



A computational approach for solving fractional integral equations based on Legendre collocation method

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Abstract

In this paper, we propose a new method for solving second-order fractional integral equations. By using of spectral collocation method based on Gauss–Legendre–Lobatto points, a system of algebraic equations is obtained and then, we get a solution based on Legendre polynomials. If L^2 norm is used, this proposed method has a favorable convergence rate, which is shown by some examples.

Keywords Legendre spectral collocation method · Fractional integral equation · Fractional order

Introduction

Fractional calculus has gained importance and popularity during the past three decades or so, due to mainly its demonstrated applications in numerous seemingly diverse fields of science and engineering. The advantage of fractional calculus becomes apparent in modeling mechanical and electrical properties of real material, as well as in the description of many fractional fields (see for details [1, 2]). In recent years, there have been many studies of fractional integral and differential equations (for instance, see [3–13]).

The theory of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. The Volterra integral equations were introduced in first by Vito Volterra and then studied by Traian Lalescu in his 1908 thesis. Throughout the last decade, some researchers have paid attention to the concept of fractional integral equations as Volterra kind [14–16]. In this paper,

we are concerned with the numerical solution of the following fractional Volterra integral equation:

$$y(x) = a(x)I^\alpha\{b(x)y(x)\} + f(x), \quad (1)$$

where I^α is the fractional integral of order $0 < \alpha < 1$, a, b and f are continuous real functions on $[0, T]$, y is the unknown function that must be determined. We directly observe that the fractional differential equation of α order [15]

$$D^\alpha y(x) = H(x, y(x)), \quad a \leq x \leq b,$$

can be written immediately as the fractional Volterra integral equation of second kind as

$$y(x) = y(a) + I^{\alpha+1}H(x, y(x)).$$

Based on the above description, we intend to use numerical methods for solving the form of the Volterra fractional integral equation as from Eq. (1). Existence and uniqueness of the solution of Eq. (1), in addition to some analytical properties and important inequalities, are investigated in [17]. Some stability results for fractional integral equation are found in [18]. The authors in [17] applied the product trapezoidal method based on Picard iteration for approximating solution at the mesh points. Also, Atangana and et al. [19] use the Simpson's rules for solving special case of Volterra fractional integral equations. Spectral methods are an emerging area in the field of applied science and engineering. These methods provide a computational approach that has achieved substantial popularity

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over the last three decades. They have been applied successfully to numerical simulations of many problems in fractional calculus [20–25]. To the best of our knowledge, fractional derivative and integral are global, i.e., they are defined by an interval over the whole interval $[0, T]$, and therefore global methods, such as spectral methods, are better suited for this equation.

We introduce a collocation method based on the Gauss–Legendre interpolation for solving Eq. (1). Inspired by the work of [26], we extend the approach to Eq. (1) and provide a rigorous convergence analysis for the Legendre spectral collocation method. We show that an approximate solution is convergent in L^2 and L^∞ norms. The results obtained in this paper may be considered as a generalization of the result in [17].

The structure of this paper is as follows: In Sect. 2, some necessary definitions and mathematical tools of the fractional calculus which are required for our subsequent developments are introduced. In Sect. 3, the Legendre spectral collocation method of the fractional integral equation of second kind is obtained. After this section, we discuss about convergence analysis and then, to show efficiency of Legendre spectral collocation method, some numerical experiments are presented in Sect. 5. Some conclusions are given in Sect. 6.

Basic definitions and notations

For the concept of fractional integral, we will adopt Riemann–Liouville definition. Throughout this paper, integral operator will be Riemann–Liouville fractional integral operator.

Definition 1.2 A real function f on $[0, T]$ is said to be in the space C_μ , $\mu \in \mathbb{R}$, if there exists a real number $p (> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1 \in C[0, T]$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2.2 The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x \in [0, T],$$

(2)

$$I^0 f(x) = f(x),$$

where $\Gamma(\cdot)$ is the Gamma function where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Some properties of the Riemann–Liouville fractional integral are

$$(i) I^\alpha I^\beta = I^{\alpha+\beta}, \quad (ii) I^\alpha I^\beta = I^\beta I^\alpha,$$

$$(iii) I^\alpha (x-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} (x-a)^{\alpha+\nu},$$

where $\alpha, \beta \geq 0$ and $\nu > -1$.

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where $\alpha, \beta \geq 0$ and $\nu > -1$.

Let $\omega^{q_1, q_2}(x) = (1-x)^{q_1}(1+x)^{q_2}$ be a weight function in the usual sense, for $q_1, q_2 > -1$. The Jacobi polynomials $\{J_n^{q_1, q_2}\}_{n=0}^\infty$ are orthogonal with respect to ω^{q_1, q_2} over $[-1, 1]$. The set of Jacobi polynomials is a complete $L^2_{\omega^{q_1, q_2}}[-1, 1]$ —orthogonal system, namely [23]

$$\int_{-1}^1 J_n^{q_1, q_2} J_m^{q_1, q_2} \omega^{q_1, q_2}(x) dx = \gamma_n^{q_1, q_2} \delta_{nm},$$

(3)

where δ_{nm} is the Kronecker function, and

$$\gamma_n^{q_1, q_2} = \begin{cases} \frac{2^{q_1+q_2+1} \Gamma(q_1+1) \Gamma(q_2+1)}{\Gamma(q_1+q_2+2)}, & n = 0, \\ \frac{2^{q_1+q_2+1} \Gamma(n+q_1+1) \Gamma(n+q_2+1)}{(2n+q_1+q_2+1)n! \Gamma(n+q_1+q_2+1)}, & n \geq 1. \end{cases}$$

In particular, we find

$$J_0^{q_1, q_2}(x) = 1, J_1^{q_1, q_2}(x) = \frac{1}{2}(q_1+q_2+2)x + \frac{1}{2}(q_1-q_2).$$

Let $S_N[-1, 1]$ be the set of all polynomials of degree $\leq N (N \geq 0)$. Thus, if we denote by $\{x_j^{q_1, q_2}, \omega_j^{q_1, q_2}\}_{j=0}^N$, the nodes and the corresponding Christoffel numbers of the standard Jacobi–Gauss interpolation on $\Lambda = (-1, 1)$, then we have

$$\int_{-1}^1 v(x) \omega_j^{q_1, q_2}(x) dx = \sum_{j=0}^N v(x_j^{q_1, q_2}) \omega_j^{q_1, q_2},$$

$$\forall v \in S_{2N-1}[-1, 1].$$

(4)

For any $u \in C(\Lambda)$, we denote by $p_{x, N}^{q_1, q_2} : C(\Lambda) \rightarrow S_N$ the Jacobi–Gauss interpolation operator, such that

$$p_{x, N}^{q_1, q_2} u(x_j^{q_1, q_2}) = u(x_j^{q_1, q_2}), \quad 0 \leq j \leq N.$$

(5)

It is clear that

$$p_{x,N}^{q_1,q_2} u(x) = \sum_{j=0}^N u_j^{q_1,q_2} J_j^{q_1,q_2}(x), \tag{6}$$

where

$$u_j^{q_1,q_2} = \frac{1}{\gamma_j^{q_1,q_2}} \sum_{j=0}^N u(x_j^{q_1,q_2}) J_j^{q_1,q_2}(x_j^{q_1,q_2}) \omega_j^{q_1,q_2}.$$

In special case, if $q_1 = q_2 = 0$, then the Jacobi polynomial is reduced to the Legendre polynomial. Thus, we recall that the Legendre polynomials $\{L_i(x); i = 0, 1, \dots\}$ are defined on Λ with the following recurrence formula [23, 24]:

$$L_{i+1}(x) = \frac{2i+1}{i+1} x L_i(x) - \frac{i}{i+1} L_{i-1}(x), \quad i = 1, 2, \dots,$$

with $L_0(x) = 1$ and $L_1(x) = x$. By using the Rodrigues formula $L_i(x) = \frac{(-1)^i}{2^i i!} D^i((1-x^2)^i)$, i.e., D is the differential operator, the Legendre polynomial has the expansion as following

$$L_i(x) = \frac{1}{2^i} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^k \frac{(2i-2k)!}{2^k k!(i-k)!(i-2k)!} x^{i-2k}. \tag{7}$$

The set $\{L_i(x); i = 0, 1, \dots\}$ is a complete orthogonal system in $L^2(\Lambda)$, and we have

$$(L_j, L_k) = \int_{-1}^1 L_j(x) L_k(x) dx = h_j \delta_{jk}, \tag{8}$$

where δ_{jk} is the Kronecker delta symbol and $h_j = \frac{2}{2j+1}$. Thus, for any $g \in L^2(\Lambda)$, we have

$$g(x) = \sum_{i=0}^{\infty} a_i L_i(x), \quad a_i = \frac{1}{h_i} \int_{-1}^1 g(x) L_i(x) dx. \tag{9}$$

For simplicity, we let $x_j = x_j^{0,0}, \omega_j = \omega_j^{0,0}$ and $p_{x,N} = p_{N,x}^{0,0}$. Thus, according to Eq. (4), we have

$$\int_{-1}^1 v(x) dx = \sum_{j=0}^N v(x_j) \omega_j, \quad \forall v \in S_{2N-1}[-1, 1], \tag{10}$$

where $x_j(0 \leq j \leq N)$, the zeros of $(1-x^2)L'_N(x)$, and $\omega_j = \frac{2}{N(N+1)(L_N(x_j))^2}$, $(0 \leq j \leq N)$ are nodes and Christoffel numbers of Gauss–Legendre–Lobatto interpolation on the classical interval $[-1, 1]$, respectively. The norm and discrete inner product in $L^2(\Lambda)$ are defined as

$$(u, v)_N = \sum_{j=0}^N u(x_j) v(x_j) \omega_j, \quad u_N = (u, u)^{\frac{1}{2}}. \tag{11}$$

In the next section, we use some special techniques for converting Eq. (1) to a problem that can be solved by the standard Legendre spectral collocation scheme. These topics will be presented in the next section.

The spectral collocation method

In this section, we shall propose a Legendre spectral collocation method for solving Eq. (1). The Legendre spectral collocation method takes advantage of both the Legendre polynomials and the Gauss–Legendre–Lobatto interpolation nodes (for more details, see [27]). The main idea is to use the method of discretizing the fractional integral equation to obtain a system of algebraic equations with unknown coefficients. As a result, the solution of the obtained system can be easily solved.

Again, let us to consider Eq. (1) by applying Definition 2.2 as

$$y(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x b(s) (x-s)^{\alpha-1} y(s) ds + f(x),$$

$$x \in I = [0, T].$$

Next, in enjambment, we let $\Lambda = [-1, 1]$. For ease of analysis, we transfer the problem (1) to an equivalent problem defined in Λ . More specifically, we apply the change of variable $t = \frac{2x}{T} - 1$ or $x = \frac{T}{2}(t+1)$, such that $t \in \Lambda$. Then, we have

$$y\left(\frac{T}{2}(t+1)\right) = \frac{a\left(\frac{T}{2}(t+1)\right)}{\Gamma(\alpha)} \int_0^{\frac{T}{2}(t+1)} b(s) \left(\frac{T}{2}(t+1) - s\right)^{\alpha-1} y(s) ds + f\left(\frac{T}{2}(t+1)\right), \quad t \in \Lambda. \tag{12}$$

Moreover, to transfer the integral interval $(0, \frac{T}{2}(t+1))$ to $(-1, t)$, we make the transformation $s = \frac{T}{2}(\mu+1)$. Then, Eq. (12) can be written as

$$y\left(\frac{T}{2}(t+1)\right) = \left(\frac{T}{2}\right)^{\alpha} \frac{a\left(\frac{T}{2}(t+1)\right)}{\Gamma(\alpha)} \int_{-1}^t b\left(\frac{T}{2}(\mu+1)\right) (t-\mu)^{\alpha-1} y\left(\frac{T}{2}(\mu+1)\right) d\mu + f\left(\frac{T}{2}(t+1)\right), \quad t \in \Lambda. \tag{13}$$

Further, if we let

$$\begin{aligned} Y(t) &= y\left(\frac{T}{2}(t+1)\right), & A(t) &= a\left(\frac{T}{2}(t+1)\right), \\ B(t) &= b\left(\frac{T}{2}(t+1)\right), & F(t) &= f\left(\frac{T}{2}(t+1)\right), \end{aligned} \tag{14}$$

then, Eq. (13) can be reduced to

$$\begin{aligned} Y(t) &= \left(\frac{T}{2}\right)^\alpha \frac{A(t)}{\Gamma(\alpha)} \int_{-1}^t B(\mu)(t-\mu)^{\alpha-1} \\ &Y(\mu)d\mu + F(t), \quad t \in \mathcal{A} = [-1, 1]. \end{aligned} \tag{15}$$

At last, under the following linear transformation

$$\mu = \mu(t, \theta) = \frac{t+1}{2}\theta + \frac{t-1}{2}, \quad \theta \in \mathcal{A}, \tag{16}$$

Eq. (15) becomes

$$\begin{aligned} Y(t) &= \left(\frac{T}{2}\right)^\alpha \left(\frac{t+1}{2}\right)^\alpha \frac{A(t)}{\Gamma(\alpha)} \int_{-1}^1 B(\mu(t, \theta))(1-\theta)^{\alpha-1} Y(\mu(t, \theta))d\theta \\ &+ F(t). \end{aligned} \tag{17}$$

The Legendre spectral collocation method for Eq. (17) is to seek $U \in S_N(\mathcal{A})$ ($N \geq 1$), such that

$$\begin{aligned} U(t) &= \frac{T^\alpha}{4^\alpha \Gamma(\alpha)} \\ &p_{t,N} \left\{ (t+1)^\alpha A(t) \int_{-1}^1 (1-\theta)^{\alpha-1} p_{\theta,N}^{\alpha-1,0}(B(\mu(t, \theta))U(\mu(t, \theta)))d\theta \right\} \\ &+ p_{t,N}(F(t)). \end{aligned} \tag{18}$$

Now, we want to describe a numerical implementation for Eq. (18). To this end, we use the following relations

$$\begin{aligned} U(t) &\cong \sum_{i=0}^N u_i L_i(t), \\ p_{t,N} \left(p_{\theta,N}^{\alpha-1,0}((t+1)^\alpha A(t)B(\mu(t, \theta))U(\mu(t, \theta))) \right) \\ &= \sum_{i=0}^N \sum_{j=0}^N d_{ij} L_i(t) J_j^{\alpha-1,0}(\theta). \end{aligned} \tag{19}$$

Then, by using (19) and then Eq. (3), a direct computation leads to the following equation

$$\begin{aligned} &\frac{T^\alpha}{4^\alpha \Gamma(\alpha)} \int_{-1}^1 (1-\theta)^{\alpha-1} p_{t,N} \left(p_{\theta,N}^{\alpha-1,0}((t+1)^\alpha A(t)B(\mu(t, \theta))U(\mu(t, \theta))) \right) d\theta \\ &= \frac{T^\alpha}{4^\alpha \Gamma(\alpha)} \sum_{i=0}^N \sum_{j=0}^N d_{ij} L_i(t) \int_{-1}^1 (1-\theta)^{\alpha-1} J_j^{\alpha-1,0}(\theta) d\theta \\ &= \frac{T^\alpha}{2^\alpha \Gamma(\alpha+1)} \sum_{i=0}^N d_{i0} L_i(t). \end{aligned} \tag{20}$$

Applying relations (3) to (6), d_{i0} can be readily obtained as

$$\begin{aligned} d_{i0} &= \frac{\alpha(2i+1)}{2^{\alpha+1}} \sum_{l_1}^N \sum_{l_2}^N (x_{l_1}+1)^\alpha A(x_{l_1}) B\left(\mu\left(x_{l_1}, \theta_{l_2}^{\alpha-1,0}\right)\right) \\ &U\left(\mu\left(x_{l_1}, \theta_{l_2}^{\alpha-1,0}\right)\right) L_i(x_{l_1}) \omega_{l_1} \omega_{l_2}^{\alpha-1,0}. \end{aligned}$$

Hence, by inserting Eqs. (19) and (20) in Eq. (18), we conclude

$$\sum_{i=0}^N u_i L_i(t) = \frac{T^\alpha}{2^\alpha \Gamma(\alpha+1)} \sum_{i=0}^N d_{i0} L_i(t). \tag{21}$$

Finally, by comparing the expansion coefficients of Eq. (21), we can get

$$u_i = \frac{T^\alpha}{2^\alpha \Gamma(\alpha+1)} d_{i0}, \quad 0 \leq i \leq N. \tag{22}$$

By replacing the above relation, we obtain the expression of $U(t)$ accordingly.

Convergence analysis

In this section, we want to carry out the error analysis for the numerical method (18) under $L^2(\mathcal{A})$ and $L^\infty(\mathcal{A})$. Here, we present some preparations as follow [20, 23]:

Let $L^2_{\omega^{\beta_1, \beta_2}}(\mathcal{A})$: the Jacobi-weighted L^2 Hilbert space with the scalar product

$$(u, v) = \int_{-1}^1 u(t)v(t)\omega(t)dt, \quad \forall u, v \in L^2_{\omega^{\beta_1, \beta_2}}(\mathcal{A}).$$

Therefore, the corresponding norm is

$$\|u\|_{L^2_{\omega^{\beta_1, \beta_2}}} = (u, u)^{\frac{1}{2}}.$$

Let

$$H^m_{\omega^{\beta_1, \beta_2}}(\mathcal{A}) = \left\{ u \in L^2_{\omega^{\beta_1, \beta_2}}(\mathcal{A}) : \frac{d^i u}{dx^i} \in L^2_{\omega^{\beta_1, \beta_2}}(\mathcal{A}), i = 0, 1, \dots, m \right\}$$

: the Jacobi-weighted Sobolev spaces with the norm and semi-norm, respectively,

$$\|u\|_{H^m_{\omega^{q_1, q_2}}}^2 = \sum_{k=0}^m \|\partial_t^k u\|_{L^2_{\omega^{q_1, q_2}}}^2,$$

and

$$|u|_{H^m_{\omega^{q_1, q_2}}} = \|\partial_t^m u\|_{\omega^{q_1+k, q_2+k}},$$

$H^m_{\omega^{q_1, q_2}}$ by its inner product is a Hilbert space. For the purpose of convenience, we write $L^2(A) = H^0_{\omega^{0,0}}$, $\|\cdot\|_2 = \|\cdot\|_{L^2(A)}$ and $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(A)}$.

Lemma 4.1 [23] *Assume that $u \in H^m_{\omega^{q_1, q_2}}$ with $q_1, q_2 > -1$, and $m \geq 1$. Then, for any integer number k such that $0 \leq k \leq m \leq N + 1$, the constant $C > 0$ exists such that the following relation holds:*

$$\|\partial_t^k (u - p_{t,N}^{q_1, q_2}(u))\|_{\omega^{q_1+k, q_2+k}} \leq CN^{k-m} \|\partial_t^m u\|_{\omega^{q_1+k, q_2+k}}. \tag{23}$$

In particular, for any $u \in H^m(A)$ with $1 \leq m \leq N + 1$, the above relation is reduced to the following result

$$\|u - p_{t,N}(u)\|_{H^1(A)} \leq CN^{\frac{3}{2}-m} \|\partial_t^m u\|_2. \tag{24}$$

Theorem 4.2 *For sufficiently large N , the Legendre spectral collocation approximation converges to the exact solution in L^2 -norm, i.e.,*

$$\|e_N\|_2 = \|Y - U\|_2 \rightarrow 0.$$

Proof Assume that $U(x)$ is obtained by using the Legendre spectral collocation method Eq. (1). Then, according to definition of $p_{t,N}$ in last section, we will have

$$\|e_N\|_2 \leq \|Y - p_{t,N}(Y)\|_2 + \|p_{t,N}(Y) - U\|_2. \tag{25}$$

By using Lemma 4.1, we can get for any integer $1 \leq m \leq N + 1$,

$$\|Y - p_{t,N}(Y)\|_2 \leq CN^{-m} \|\partial_t^m Y\|_{\omega^{m,m}}. \tag{26}$$

The rest of our proof is to show $\|p_{t,N}(Y) - U\|_2 \rightarrow 0$ as $N \rightarrow \infty$. To this end, along with definition Y in Eq. (17), for $N \geq 1$, we have

$$p_{t,N}Y(t) = \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} p_{t,N} \left(A(t) \int_{-1}^t (t-\mu)^\alpha F(B(\mu), Y(\mu)) d\mu \right) + p_{t,N}(F), \tag{27}$$

where,

$$G(B(\mu), Y(\mu)) = B(\mu(t, \theta))Y(\mu(t, \theta)).$$

Similarly, we can write

$$U(t) = \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} p_{t,N} \left\{ A(t) \int_{-1}^t (t-\mu)^{\alpha-1} p_{\mu,N}^{\alpha-1,0}(G(B(\mu), U(\mu))) d\mu \right\} + p_{t,N}(F), \tag{28}$$

where

$$p_{\mu,N}^{\alpha-1,0} v(\mu) = p_{\theta,N}^{\alpha-1,0} v(\mu(t, \theta))|_{\theta=\frac{2\mu-t-1}{t+1}}. \tag{29}$$

Before we proceed to the rest of proof, we present some relations that will be used later. By virtue of (29) and then using the standard Jacobi–Gauss quadrature formula (4), we obtain

$$\begin{aligned} \int_{-1}^t (t-\mu)^{\alpha-1} p_{\mu,N}^{\alpha-1,0} v(\mu) d\mu &= \left(\frac{1+t}{2}\right)^\alpha \int_{-1}^1 (1-\theta)^{\alpha-1} p_{\theta,N}^{\alpha-1,0} \\ &\quad (\mu(t, \theta)) d\theta \\ &= \left(\frac{1+t}{2}\right)^\alpha \sum_{j=0}^N v(\mu(t, \theta_j^{x-1,0})) \omega_j^{\alpha-1,0} \\ &= \left(\frac{1+t}{2}\right)^\alpha \sum_{j=0}^N v(\mu_j^{x-1,0}) \omega_j^{\alpha-1,0}. \end{aligned} \tag{30}$$

Again, in the same scheme, we have

$$\begin{aligned} \int_{-1}^t (t-\mu)^{\alpha-1} (p_{\mu,N}^{\alpha-1,0} v(\mu))^2 d\mu \\ = \left(\frac{1+t}{2}\right)^\alpha \sum_{j=0}^N v^2(\mu_j^{x-1,0}) \omega_j^{\alpha-1,0}. \end{aligned} \tag{31}$$

At last, by applying Lemma 4.1 and Eq. (29) together, for any integer $m(1 \leq m \leq N + 1)$, we conclude that

$$\begin{aligned} \int_{-1}^t (t-\mu)^{\alpha-1} (v(\mu) - p_{\mu,N}^{\alpha-1,0} v(\mu))^2 d\mu \\ = \left(\frac{1+t}{2}\right)^\alpha \int_{-1}^1 (1-\theta)^{\alpha-1} (v(\mu(t, \theta)) - p_{\theta,N}^{\alpha-1,0} v(\mu(t, \theta)))^2 d\theta \\ \leq CN^{-m} \left(\frac{1+t}{2}\right)^\alpha \int_{-1}^1 (\partial_\theta^m v(\mu(t, \theta)))^2 (1-\theta)^{\alpha+m-1} (1+\theta)^m d\theta \\ = CN^{-m} \int_{-1}^x (\partial_\mu^m v(\mu))^2 (x-\mu)^{\alpha+m-1} (1+\mu)^m d\mu. \end{aligned} \tag{32}$$

Now, let us explain the rest of proof. By subtracting Eq. (28) from (27), we derive that

$$p_{t,N}Y(t) - U(t) = \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \times p_{t,N} \left(A(t) \int_{-1}^t (t - \mu)^{\alpha-1} \left(G(B(\mu), Y(\mu)) - p_{\mu,N}^{\alpha-1,0}(G(B(\mu), U(\mu))) \right) d\mu \right).$$

The above formula can be rewritten as

$$p_{t,N}Y(t) - U(t) = \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \times p_{t,N} \left(A(t) \int_{-1}^t (t - \mu)^{\alpha-1} \left(G(B(\mu), Y(\mu)) - p_{\mu,N}^{\alpha-1,0}(G(B(\mu), Y(\mu))) \right) d\mu \right) + \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \times p_{t,N} \left(A(t) \int_{-1}^t (t - \mu)^{\alpha-1} \left(p_{\mu,N}^{\alpha-1,0}(G(B(\mu), Y(\mu))) - G(B(\mu), U(\mu)) \right) d\mu \right). \tag{33}$$

We let

$$E_1(t) = \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \times p_{t,N} \left(A(t) \int_{-1}^t (t - \mu)^{\alpha-1} \left(G(B(\mu), Y(\mu)) - p_{\mu,N}^{\alpha-1,0}(G(B(\mu), Y(\mu))) \right) d\mu \right),$$

$$E_2(t) = \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \times p_{t,N} \left(A(t) \int_{-1}^t (t - \mu)^{\alpha-1} \left(p_{\mu,N}^{\alpha-1,0}(G(B(\mu), Y(\mu))) - G(B(\mu), U(\mu)) \right) d\mu \right).$$



Fig. 2 The error function of Example 5.1 versus the number of interpolation points

Next, by using the standard Legendre–Gauss quadrature formula, we obtain

$$\|E_1\|_2 = \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left\{ \sum_{j=0}^N \omega_j A(x_j) \left(\int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} \left(G(B(\mu), Y(\mu)) - p_{\mu,N}^{\alpha-1,0} \Big|_{x_j} (G(B(\mu), Y(\mu))) \right) d\mu \right)^2 \right\}^{\frac{1}{2}}.$$

By using the Cauchy–Schwartz inequality and Eq. (32), we get

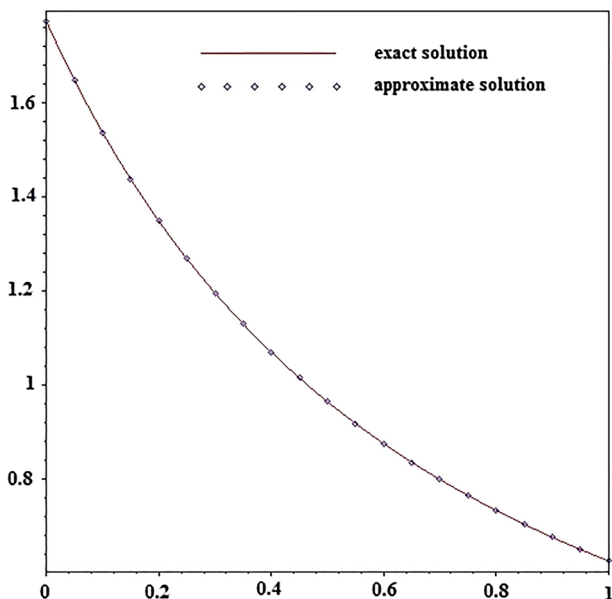


Fig. 1 Comparison between exact solution and approximate solution of Example 5.1 for $N = 10$

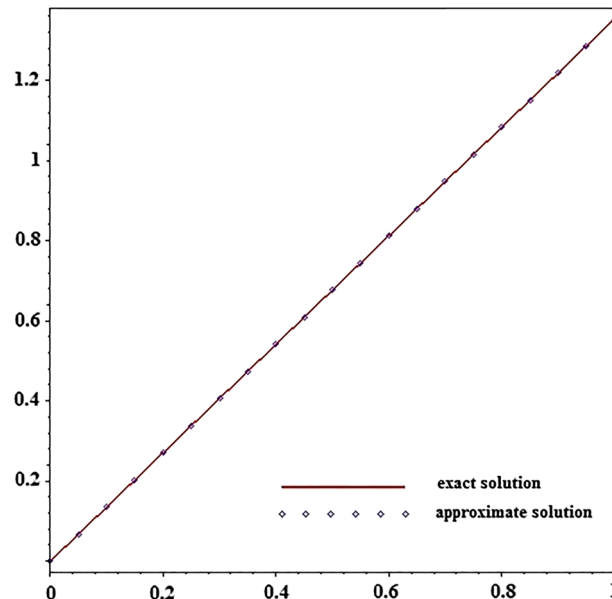


Fig. 3 Comparison between exact solution and approximate solution of Example 5.2 for $N = 6$

$$\begin{aligned} \|E_1\|_2 &\leq \max_{t \in A} A(t) \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left\{ \sum_{j=0}^N \omega_j \int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} d\mu \int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} \right. \\ &\quad \left. \left(G(B(\mu), Y(\mu)) - p_{\mu,N}^{\alpha-1,0} \Big|_{x_j} (G(B(\mu), Y(\mu))) \right)^2 d\mu \right\}^{\frac{1}{2}} = \max_{t \in A} A(t) \frac{T^\alpha}{2^\alpha \Gamma(\alpha+1)} \\ &\quad \times \left\{ \sum_{j=0}^N \omega_j (x_j + 1)^\alpha \int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} \right. \\ &\quad \left. \times \left(G(B(\mu), Y(\mu)) - p_{\mu,N}^{\alpha-1,0} \Big|_{x_j} (G(B(\mu), Y(\mu))) \right)^2 d\mu \right\}^{\frac{1}{2}} \\ &\leq C \max_{t \in A} A(t) \frac{T^\alpha}{2^\alpha \Gamma(\alpha+1)} N^{-m} \times \left\{ \sum_{j=0}^N \omega_j (x_j + 1)^\alpha \right. \\ &\quad \left. \times \int_{-1}^{x_j} \left(\partial_\mu^m G(B(\mu), Y(\mu)) \right)^2 (x_j - \mu)^{\alpha+m-1} (1 + \mu)^m d\mu \right\}^{\frac{1}{2}} \\ &= C_1 N^{-m} \partial_\mu^m G(B(\cdot), Y(\cdot))_{\omega^{\alpha+m-1,m}}. \end{aligned} \tag{34}$$

Similarly, by applying Eqs. (4) and (31) and the Cauchy–Schwartz inequality, we can get

$$\begin{aligned} \|E_2\|_2 &= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left\{ \sum_{j=0}^N \omega_j \left[A(x_j) \int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} \right. \right. \\ &\quad \left. \left. \left(p_{\mu,N}^{\alpha-1,0} (G(B(\mu), Y(\mu))) - G(B(\mu), U(\mu)) \right) d\mu \right]^2 \right\}^{\frac{1}{2}} \\ &\leq \max_{t \in A} A(t) \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left\{ \sum_{j=0}^N \omega_j \int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} d\mu \right. \\ &\quad \left. \times \int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} \left(p_{\mu,N}^{\alpha-1,0} \Big|_{x_j} (G(B(\mu), Y(\mu))) - G(B(\mu), U(\mu)) \right)^2 d\mu \right\}^{\frac{1}{2}} \\ &= \max_{t \in A} A(t) \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left\{ \sum_{j=0}^N \omega_j \frac{(x_j + 1)^\alpha}{\alpha} \sum_{i=0}^N \omega_i^{\alpha-1,0} \right. \\ &\quad \left. \left(G(B(\mu_i^{\alpha-1,0}), Y(\mu_i^{\alpha-1,0})) - G(B(\mu_i^{\alpha-1,0}), U(\mu_i^{\alpha-1,0})) \right)^2 \right\}^{\frac{1}{2}}. \end{aligned} \tag{35}$$

On the other hand, Jensen’s inequality for any convex function g on interval (a, b) is

$$g(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda g(t_1) + (1 - \lambda)g(t_2), \quad \forall t_1, t_2 \in (a, b), \quad \forall \lambda \in [0, 1].$$

Thus, since

$$\frac{d^2}{dt^2} (x_j + 1)^t = (x_j + 1)^t \ln^2(x_j + 1) > 0, \quad x_j \text{ is constant,}$$

then, for any given $x_j \in (-1, 1)$, $g(t) = (x_j + 1)^t$ is convex function and according to Jensen’s inequality on interval $[0, 1]$, we will have

$$g(t) = g((1 - t) \times 0 + t \times 1) \leq (1 - t)g(0) + tg(1). \tag{36}$$

Therefore, by applying the previous inequality, we yield the following inequality

$$\begin{aligned} \sum_{j=0}^N \omega_j (x_j + 1)^\alpha &\leq \sum_{j=0}^N \omega_j \{ (1 - \alpha) + \alpha(x_j + 1) \} \\ &= (1 - \alpha) \sum_{j=0}^N \omega_j + \alpha \sum_{j=0}^N \omega_j (x_j + 1) \\ &= (1 - \alpha) \int_A dt + \alpha \int_A (x + 1) dt \\ &= 2, \quad \forall \alpha \in [0, 1]. \end{aligned} \tag{37}$$

Next, by using relation (37), Eq. (35) will be reduced to

$$\begin{aligned} \|E_2\|_2 &\leq \max_{t \in A} A(t) \frac{T^\alpha}{2^{\alpha-1} \Gamma(\alpha+1)} \\ &\quad \times \max_{0 \leq j \leq N} \left\{ \int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} \right. \\ &\quad \left. \times \left(p_{\mu,N}^{\alpha-1,0} \Big|_{x_j} (G(B(\mu), Y(\mu))) - G(B(\mu), U(\mu)) \right)^2 d\mu \right\}^{\frac{1}{2}} \\ &\leq \max_{t \in A} A(t) \max_{t \in A} B(t) \frac{T^\alpha}{2^{\alpha-1} \Gamma(\alpha+1)} \\ &\quad \times \max_{0 \leq j \leq N} \left\{ \int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} \left(p_{\mu,N}^{\alpha-1,0} \Big|_{x_j} (Y(\mu)) - U(\mu) \right)^2 d\mu \right\}^{\frac{1}{2}} \\ &\leq \max_{t \in A} A(t) \max_{t \in A} B(t) \frac{T^\alpha}{2^{\alpha-1} \Gamma(\alpha+1)} \\ &\quad \times \max_{0 \leq j \leq N} \left\{ \left(\int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} \left(p_{\mu,N}^{\alpha-1,0} \Big|_{x_j} (Y(\mu)) - Y(\mu) \right)^2 d\mu \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\int_{-1}^{x_j} (x_j - \mu)^{\alpha-1} (Y(\mu) - U(\mu))^2 d\mu \right)^{\frac{1}{2}} \right\}. \end{aligned} \tag{38}$$

At last, along with Lemma 4.1 and relations (32), (37), the previous result yields

$$\begin{aligned} \|E_2\|_2 &\leq C_2 \left\{ CN^{-m} \max_{0 \leq j \leq N} \left(\int_{-1}^{x_j} \left(\partial_\mu^m Y(\mu) \right)^2 (x_j - \mu)^{\alpha+m-1} (1 + \mu)^m d\mu \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \frac{2}{\alpha} \left(\int_{-1}^1 (Y(\mu) - U(\mu))^2 d\mu \right)^{\frac{1}{2}} \right\} \leq C_2 CN^{-m} \partial_\mu^m Y_{\omega^{\alpha+m-1,m}} + \frac{2C_2}{\alpha} e_N. \end{aligned} \tag{39}$$

Finally, relations (33), (34), (39) prove that the approximation is convergent in L^2 -norm. Hence, the theorem is proved.

Numerical example

To show efficiency of our numerical method, the following examples which have exact solutions are considered.

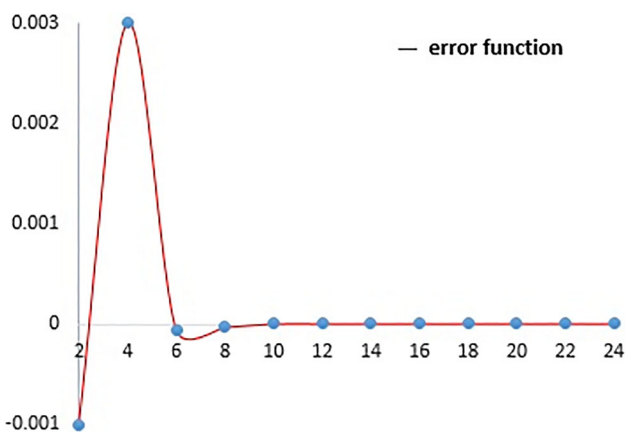


Fig. 4 The error function of Example 5.1 versus the number of interpolation points

Example 5.1 Consider the following fractional integral equation [17]

$$y(t) = \frac{0.01}{\Gamma(0.5)} t^{\frac{5}{2}} \int_0^t (t-s)^{-\frac{1}{2}} y(s) ds + \sqrt{\pi}(1+t)^{-\frac{3}{2}} - 0.02 \frac{t^3}{1+t},$$

$t \in [0, 1]$.

The corresponding exact solution is given by $y(t) =$

$\sqrt{\pi}(1+t)^{-\frac{3}{2}}$. Fig. 1 presents the approximate and exact solutions which are found in very good agreement. In Fig. 2, the numerical errors are plotted in L^2 -norm for $2 \leq N \leq 24$.

Example 5.2 Our second example is the following fractional integral equation [17]

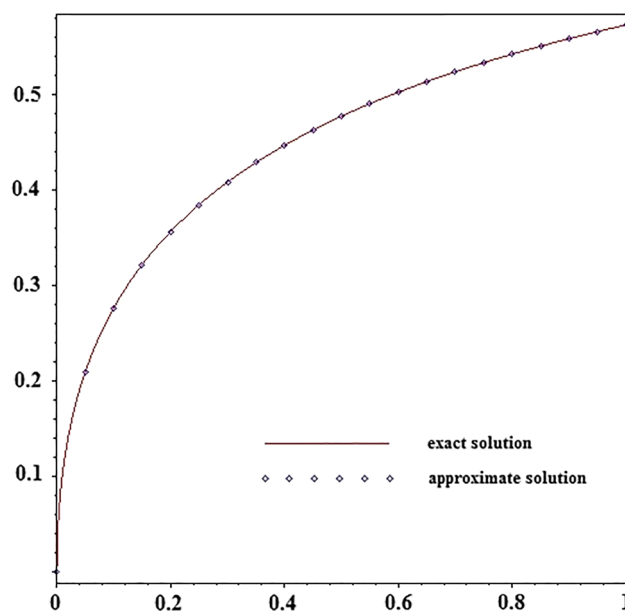


Fig. 5 Comparison between exact solution and approximate solution of Example 5.3 for $N = 16$

$$y(t) = \frac{1}{27\Gamma(\frac{2}{3})} \int_0^t s(t-s)^{-\frac{1}{3}} y(s) ds + \Gamma\left(\frac{2}{3}\right)t - \frac{1}{40}t^{\frac{8}{3}},$$

$t \in [0, 1]$.

The exact solution is $y(x) = \Gamma(\frac{2}{3})t$. We get the numerical solution of the above fractional integral equation as

$$y_1(t) = \Gamma\left(\frac{2}{3}\right)t,$$

$$y_2(t) = 1.21 \times 10^{-2} + \Gamma\left(\frac{2}{3}\right)t,$$

Table 1 Comparison of approximate solutions and exact solutions for various numbers N

t	Proposed method at $N = 8$	Proposed method at $N = 10$	The method in [17] for 10 iterations and 24 nodes	Exact solution
0	.0000000000	.0000000000	.0000000000	.0000000000
0.1	.2764215215	.2764215613	.2764220436	.2764215616
0.2	.3562117262	.3562117272	.3562107843	.3562117278
0.3	.4079815809	.4079815886	.4079882463	.4079815886
0.4	.4463937357	.4463937452	.4463885785	.4463937458
0.5	.4768434165	.4768434157	.4768335412	.4768434159
0.6	.5019754248	.5019754315	.5019745514	.5019754315
0.7	.5232972359	.5232972687	.5232971801	.5232972687
0.8	.5417539715	.5417539771	.5417553840	.5417539772
0.9	.5579785802	.5579785884	.5579785412	.5579785885
1	.5724164231	.5724164243	.5724161242	.5724164241

$$y_4(t) = 2.15 \times 10^{-9} + \Gamma\left(\frac{2}{3}\right)t + 3.12 \times 10^{-10}t^2 + 2.39 \\ \times 10^{-7}t^3 + 4.81 \times 10^{-9}t^4.$$

The numerical results of proposed method can be seen from Figs. 3 and 4. These results indicate that the spectral accuracy is obtained for this problem, although the given function f is not very smooth.

These results indicate that the spectral accuracy is obtained by using Legendre collocation method for this problem.

Example 5.3 Consider the following fractional integral equation [17]

$$y(t) = -\frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-\frac{1}{2}} y(s) ds + 2\sqrt{\frac{t}{\pi}}, \quad t \in [0, 1].$$

The exact solution is $y(t) = 1 - e^t \operatorname{erfc}(\sqrt{t})$, such that erfc is the complementary error function defined by

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-s^2} ds.$$

We list the numerical results in Table 1 to be able to test the predicted convergence rate. The comparison of the results for $N = 8$ and $N = 10$ with the iterative numerical method [17] is investigated in Table 1. In Fig. 5, we can observe that our numerical solutions coincide closely with the exact ones. At a glance, we can find out the results of Legendre spectral collocation method are satisfactory in the few steps.

Conclusion

In this paper, we present a spectral Legendre collocation approximation of a class of fractional integral equations of the second kind. The most important contribution of this paper is that the errors of approximations decay exponentially in L^2 -norm. We prove that our proposed method is effective and has high convergence rate and is better than some new methods introduced in this paper. The numerical experiments show that the proposed method is efficient and powerful numerical scheme for solving many fractional problems.

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