)}} t^{-1/2} I_1(2n\sqrt{t}) f(t) dt + f(0) \right\}, \end{aligned} \quad (6)$$

such that

$$e^{Ax} = (\tilde{T}_n \exp_A)(x) = \int_0^\infty k_n(a_n(x), t) e^{At} dt,$$

which is the case if

$$a_n(x) = \frac{x(2n^2 + A^2x) + Ax\sqrt{A^2x^2 + 4xn^2}}{2n^2}.$$

This modified form of operators preserves constants as well as the exponential function $\exp_A$, but loose to preserve the linear functions. We have the following limits:

$$\lim_{n \rightarrow \infty} a_n(x) = x = \lim_{A \rightarrow 0} a_n(x).$$

Also, we have

$$\lim_{n \rightarrow \infty} a'_n(x) = 1.$$

After simple computations, we derive the following representation.

Lemma 4 For real numbers C , it holds

$$(\tilde{T}_n \exp_C)(x) = \exp\left(\frac{nCa_n(x)}{n - C\sqrt{a_n(x)}}\right).$$

We may observe here that the r -th order moment $\tilde{T}_n e_r$ is given by

$$(\tilde{T}_n e_r)(x) = \left[\left(\frac{\partial}{\partial C} \right)^r \exp\left(\frac{nCa_n(x)}{n - C\sqrt{a_n(x)}}\right) \right]_{C=0}.$$

In particular, we have

$$\begin{aligned} (\tilde{T}_n e_0)(x) &= 1, \quad (\tilde{T}_n e_1)(x) = a_n(x), \\ (\tilde{T}_n e_2)(x) &= [a_n(x)]^2 + \frac{2[a_n(x)]^{3/2}}{n}, \\ (\tilde{T}_n e_3)(x) &= [a_n(x)]^3 + \frac{6[a_n(x)]^{5/2}}{n} + \frac{6[a_n(x)]^2}{n^2}, \\ (\tilde{T}_n e_4)(x) &= [a_n(x)]^4 + \frac{12[a_n(x)]^{7/2}}{n} \\ &\quad + \frac{36[a_n(x)]^3}{n^2} + \frac{24[a_n(x)]^{5/2}}{n^3}. \end{aligned}$$

Remark 1 The central moments of the operators $\tilde{T}_n$ have the representation

$$(\tilde{T}_n \psi_x^s)(x) = \left[\left(\frac{\partial}{\partial C} \right)^s \exp\left(-Cx + \frac{nCa_n(x)}{n - C\sqrt{a_n(x)}}\right) \right]_{C=0}.$$

Direct computation yields

$$\begin{aligned} \lim_{n \rightarrow \infty} n(\tilde{T}_n \psi_x)(x) &= Ax^{3/2}, \\ \lim_{n \rightarrow \infty} n(\tilde{T}_n \psi_x^2)(x) &= 2x^{3/2}, \\ \lim_{n \rightarrow \infty} n^2(\tilde{T}_n \psi_x^3)(x) &= 6x^2(1 + Ax), \\ \lim_{n \rightarrow \infty} n^2(\tilde{T}_n \psi_x^4)(x) &= 12x^3. \end{aligned}$$

Hence, for $s = 0, 1, 2, 3, 4$, the central moments satisfy

$$(\tilde{T}_n \psi_x^s)(x) = O_x(n^{-(s+1)/2}), \quad n \rightarrow \infty.$$

The proof of the next lemma uses standard techniques.

Corollary 2 Let γ and δ be any two positive real numbers and $[a, b] \subset (0, \infty)$ be any bounded interval. Then, for any $m > 0$ there exists a constant M depending on m only such that

$$\left(\int_{|t-x| \geq \delta} k_n(a_n(x), t) e^{\gamma t} dt \right) \leq Mn^{-m},$$

where $\|\cdot\|$ is the sup-norm over $[a, b]$.

By Remark 1 and Corollary 2, we have the following asymptotic formula for the modified operators $\tilde{T}_n$.

Theorem 2 Let $f \in C_\gamma(0, \infty)$ for some $\gamma > 0$. If f'' exists at a point $x \in (0, \infty)$, then we have

$$\lim_{n \rightarrow \infty} n \left((\tilde{T}_n f)(x) - f(x) \right) = Ax^{3/2}f'(x) + x^{3/2}f''(x).$$

Without a proof, we state the following formula for the first derivative.

Theorem 3 Let $f \in C_\gamma(0, \infty)$ admitting the derivative of 3-rd order at a fixed point $x \in (0, \infty)$, then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left((\tilde{T}_n f)'(x) - f'(x) \right) &= \frac{3Ax^{1/2}}{2} f^{(1)}(x) \\ &\quad + \left(\frac{3}{2}x^{1/2} + Ax^{3/2} \right) f^{(2)}(x) \\ &\quad + x^{3/2} f^{(3)}(x). \end{aligned}$$

The above result can be derived by using the fact that the kernel of the operator $(\tilde{T}_n f)(x)$ satisfies the partial differential equation

$$\begin{aligned} \frac{\partial}{\partial x} k_n(a_n(x), t) &= \frac{n(t - a_n(x))}{2[a_n(x)]^{3/2}} k_n(a_n(x), t) \\ &\quad \left(1 + \frac{A^2x}{n^2} + \frac{A^3x^2 + 3n^2Ax}{n^2\sqrt{A^2x^2 + 4n^2x}} \right). \end{aligned}$$

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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